

## A class of special Hardy–Orlicz spaces and the space of *BMOA* functions

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**Abstract.** In this paper we discuss a class of special Hardy–Orlicz spaces  $Ba(H^{Ba})$  in which a “Young function” arises from an entire function of finite order. We discuss some properties of spaces  $Ba$  (or  $H^{Ba}$ ) and give an equivalent definition of *BMOA* functions, which exhibits interrelation between the spaces  $H^{Ba}$  and *BMOA*.

1. It is well known that many results on  $L_p$  spaces (or  $H_p$  spaces) reveal a large gap between  $L_\infty$  and  $L^* = \prod_{p < \infty} L_p$ . There are many important spaces in this gap, e.g. some classes of Orlicz spaces, *BMO*, etc. In most cases the properties of  $L_p$  ( $1 < p < \infty$ ) are entirely different from those of  $L_\infty$ . For example, the Hilbert transform is bounded in  $L_p$  ( $1 < p < \infty$ ), but not in  $L_\infty$ ; it is natural to ask whether this is true in the gap spaces and what conditions imposed on the space ensure that the conclusion holds. The interrelations between the gap spaces and their properties have fallen into the field of interest of several authors ([2], [3], [7]). In this paper we consider a class of special Orlicz spaces in which a “Young function” arises from an entire function  $E(z)$  of finite order. A space of that class is called a *Ba space* (or an  $H^{Ba}$  space); this terminology was introduced by Ding and Luo in [3] in a general form. This class of spaces is a very natural generalization of  $L_p$  and has been effectively applied to obtain prior estimates for linear partial differential equations [3] and to the calculus of variations with strong non-linearity [4]. In particular, if  $E(z)$  is a transcendental entire function, then the corresponding *Ba* space lies in the gap between  $L_\infty$  and  $L^*$ . We discuss some properties of *Ba* (or  $H^{Ba}$ ) spaces and give an equivalent definition of *BMOA*, which points of some relations between  $H^{Ba}$  and *BMOA*.

2. Let  $E(z) = \sum_{n=2}^{\infty} a_n z^n$  be an entire function with non-negative coefficients  $a_n \geq 0$  and is of finite order  $\rho (< +\infty)$  and mean type  $\sigma (< +\infty)$ . Let

$\{P_n\}$  be a sequence which satisfies  $1 \leq p_1 < p_2 < \dots < p_n < \dots$  and

$$(1) \quad p = \lim_{n \rightarrow \infty} (p_n/n^{1/q}) < +\infty.$$

Set  $D = \{z, |z| < 1\}$ , for  $u(\theta) \in \prod_{p_n} L_{p_n}(\partial D)$ , form the power series

$$(2) \quad I(\alpha, u) = \sum a_n \|u\|_{L_{p_n}}^n \alpha^n,$$

where  $\|\cdot\|_{L_{p_n}}$  is the norm in  $L_{p_n}$ . Let  $d_u$  denote the radius of convergence of (2) and let  $Ba$  be the following set of functions:

$$(3) \quad Ba = \{u(\theta): u(\theta) \in \prod_{p_n} L_{p_n}, d_u > 0\};$$

the norm of  $u(\theta)$  in  $Ba$  is defined by  $\|u\|_{Ba} = \inf_{\{\alpha, I(\alpha, u) \leq 1\}} \{1/|\alpha|\}$ . It is easy to show that  $\|u\|_{Ba} = \|u\|$  satisfies the three norm axioms and so the set  $Ba$ , with the usual addition and multiplication by scalars, becomes a normed linear function space. Let us prove the triangle inequality  $\|u_1 + u_2\|_{Ba} \leq \|u_1\|_{Ba} + \|u_2\|_{Ba}$ . To do this, it is enough to show that

$$I\left(\frac{1}{\|u_1\| + \|u_2\|}, u_1 + u_2\right) \leq 1.$$

In fact, by the definition and by properties of convex functions, we have

$$\begin{aligned} I\left(\frac{1}{\|u_1\| + \|u_2\|}, u_1 + u_2\right) &= \sum a_n \|u_1 + u_2\|_{L_{p_n}}^n \left(\frac{1}{\|u_1\| + \|u_2\|}\right)^n \\ &\leq \sum a_n \left(\frac{\|u_1\|_{L_{p_n}} + \|u_2\|_{L_{p_n}}}{\|u_1\| + \|u_2\|}\right)^n \\ &= \sum a_n \left\{ \frac{\|u_1\|}{\|u_1\| + \|u_2\|} \left(\frac{\|u_1\|_{L_{p_n}}}{\|u_1\|}\right) + \frac{\|u_2\|}{\|u_1\| + \|u_2\|} \left(\frac{\|u_2\|_{L_{p_n}}}{\|u_2\|}\right) \right\}^n \\ &\leq \frac{\|u_1\|}{\|u_1\| + \|u_2\|} I\left(\frac{1}{\|u_1\|}, u_1\right) + \frac{\|u_2\|}{\|u_1\| + \|u_2\|} I\left(\frac{1}{\|u_2\|}, u_2\right) \leq 1. \end{aligned}$$

Similarly,  $H^{Ba}$  is defined as follows. Let  $f(z) \in \prod_{p_n} H_{p_n}(D)$ ; set

$$I(\alpha, f) = \sum a_n \|f\|_{H_{p_n}}^n \alpha^n,$$

where  $\|f\|_{H_{p_n}}$  is the  $H_{p_n}$ -norm of  $f$ . Let  $d_f$  denote the radius of convergence of  $I(\alpha, f)$  and let  $H^{Ba}$  be the set

$$(4) \quad H^{Ba} = \{f: f \in \prod_{p_n} H_{p_n}, d_f > 0\};$$

the norm of an element  $f$  is defined by

$$(5) \quad \|f\|_{H^{Ba}} = \inf_{|\alpha, I(|\alpha|, f)| \leq 1} \{1/|\alpha|\}.$$

Further, we have the following propositions:

(i) Suppose that  $f(z)$  is analytic in  $D$ ; if

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta,$$

we say that  $f$  belongs to  $N^+$ ; this is a subclass of Nevanlinna's class  $N$ . Let  $N^+ \cap Ba$  denote the set of functions  $f(z) \in N^+$  with boundary values  $\hat{f}(e^{i\theta}) \in Ba$ ; then

$$(6) \quad N^+ \cap Ba \subset H^{Ba}.$$

In fact, noting that  $\|f\|_{H_p} \leq \|\hat{f}\|_{L_p}$  we get

$$\sum a_n \|f\|_{H_{p_n}}^n \left(\frac{1}{\|f\|_{Ba}}\right)^n \leq \sum a_n \|\hat{f}\|_{L_{p_n}}^n \left(\frac{1}{\|f\|_{Ba}}\right)^n \leq 1,$$

and so  $\|f\|_{H^{Ba}} \leq \|\hat{f}\|_{Ba}$ .

(ii) If  $f(z) \in N^+$ , then  $f(z) \in H^{Ba}$  if and only if  $F(z) \in H^{Ba}$ , where  $F(z)$  is the out-function of  $f(z)$ .

Indeed, for  $f(z) \in N^+$  we have [6]

$$f(z) = e^{i\alpha} B(z) F(z) S(z),$$

where  $B(z)$  is the Blaschke product of the zeros of  $f(z)$ ,  $F(z)$  is the out-function, and  $S(z)$  is a singular function. Since  $\|F\|_{H_p} \leq \|f\|_{H_p}$ , we have  $\|F\|_{H^{Ba}} \leq \|f\|_{H^{Ba}}$ . The inverse is obvious.

(iii) Suppose that  $f(z)$  is analytic in  $D$ ,  $G(z)$  is univalent and  $G(0) = f(0)$ . If  $f(D) \subset G(D)$ , we say that  $f(z)$  is subordinate to  $G(z)$ , in symbols  $f(z) \prec G(z)$ . If  $G(z) \in H^{Ba}$ , then  $f(z) \in H^{Ba}$ .

In fact, since  $f(z) \prec G(z)$ , we have  $f(z) = G \circ \varphi(z)$ , where  $\varphi(z)$  is a function which satisfies the conditions at the Schwarz lemma. Since  $\|f\|_{H_p} \leq \|G\|_{H_p}$ , we get  $\|f\|_{H^{Ba}} \leq \|G\|_{H^{Ba}}$ .

Now, let  $f(\theta)$  be a complex-valued function of bounded mean oscillation on  $\partial D$ . If its harmonic extension to  $D$ ,  $f(z) = \int_0^{2\pi} f(\theta) d\mu_z(\theta)$ , is analytic, we say that  $f$  is an analytic function of bounded mean oscillation, or  $f \in BMOA$ . Suppose that  $f(z)$  is a function defined on  $D$ . We define a set  $\mathfrak{M}(f)$  by

$$\mathfrak{M}(f) = \{g(z): g(z) = f \circ S(z) - f \circ S(0), S \in \mathfrak{M}\},$$

where

$$\mathfrak{M} = \left\{ S(z): S(z) = e^{i\alpha} \frac{\beta + a}{1 + \bar{a}\beta}, a \in D, e^{i\alpha} \in \partial D \right\}$$

is the group of all conformal automorphisms of  $D$ . We have

THEOREM 1. *There exists a constant  $c$  such that*

$$(7) \quad c^{-1} \|f\|_* \leq \|f\|_{Ba} \leq c \|f\|_*,$$

where  $\|f\|_{Ba} = \sup_{g \in \mathfrak{M}(f)} \{\|g\|_{Ba}\}$ ,  $\|f\|_*$  is the norm of  $f$  in  $BMOA$ , i.e.,

$$\|f\|_* = \sup_{g \in \mathfrak{M}(f)} \{\|g\|_{H_1}\} = \sup_{a \in D} \left\{ \int_0^{2\pi} |f(\theta) - f(a)| d\mu_a(\theta), f(a) = \int_0^{2\pi} f(\theta) d\mu_a(\theta) \right\}.$$

Proof. In [1], Baernstein proved the John-Nirenberg theorem in the following form: if  $f \in BMOA$ , then for  $g \in \mathfrak{M}(f)$

$$(8) \quad \lambda(t) = \mu_a(\{e^{i\theta} \in D, |g(e^{i\theta})| > t\}) \leq K e^{-\beta t},$$

where  $\lambda(t) = \mu_a(E_t)$  is the harmonic measure of  $E_t = \{e^{i\theta} \in \partial D, |g(e^{i\theta})| > t\}$  at  $a$  with respect to  $D$ , i.e.,  $\mu_a(E_t) = \int_{E_t} d\mu_a(\theta)$ ,  $\beta = c \|f\|_*^{-1}$ ,  $c$  and  $K$  are two

constants independent of  $f$ . By definition and (8), we get

$$\int_0^{2\pi} |g(e^{i\theta})|^p d\mu_z(\theta) = p \int_0^{\infty} t^{p-1} \lambda(t) dt \leq K p \int_0^{\infty} t^{p-1} e^{-\beta t} dt = K \beta^{-p} \Gamma(p+1).$$

Noting that  $g(z) = \int_0^{2\pi} g(e^{i\theta}) d\mu_z(\theta)$  and applying the Jensen inequality

$$|g(z)|^p \leq \int_0^{2\pi} |g(e^{i\theta})|^p d\mu_z(\theta) \leq K \beta^{-p} \Gamma(p+1),$$

we obtain

$$\|g\|_{H_p} \leq \frac{K \Gamma(p+1)}{c^p} \|f\|_*^p.$$

We set

$$\begin{aligned} I\left(\frac{\alpha}{\|f\|_*}, g\right) &= \sum a_n \|g\|_{H_{p_n}}^n \left(\frac{\alpha}{\|f\|_*}\right)^n \\ &\leq \sum a_n K^{n/p_n} (\Gamma(p_n+1))^{n/p_n} c^{-n} \alpha^n = \sum A_n \alpha^n, \end{aligned}$$

where  $A_n = a_n K^{n/p_n} (\Gamma(p_n+1))^{n/p_n} c^{-n}$ . We now prove that the power series  $\sum A_n \alpha^n$  has a positive radius of convergence. Indeed, according to Stirling's

formula

$$\Gamma(p+1) = p^p e^{-p} \sqrt{2\pi p} (1+o(1))$$

and the relation

$$(\varrho\sigma e)^{1/\varrho} = \overline{\lim}_{n \rightarrow \infty} (n^{1/\varrho} \sqrt[n]{a_n}),$$

where  $\varrho$  and  $\sigma$  are the order and the type of  $E(z)$ , respectively. We have

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{A_n} &= \overline{\lim}_{n \rightarrow \infty} \left\{ (\sqrt[n]{a_n} n^{1/\varrho}) \left( \frac{p_n}{n^{1/\varrho}} \right) K^{1/p_n} (2\pi p_n)^{1/2 p_n} c^{-1} e^{-1} \right\} \\ &\leq (\varrho\sigma e)^{1/\varrho} p e^{-1} c_1^{-1} < +\infty. \end{aligned}$$

Therefore the radius of convergence is

$$d = \frac{1}{\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{A_n}} > \frac{c_1 e}{p(\varrho\sigma e)^{1/\varrho}} = \hat{d} > 0.$$

This shows that there exists  $\alpha_0 \in (0, \hat{d})$ , say

$$\alpha_0 = \frac{1}{2} \frac{c_1 e}{p(\varrho\sigma e)^{1/\varrho}}, \quad \text{such that} \quad I\left(\frac{\alpha_0}{\|f\|_*}, g\right) \leq K_{\alpha_0} < +\infty$$

in which  $K_{\alpha_0}$  is a constant depending only on  $\varrho$ ,  $\sigma$  and  $p$ . If  $K_{\alpha_0} \leq 1$ , then

$$\|g\|_{HBa} = \inf_{|\alpha|, I(|\alpha|, f) \leq 1} \left\{ \frac{1}{|\alpha|} \right\} \leq \frac{1}{\alpha_0} \|f\|_* = \frac{2p(\varrho\sigma e)^{1/\varrho}}{c_1 e} \|f\|_*$$

hence  $\|f\|_{Ba} \leq c \|f\|_*$ . If  $K_{\alpha_0} > 1$ , we first consider  $\|f\|_* = 1$  and take  $g(z)/K_{\alpha_0}$  instead of  $g(z)$ . By a homogeneity argument, we have

$$\|g\|_{HBa} \leq K_{\alpha_0}/\alpha_0 = \frac{K_{\alpha_0}}{\alpha_0} \|f\|_* = c \|f\|_*.$$

Thus  $\|f\|_{Ba} \leq c \|f\|_*$ . If  $\|f\|_* \neq 1$ , we consider  $f/\|f\|_*$  instead of  $f$ . Since the norm  $\|\cdot\|_{HBa}$  is homogeneous, we get  $\|g\|_{HBa} \leq c \|f\|_*$ , and thus again  $\|f\|_{Ba} \leq c \|f\|_*$ .

To prove the first inequality of (7), we assume that  $a_{n_0}$  is a non-zero coefficient of  $E(z)$  such that

$$m = \text{Min}_n \left\{ \frac{1}{\sqrt[n]{a_n}} \right\} = \frac{1}{\sqrt[n_0]{a_{n_0}}}.$$

We have

$$1 \geq I\left(\frac{1}{\|g\|_{HBa}}, g\right) \geq a_{n_0} \left(\frac{1}{\|g\|_{HBa}}\right)^{n_0} \|g\|_{H_{p_n}}^{n_0};$$

hence

$$\|g\|_{H_{p_{n_0}}} \leq \frac{1}{\sqrt[n_0]{a_{n_0}}} \|g\|_{H_{Ba}} \leq m \|f\|_{Ba}.$$

On the other hand, by Hölder's inequality,

$$\|f\|_* = \sup_{g \in \mathfrak{M}(f)} \{\|g\|_{H_1}\} \leq \sup_{g \in \mathfrak{M}(f)} \{\|g\|_{H_{p_{n_0}}}\} \leq m \|f\|_{Ba}.$$

This proves the first inequality of (7).

Now let us consider the operator  $Hu(\theta)$  defined by the Cauchy principal value integral

$$\tilde{u}(\theta) = Hu(\theta) = \lim_{\varepsilon \rightarrow 0} \int_{|\varphi - \theta| < \varepsilon} \cot \frac{\theta - \varphi}{2} u(\varphi) d\varphi$$

and call  $Hu$  the *Hilbert transform*. It is well known that  $H$  is bounded in  $L_p$ ,  $1 < p < +\infty$ . A natural question arises: Is the operator  $H$  is bounded in  $Ba$ ? Recently Deng and Gu [2] proved that  $H$  is bounded in  $Ba$  if and only if there exist  $\alpha$  and  $\beta$  with  $1 < \alpha < \beta < +\infty$  such that, for all  $n$ ,  $\alpha \leq p_n \leq \beta$ . We obtain the following results as the consequence of Theorem 1.

**COROLLARY 1.** *If  $u(\theta) \in L_\infty(\partial D)$ , then  $\tilde{u}(\theta) \in Ba$ ; i.e., there exists a constant  $c$  such that*

$$(9) \quad \|\tilde{u}\|_{Ba} = \|Hu\|_{Ba} \leq c \|u\|_{L_\infty}.$$

In particular, taking  $p_n = n$ ,  $a_n = 1/n!$ , we have [6]

$$\int_0^{2\pi} e^{\alpha|\tilde{u}(\theta)|} d\mu_0(\theta) \leq c_\alpha \|u\|_{L_\infty}.$$

Indeed, set  $f(\theta) = u(\theta) + i\tilde{u}(\theta)$  and denote the harmonic extensions of  $f(\theta)$ ,  $u(\theta)$  and  $\tilde{u}(\theta)$  by  $f(z)$ ,  $u(z)$  and  $\tilde{u}(z)$ . By [1], we have

$$u(\theta) \in BMO \Leftrightarrow f(\theta) \in BMOA \Leftrightarrow \tilde{u}(\theta) \in BMO$$

and  $\|f\|_* \leq c \|u\|_* \leq c_1 \|u\|_{L_\infty}$ , where

$$(10) \quad \|u\|_* = \sup_{v \in \mathfrak{M}(u)} \{\|v\|\}_{L_1}$$

and

$$\mathfrak{M}(u) = \{v: v = u \circ s - u \circ s(0), s \in \mathfrak{M}\}.$$

Since  $\tilde{u}(0) = \int_0^{2\pi} u(\theta) d\mu_0(\theta) = 0$ , we have

$$\mu_0(\{e^{i\theta} \in \partial D, |\tilde{u}(\theta)| > t\}) \leq \mu_0(\{e^{i\theta} \in \partial D, |f(\theta) - f(0)| > t\}) \leq Ke^{-\beta_1 t},$$

where  $\beta_1 = c_1/\|u\|_{L_\infty}$ . Similarly to the proof of Theorem 1, we deduce (9).

COROLLARY 2. Let

$$\|u\|_{Ba} = \sup_{v \in \mathfrak{M}(u)} \{\|v\|_{Ba}\} = \sup_{s \in \mathfrak{M}} \{\|u \circ s - u \circ s(0)\|_{Ba}\}.$$

Then there exists a constant  $c$  such that

$$\|Hu\|_{Ba} \leq c \|u\|_{Ba}.$$

In fact, by John-Nirenberg's theorem

$$\mu_0(\{e^{i\theta} \in \partial D, |v(\theta)| > t\}) \leq \mu_0(\{e^{i\theta} \in \partial D, |g(\theta)| > t\}) \leq K_1 e^{-\beta_2 t},$$

where  $\beta_2 = c_2/\|u\|_*$ ,  $v \in \mathfrak{M}(u)$  and  $g \in \mathfrak{M}(f)$ ,  $f = u + i\tilde{u}$ . As in the proof of Theorem 1, we get

$$\|v\|_{Ba} \leq c(\varrho, \sigma, p) \|u\|_*;$$

hence

$$(11) \quad \|u\|_{Ba} \leq c \|u\|_*.$$

On the other hand, suppose that  $a_{n_0}$  is the coefficient of  $E(z)$  for which

$$\frac{1}{\sqrt[n]{a_{n_0}}} = \text{Min}_n \left\{ \frac{1}{\sqrt[n]{a_n}} \right\} = m.$$

Then we have for each  $v \in \mathfrak{M}(u)$

$$1 \geq I\left(\frac{1}{\|v\|_{Ba}}, v\right) \geq a_{n_0} (\|v\|_{L_{p_{n_0}}}/\|v\|_{Ba})^n$$

and so  $\|v\|_{L_{p_{n_0}}} \leq m \|u\|_{Ba}$ . By (10) and the Schwarz inequality,

$$(12) \quad \|u\|_* \leq \sup_{v \in \mathfrak{M}(u)} \{\|v\|_{L_{p_{n_0}}}\} \leq m \|u\|_{Ba}.$$

(11) and (12) show that  $u \in BMO \Leftrightarrow \mathfrak{M}(u)$  is a bounded set in  $Ba$ . Therefore,  $\tilde{u} \in BMO \Leftrightarrow \mathfrak{M}(u)$  is a bounded set in  $Ba$ , i.e.,  $\|\tilde{u}\|_{Ba} \sim \|\tilde{u}\|_*$ . Since  $u \in BMO \Leftrightarrow \tilde{u} \in BMO$ , we get

$$\|\tilde{u}\|_{Ba} \leq \tilde{c} \|u\|_* \leq c_1 \|u\|_* \leq c \|u\|_{Ba}.$$

This proves Corollary 2.

In [1], Baernstein proved that if  $f(z)$  is univalent and zero free, then  $\log f \in BMOA$ . By Theorem 1, we have

COROLLARY 3. Let  $f$  be a function univalent in  $D$  and zero free. Then  $\log f \in H^{Ba}$ .

3. In this section we concentrate attention on the  $BMOA$  and  $Ba$  function spaces on  $\mathcal{A} = \{z, \text{Im } z > 0\}$ . Let  $E(z) = \sum_{n=2}^{\infty} a_n z^n$  be an entire

function and  $\{p_n\}$  be a sequence as before. Suppose that  $u(t) \in L_{loc}(\mathbf{R})$  and for all  $1 \leq p < +\infty$ . Consider the power series

$$(13) \quad I(\alpha, u) = \sum a_n \|u\|_{L_{p_n}}^n \alpha^n.$$

If (13) has a positive radius of convergence, we say that  $u \in Ba$ , and we define the norm of  $u(t)$  in  $Ba$  by

$$\|u\|_{Ba} = \inf_{|\alpha|, I(\alpha, u) \leq 1} \left\{ \frac{1}{|\alpha|} \right\}.$$

Similarly, let  $f(z)$  be an analytic function in  $\mathcal{A}$  such that the boundary value function  $f(t)$  belongs to  $L_{loc}(\mathbf{R})$  and

$$\int_{\mathbf{R}} \frac{|f(t)|}{1+t^2} dt < +\infty.$$

For any  $p_n$ , let

$$\|f\|_{\hat{H}_{p_n}} = \sup_{z \in \mathcal{A}} \left\{ \int_{\mathbf{R}} |f(t) - f(z)|^{p_n} d\mu_z(t) \right\}^{1/p_n} < +\infty,$$

where

$$d\mu_z(t) = \frac{1}{\pi} \cdot \frac{y dt}{(x-t)^2 + y^2}, \quad z = x + iy.$$

We form the power series

$$(14) \quad I(\alpha, f) = \sum a_n \|f\|_{\hat{H}_{p_n}}^n \alpha^n.$$

If (14) has a positive radius of convergence, we say that  $f(z) \in H^{Ba}$ , and define the norm of  $f(z)$  in  $H^{Ba}$  by

$$\|f\|_{\hat{H}^{Ba}} = \inf_{|\alpha|, I(\alpha, f) \leq 1} \left\{ \frac{1}{|\alpha|} \right\}.$$

**THEOREM 2.** Let  $f(t) \in L_{loc}(\mathbf{R})$ ; then  $f \in BMOA(\mathbf{R})$  if and only if

$$(15) \quad \|f\|_{L_p} < +\infty, \quad 1 \leq p < +\infty$$

and

$$(16) \quad \|f\|_{\hat{H}_p} < +\infty, \quad 1 \leq p < +\infty.$$

**Proof.** Set  $I_0 = [-\frac{1}{2}, \frac{1}{2}]$ ,  $I_k = 2^k I_0 = [-2^{k-1}, 2^{k-1}]$ ,  $k = 0, 1, 2, \dots$ , and

$$f_I = \frac{1}{|I|} \int_I f(t) dt.$$



Noting that

$$\int_{\mathbf{R}} d\mu_i(t) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{dt}{1+t^2} = 1,$$

we have

$$\begin{aligned} \int_{\mathbf{R}} |f(t)|^p d\mu_i(t) &\leq 2^p \left\{ \int_{\mathbf{R}} |f(t)-f_{I_0}|^p d\mu_i(t) + |f_{I_0}|^p \right\} \\ &= 2^p \left\{ \left( \int_{I_0} + \sum_{k=1}^{\infty} \int_{I_k \setminus I_{k-1}} \right) |f(t)-f_{I_0}|^p d\mu_i(t) + |f_{I_0}|^p \right\}. \end{aligned}$$

By using the John-Nirenberg theorem [6]

$$\{|t \in I, |f(t)-f_I| > t\} \leq K \exp(-ct/\|f\|_*) |I|,$$

where  $\|f\|_* = \sup_{z \in \mathcal{A}} \left\{ \int_{\mathbf{R}} |f(t)-f(z)| d\mu_z(t) \right\}$ , we get

$$\int_{I_0} |f(t)-f_{I_0}|^p d\mu_i(t) \leq \frac{1}{|I_0|} \int_{I_0} |f(t)-f_{I_0}|^p dt \leq \frac{K\Gamma(p+1)}{c^p} \|f\|_*^p.$$

Since  $\|f\|_{**} \leq c_1 \|f\|_*$  (cf. [6], p. 224), we have

$$|f_{I_k}-f_{I_0}| \leq 2k \|f\|_{**} \leq 2kc_1 \|f\|_*, \quad \text{where} \quad \|f\|_{**} = \sup_{I \in \mathbf{R}} \left\{ \frac{1}{|I|} \int_I |f(t)-f_I| dt \right\}.$$

In view of the equality  $|I_k| = 2^k$  we get

$$\int_{I_k \setminus I_{k-1}} |f(t)-f_{I_0}|^p d\mu_i(t) \leq \left\{ \frac{K2^{p+4} \Gamma(p+1) c^{-p}}{2^k} + \frac{2^{2p+4} c_1^p k^p}{2^k} \right\} \|f\|_*^p.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^p/2^k \leq 2(\log 2)^{p+1} \Gamma(p+1),$$

we have

$$\int_{\mathbf{R}} |f(t)|^p d\mu_i(t) \leq 2^p |f_{I_0}|^p + c_p \|f\|_*^p < +\infty,$$

where  $c_p = K_0 2^{3p} \Gamma(p+1)$  and  $K_0$  is a constant independent of  $p$ .

To prove (16), we need the John-Nirenberg theorem in the following form:

$$(17) \quad \mu_z(\{t \in \mathbf{R}, |f(t)-f(z)| > t\}) \leq K \exp(-ct/\|f\|_*),$$

where

$$\mu_z(E) = \int_E d\mu_z(t) = \frac{1}{\pi} \int_E \frac{y dt}{(x-t)^2 + y^2}, \quad z = x + iy,$$

is the harmonic measure of  $E$  at  $z$  with respect to  $\mathcal{A}$ . (17) results from (8),

because  $BMOA(\mathbf{R})$  and  $BMOA(\partial D)$  are transformed into each other under the conformal mapping

$$z = \frac{1+w}{1-w}, \quad w < 1.$$

Thus

$$\int_{\mathbf{R}} |f(t) - f(z)|^p d_z(t) \leq \frac{K\Gamma(p+1)}{c^p} \|f\|_*^p$$

and (16) follows.

Conversely, if (15) and (16) hold, then the Hölder inequality shows that

$$\|f\|_* = \sup_{z \in \mathcal{A}} \left\{ \int_{\mathbf{R}} |f(t) - f(z)| d\mu_z(t) \right\} \leq \sup_{z \in \mathcal{A}} \left\{ \left( \int_{\mathbf{R}} |f(t) - f(z)|^p d\mu_z(t) \right)^{1/p} \right\}$$

and consequently  $f \in BMOA(\mathbf{R})$ .

**Remark.** In the case of  $p = 2$ , Theorem 2 reduces to a result of [6].

Similarly to the proof of Theorem 1, we have

**THEOREM 3.**  $f \in BMOA(\mathbf{R})$  if and only if  $f \in Ba \cap H^{Ba}$ , i.e.,  $\|f\|_{Ba} < +\infty$  and  $\|f\|_{\tilde{H}Ba} < +\infty$ .

**Remark.** One can also consider the space  $Ba$  or  $H^{Ba}$  in  $\mathbf{R}_+^n$ , or in the unit ball in  $\mathbf{C}^n$ , or in spaces of homogeneous type, and obtain some relations between  $BMO$  and  $Ba$ .

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