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On the non-existence of maximal solutions for hyperbolic differential equations*

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1. Introduction. The initial value problem for an ordinary differential equation

(1) $$u' = f(t, u) \quad \text{in} \quad J\colon\ 0 \leqslant t \leqslant T, \quad u(0) = u_0 .$$

and the characteristic initial value problem for the hyperbolic differential equation

(2) $$\begin{cases} u_{xy} = f(x, y, u) & \text{in} \quad R\colon\ 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b ,\\ u(x, 0) = \sigma(x) \text{ for } 0 \leqslant x \leqslant a, & u(0, y) = \tau(y) \text{ for } 0 \leqslant y \leqslant b \end{cases}$$

have many aspects in common. If $f(t, z)$ is bounded and continuous in $J\times(-\infty, \infty)$, then there is at least one solution of (1), existing in J (Peano's theorem). Moreover there exists a maximal solution u^* and a minimal solution u_* with the property that for every other solution u we have

$$u_*(t) \leqslant u(t) \leqslant u^*(t) \quad \text{in } J$$

(note that under our conditions every solution can be extended to J). Similarly, if $f(x, y, z)$ is bounded and continuous in $R\times(-\infty, \infty)$ and if the functions $\sigma(x)$ and $\tau(y)$ are of class C^1 in $[0, a]$ resp. $[0, b]$ with $\sigma(0) = \tau(0)$, then the problem (2) has at least one solution $u(x, y)$ in R with continuous derivatives u_x, u_y, u_{xy}. Apparently Satō [2] was the first one to prove this theorem. He actually treated a more general case

(3) $$u_{xy} = f(x, y, u, u_x, u_y)$$

as many other authors did (for a bibliography see [3]). A solution u^* of the problem (2) existing in R is called a *maximal solution*, if for any other solution u, existing in R, we have $u \leqslant u^*$ in R. As long as f is bounded and continuous, which will always be assumed here, a solution u, existing in a subrectangle $R(x_0, y_0) = [0, x_0]\times[0, y_0]$ with $(x_0, y_0) \in R$, can be

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extended to R. Therefore the given definition is equivalent with the following: u^* is a maximal solution in R, if for any solution u in $R(x_0, y_0)$, $(x_0, y_0) \in R$, we have $u \leqslant u^*$ in $R(x_0, y_0)$. A similar definition applies to the minimal solution. The existence of maximal and minimal solutions for the problem (2) is easily established, when $f(x, y, z)$ is increasing in z; see e.g. Walter ([3], p. 122). Zwirner [4] has stated that this result is true without a monotonicity condition, i.e. for bounded continuous f. For the characteristic initial value problem with respect to the equation (3), maximal and minimal solutions do exist if $f = f(x, y, z, p, q)$ is (in addition to some requirements which guarantee existence) monotonically increasing in z, p and q. It was announced by Santoro [1] that the same holds without a monotonicity condition.

On the contrary, we shall give here an example of a problem (2) with bounded and continuous f, which has no maximal and no minimal solution. A Cauchy problem with a similar behaviour will also be given. These examples establish a significant difference between the one-dimensional problem (1) and multi-dimensional problems like (2).

A better understanding of this situation is obtained if one compares the theorems on differential inequalities. In the one-dimensional case the inequalities

$$v' - f(t, v) < w' - f(t, w) \quad \text{in } J, \quad v(0) < w(0)$$

imply

$$v(t) < w(t) \quad \text{in } J$$

([3], p. 57). This fact is used in the standard existence proof for the maximal and minimal solution ([3], p. 59).

The corresponding theorem for the problem (2) reads as follows: If the inequalities

$$v_{xy} - f(x, y, v) < w_{xy} - f(x, y, w)$$

and

$$v(x, 0) + v(0, y) - v(0, 0) < w(x, 0) + w(0, y) - w(0, 0)$$

hold in R, then

$$v(x, y) < w(x, y) \quad \text{in } R\,.$$

This latter theorem holds—contrary to the first one—only for functions $f(x, y, z)$ which are increasing in z; see [3], p. 135 and the counter-example in [3], p. 140.

Since the result of Zwirner is not valid, the remarks in [3] in 20 III (p. 134) about one-sided uniqueness conditions for the problem (2) and at the end of 20 X (p. 140) are not correct.

2. A second order ordinary differential equation without a maximal solution. It is well known that the initial value problem for a second order equation

$$u'' = f(t, u) \quad \text{in} \quad J\colon\ 0 \leqslant t \leqslant T, \quad u(0) = u_0, \quad u'(0) = u_1$$

has, in general, no maximal solution. A maximal solution does exist, however, when $f(t, z)$ is increasing in z ([3], p. 97). We shall give here an example where no maximal and no minimal solution exists (to the author's knowledge no such example has been published). It will be needed later.

The linear differential equation

$$u'' = h(t)u, \quad \text{where} \quad h(t) = \frac{12}{t^2}\cdot\frac{5t-2}{3t-4} \tag{4}$$

has a polynomial solution

$$\varphi(t) = t^3(t-1)(3t-4)\,. \tag{5}$$

Since $h(t)$ is continuous in $(0, 4/3)$, there are, in this interval, two linearly independent solutions φ, ψ, and by d'Alembert's reduction method we get

$$\psi(t) = \varphi(t)\int_a^t \varphi^{-2}(\tau)\,d\tau \qquad (0 < a < 1)\,.$$

For $t \to +0$, φ behaves like t^3, ψ like t^{-2}. Therefore the only functions satisfying

$$u'' = h(t)u \quad \text{in } (0, 4/3) \quad \text{and} \quad \lim_{t\to+0} u(t) = 0 \tag{6}$$

are the functions $u = c\varphi(t)$ (c real).

Let $g(t, z)$ be the continuous function which is, for fixed t, equal to $h(t)z$ between the t-axis and the curve $\varphi(t)$ and constant otherwise, i.e.

$$g(t, z) = \begin{cases} 0 \text{ for } 0 \leqslant t \leqslant 1,\ z \leqslant 0 \text{ and for } 1 < t < 4/3,\ z \geqslant 0\,, \\ h(t)z \text{ for } 0 < t < 1,\ 0 < z < \varphi(t) \ldots 1 < t < 4/3,\ \varphi(t) < z < 0\,, \\ 12t(t-1)(5t-2) \text{ for } 0 \leqslant t \leqslant 1,\ z \geqslant \varphi(t) \ldots 1 < t < 4/3,\ z \leqslant \varphi(t)\,. \end{cases} \tag{7}$$

The function $g(t, z)$ is bounded and continuous in $[0, 6/5]\times(-\infty, \infty)$ and it satisfies a uniform Lipschitz condition with respect to z in $[\delta, 6/5]\times$ $\times(-\infty, \infty)$ for any $\delta > 0$. The initial value problem

$$u'' = g(t, u) \quad \text{in } J = [0,\ 6/5], \quad u(0) = u'(0) = 0 \tag{8}$$

has the solutions

$$u = c\varphi(t), \qquad 0 \leqslant c \leqslant 1$$

and only these.

In proving this statement we make use of the fact that $g(t, z)$ is increasing in z for $0 \leqslant t \leqslant 2/5$. This implies that maximal and minimal solutions exist and that a corresponding theorem on differential inequalities holds, as long as $0 \leqslant t \leqslant 2/5$; see [3], p. 97. Let $v(t)$ resp. $w(t)$ be a solution of the initial value problem

$$u'' = g(t, u) + \varepsilon \quad \text{for} \quad 0 \leqslant t \leqslant 2/5, \quad u(0) = u'(0) = \varepsilon$$

for $\varepsilon < 0$ resp. $\varepsilon > 0$, i.e. $v(t) = \varepsilon(1+t+t^2/2)$, $\varepsilon < 0$, and $w(t) = \varphi(t) + \varepsilon(1+t+t^2/2)$, $\varepsilon > 0$. For every solution u of (8) we have $v < u < w$; it follows that 0 and $\varphi(t)$ are the minimal and maximal solution respectively and that

$$(9) \qquad 0 \leqslant u(t) \leqslant \varphi(t) \quad \text{for} \quad 0 \leqslant t \leqslant 2/5 .$$

But every solution of (8) for which (9) holds, is also a solution of (6) in $(0, 2/5)$ and is therefore equal to $c\varphi(t)$, $0 \leqslant c \leqslant 1$. Since $g(t, z)$ satisfies a uniform Lipschitz condition in $[2/5, 6/5] \times (-\infty, \infty)$, the extension of the solution $c\varphi(t)$ to the interval $[0, 6/5]$ is uniquely given by $c\varphi(t)$; q.e.d.

It follows that the problem (8) has no maximal solution and no minimal solution in $J = [0, 6/5]$: neither $u^*(t) = \max\{\varphi(t), 0\}$ nor $u_*(t)$ $\min\{\varphi(t), 0\}$ is a solution in J.

3. A Cauchy problem without maximal and minimal solutions. Let, for $x_0 + y_0 > 0$, $R(x_0, y_0)$ be the triangle defined by the inequalities $x + y \geqslant 0$, $x \leqslant x_0$, $y \leqslant y_0$. Furthermore let $a > 0$ and $R = R(a, a)$.

If $f(x, y, z)$ is bounded and continuous in $R \times (-\infty, \infty)$, the Cauchy problem

$$(10) \quad u_{xy} = f(x, y, u) \quad \text{in} \quad R, \quad u = u_x = u_y = 0 \quad \text{for} \quad x + y = 0$$

has at least one solution in R ([3], p. 134). We call the solution u^* a *maximal solution in* R, if for any other solution u, existing in $R(x_0, y_0)$, where $(x_0, y_0) \in R$, the inequality

$$u(x, y) \leqslant u^*(x, y) \quad \text{in } R(x_0, y_0)$$

holds. Since a solution u, existing in $R(x_0, y_0)$, can be extended to R, the following definition is equivalent: u^* is a maximal solution in R, if for every solution u, existing in R, we have $u \leqslant u^*$ in R.

We shall show that the Cauchy problem

$$(11) \quad u_{xy} = g(x + y,\ u) \quad \text{in } R, \quad u = u_x = u_y = 0 \quad \text{for} \quad x + y = 0 ,$$

where $g(t, z)$ is given by (7) and $R = R(3/5, 3/5)$, has no maximal solution and no minimal solution.

Suppose that u^* is the maximal solution. We will show first that u^*

is constant along the lines $x+y=$ constant. Assume that (x_0, y_0) and (x_0+h, y_0-h) $(h \neq 0)$ are in R and that

$$u^*(x_0, y_0) < u^*(x_0+h, y_0-h) .$$

Since $v(x, y) = u^*(x+h, y-h)$ is a solution of (11) in $R(x_0, y_0)$, we have

$$v(x_0, y_0) > u^*(x_0, y_0) ,$$

which is a contradiction since u^* is the maximal solution. Therefore there is a function $v(t)$ such that $u^*(x, y) = v(x+y)$. It follows that $v(t)$ is a solution of the problem (8). On the other side, if $u(t)$ is an arbitrary solution of (8), then $u(x+y)$ is a solution of (11). But we have seen in section 2 that (8) has no maximal solution. This argument proves that no maximal solution for (11) exists.

A similar reasoning applies to the minimal solution.

4. A characteristic initial value problem without maximal and minimal solutions. Let $g(t, z)$ be defined by (7) for $0 \leqslant t \leqslant 6/5$, $g = 0$ for $-6/5 \leqslant t < 0$, and let $R = [-3/5, 3/5] \times [-3/5, 3/5]$. The function $g(t, z)$ is bounded and continuous in $[-6/5, 6/5] \times (-\infty, \infty)$, and the characteristic inital value problem

$$(12) \qquad u_{xy} = g(x+y, u) \quad \text{in } R, \quad u(x, -3/5) = u(-3/5, y) = 0$$

has no maximal and no minimal solution in R.

This follows immediately from the fact that any solution of (12) is zero below the diagonal $x+y=0$ and is a solution of the Cauchy problem (11) above that diagonal.

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