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Asymptotic stability of an integro-differential equation of parabolic type

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Abstract. It is shown that for a large class of stochastic kernels the solutions of a linear integro-differential equation of parabolic type converge to the same limit independent of the initial conditions. The solutions are constructed by the use of the semigroup theory of linear operators; the proofs of the convergence are based on a Lasota criterion for asymptotical stability of stochastic semigroups [3].

Introduction. The purpose of the present paper is to study the behaviour of solutions of the integro-differential equation of the form

$$\frac{\partial u(t, x)}{\partial t} + u(t, x) = c^2 \frac{\partial^2 u(t, x)}{\partial x^2} + \int_{\Delta} k(x, y) u(t, y)\, dy, \qquad x \in \Delta,\ t \geqslant 0, \tag{0.1}$$

where $k\colon \Delta \times \Delta \to R$ is a measurable stochastic kernel, i.e.,

$$k(x, y) \geqslant 0, \quad \int_{\Delta} k(x, y)\, dx = 1 \quad \text{for } y \in \Delta.$$

We shall study separately two cases: $\Delta = R$ (real line) and $\Delta = [0, \pi]$. In the last case, equation (0.1) will be combined with the boundary conditions

$$u(t, 0) = u(t, \pi) = 0 \quad \text{for } t \geqslant 0. \tag{0.2}$$

Equations similar to (0.1) appear as models of some biological processes (see [2] and [5] for extensive literature). They also give perfect examples of applications of the semigroup theory (see [4], Chapter 5, 7). In fact, when k is only a measurable stochastic kernel and does not satisfy additional regularity condition, the classical treatment of (0.1) is inconvenient.

The paper consists of five section. In Section 1 we start with rewriting equation (0.1) as an evolution equation in L^1 space and we construct the solutions of the corresponding abstract Cauchy problem. In Section 2 we recall the recent results of A. Lasota [3] concerning the asymptotical stability of stochastic semigroups and we show that the semigroup $\{S(t)\}$ generated by equation (0.1) is stochastic. Section 3 contains a proof of asymptotical stability of $\{S(t)\}$ in the case of a bounded interval $\Delta = [0, \pi]$ and Section 4 is devoted to analogous problem for $\Delta = R$. Finally, in Section 5, we show some properties of the limiting function

$$u_* = \lim_{t\to\infty} S(t)f.$$

1. A linear evolution equation. In order to rewrite the integro-differential equation (0.1) as an evolution equation in L^1 space we first must replace the operator d^2/dx^2 by its closure A in L^1. To be more specific in the case $\Delta = R$ we define

$$D_A = \{f \in L^1(\Delta) \cap C^{1+a}(\Delta)\colon f'' \in L^1(\Delta)\}$$

and $Af = c^2 f''$. In the case $\Delta = [0, \pi]$ we admit

$$D_A = \{f \in L^1(\Delta) \cap C^{1+a}(\Delta)\colon f'(0) = f'(\pi) = 0 \text{ and } f'' \in L^1(\Delta)\}$$

and $Af = c^2 f''$. Here $C^{1+a}(\Delta)$ denotes the set of all differentiable functions having absolutely continuous derivatives in Δ. Further, set

$$(1.1)\qquad If = f \quad \text{and} \quad Kf(x) = \int_\Delta k(x, y) f(y)\,dy.$$

It is well known that A is the infinitesimal operator of a semigroup $\{T_0(t)\}$ such that $u(t, x) = T_0(t) f(x)$ is a classical solution of equation $u_t = Au$ for sufficiently smooth f. This semigroup is given by the formula

$$(1.2)\qquad T_0(t) f(x) = \int_\Delta \Gamma_0(t, x, y) f(y)\,dy \quad \text{for } t > 0,$$

where

$$(1.3)\qquad \Gamma_0(t, x, y) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2c^2t} \cos nx \cos ny \quad \text{if} \quad \Delta = [0, \pi],$$

$$(1.4)\qquad \Gamma_0(t, x, y) = \frac{1}{2c\sqrt{\pi t}} e^{-(x-y)^2/4c^2t} \quad \text{if} \quad \Delta = R.$$

Analogously, $A - I$ is the infinitesimal operator of the semigroup $e^{-t} T_0(t)$ and $u(t, x) = e^{-t} T_0(t) f(x)$ for smooth f is the solution of $u_t = (A - I)u$. Finally, $A - I + K$ is the infinitesimal operator for the semigroup $S(t)$ such that $u(t, x) = S(t) f(x)$ satisfies

$$(1.5)\qquad u_t = (A - I + K)u$$

for $f \in D_A$. According to the well-known Phillips perturbation theorem, $S(t)$ is given by

$$(1.6)\qquad S(t) = \sum_{n=0}^{\infty} S_n(t)$$

with

(1.7) $$S_0(t) = T(t) \quad \text{and} \quad S_n(t)f = \int_0^t T(t-s)KS_{n-1}(s)f\,ds.$$

Thus instead of studing solutions of equations (0.1) we shall study the behaviour of the semigroup $\{S(t)\}$.

The function $u(t) = S(t)f$ may be considered as a generalized solution of equation (0.1) or (1.5). In fact, if $f \in D_A$ (domain of A), then $u(t)$ is a strong solution of (1.5) and for sufficiently smooth k and f the formula $u(t, x) = S(t)f(x)$ gives a classical solution of (0.1). We shall not use these facts in the sequel and we mention them only in support of our desire to study the semigroup $\{S(t)\}$ instead of the original equation (0.1).

2. Stochastic semigroups. Let (X, Q, μ) be a σ-finite measure space and let $L^1 = L^1(X, Q, \mu)$ denote the space of all integrable functions on X. A linear mapping $P\colon L^1 \to L^1$ is called a *Markov operator* if it satisfies the following two conditions

(a) $$Pf \geqslant 0 \quad \text{for } f \geqslant 0,\ f \in L^1,$$

(b) $$\|Pf\| = \|f\| \quad \text{for } f \geqslant 0,\ f \in L^1,$$

where $\|\cdot\|$ denotes the norm in L^1.

From conditions (a) and (b) it is easy to derive that

(c) $$\|Pf\| \leqslant \|f\| \quad \text{for every } f \in L^1.$$

A family of Markov operators $\{P(t)\}_{t \geqslant 0}$ is called a *stochastic semigroup* if

$$P(t_1+t_2) = P(t_1)P(t_2) \quad \text{and} \quad P(0) = I \quad \text{for } t_1, t_2 \geqslant 0.$$

By $D = D(X, Q, \mu)$ we denote the set of all nonnegative normalized elements of L^1, i.e.,

$$D = \{f \in L^1 : f \geqslant 0 \text{ and } \|f\| = 1\}.$$

The elements of D will be called *densities*.

Thus a semigroup $\{P(t)\}$ is stochastic if D is invariant with respect to $\{P(t)\}_{t \geqslant 0}$. A density f is called *stationary* if $P(t)f = f$ for all $t \geqslant 0$.

A stochastic semigroup is called *asymptotically stable* if there exists $f_* \in D$ such that

(2.1) $$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for all } f \in D.$$

From (2.1) it follows that f_* is unique and is a stationary density.

A nonnegative function $h \in L^1$ is called a *lower function* for $\{P(t)\}_{t \geqslant 0}$ if

$$\lim_{t\to\infty} \|(P(t)f - h)^-\| = 0 \quad \text{for every } f \in D.$$

A lower function h is called *nontrivial* if $h \geqslant 0$ and $\|h\| > 0$.

Our main tool in proving the stability of the semigroup generated by equation (1.5) will be the following result of A. Lasota [3] (see also Lasota–Yorke [4]).

THEOREM 2.1 (Lasota). *A stochastic semigroup* $\{P(t)\}$ *is asymptotically stable if and only if it has a nontrivial lower function.*

Now let us come back to equation (1.5) or rather to the semigroup $S(t)\colon L^1(\Delta)\to L^1(\Delta)$ generated by the operator $A-I+K$. Our first step in studing $\{S(t)\}_{t\geqslant 0}$ is the following

PROPOSITION 2.1. *The semigroup* $\{S(t)\}_{t\geqslant 0}$ *is stochastic.*

Proof. It is well known that the semigroup $T_0(t)$ which gives the solution of the heat equation

$$u_t = c^2 u_{xx}$$

is stochastic. Thus, in formula (1.6) all the terms $S_n(t)f$ are nonnegative for $f\geqslant 0$ and consequently $S(t)f\geqslant 0$ for $f\geqslant 0$.

Further for $f\geqslant 0$ we have

$$\|S(t)f\| = \sum_{n=0}^{\infty} \|S_n(t)\|$$

and

$$\text{(2.2)}\qquad \|S_n(t)f\| = \int_0^t \|T(t-s)KS_{n-1}(s)f\|\,ds = \int_0^t e^{-(t-s)}\|KS_{n-1}(s)f\|\,ds$$
$$= \int_0^t e^{-(t-s)}\|S_{n-1}(s)f\|\,ds.$$

Since

$$\|S_0(s)f\| = \|T(s)f\| = e^{-s}\|f\|,$$

we obtain by induction from (2.2)

$$\|S_n(t)f\| = e^{-t}\frac{t^n}{n!}\|f\|,$$

and consequently

$$\|S(t)f\| = \|f\|.$$

This completes the proof.

3. Stability in the case of a bounded interval. The stability of the semigroup $\{S(t)\}$ in the case of the bounded interval Δ is almost an immediate consequence of the fact that $\inf_{x,y}\Gamma_0(t_0, x, y)$ is strictly positive for

$t_0 > 0$. This fact may be obtained as a consequence of some general properties of parabolic equations. For the convenience of further considerations we give an independent proof.

PROPOSITION 3.1. *A function $\Gamma_0(t, x, y)$ given by* (1.3) *satisfies the inequality*

$$\int_0^\pi \Gamma_0(t, x, y) f(y)\,dy \geqslant \frac{1}{\pi}\Big(1 - 2\sum_{n=1}^{\infty} e^{-n^2c^2t'}\Big) \quad \textit{for } f \in D([0, \pi]),$$

where $t' = \max(t, t_1)$ and t_1 is the unique real number such that

$$1 = 2\sum_{n=1}^{\infty} e^{-n^2c^2t_1}.$$

Proof. Define

$$u_f(t, x) = \int_0^\pi \Gamma_0(t, x, y) f(y)\,dy \quad \text{for } f \in L^1([0, \pi]). \tag{3.1}$$

Using (1.3), we may rewrite u_f in the form

$$u_f(t, x) = \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nx e^{-n^2c^2t}, \tag{3.2}$$

where

$$\alpha_n = \frac{2}{\pi}\int_0^\pi f(y)\cos ny\,dy, \qquad n = 0, 1, 2, \ldots$$

For $f \in D([0, \pi])$ we have

$$|\alpha_n| \leqslant \frac{2}{\pi}\|f(x)\|.$$

Consequently from (3.2) it follows that

$$u_f(t, x) \geqslant \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2c^2t} \quad \text{for } f \in D([0, \pi]). \tag{3.3}$$

If $f \geqslant 0$ is [illegible] ction such that $f'(0) = f'(\pi) = 0$, then the function $u_f(t, x)$ exter[illegible] $t = 0$ by continuity is a classical solution of the mixed problem

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \quad u(t, 0) = u(t, \pi) = 0, \quad u(0, x) = f(x).$$

Thus, by the maximum principle,

$$u_f(t, x) = \int_0^\pi \Gamma_0(t, x, y) f(y)\,dy \geqslant 0. \tag{3.4}$$

Since $\Gamma_0(t, x, y)$ is bounded for every fixed $t > 0$, using an approximation argument it is easy to verify the same inequality for every $f \geqslant 0$, $f \in L^1([0, \pi])$. Combining (3.3) and (3.4), we obtain

$$\begin{aligned} u_f(t, x) &\geqslant \max\left(0, \frac{1}{\pi}\Big(1 - 2\sum_{n=1}^{\infty} e^{-n^2c^2t}\Big)\right) \\ &= \frac{1}{\pi}\Big(1 - 2\sum_{n=1}^{\infty} e^{-n^2c^2t'}\Big) \quad \text{for } f \in D([0, \pi]) \end{aligned}$$

which completes the proof.

THEOREM 3.1. *If $\Delta = [0, \pi]$, then for every stochastic kernel k the semigroup $\{S(t)\}$ generated by equation* (0.1) *with the boundary conditions* (0.2) *is asymptotically stable.*

Proof. From (1.6), (1.7) it follows that

$$\begin{aligned} S(t)f &= \sum_{n=0}^{\infty} S_n(t)f = S_0(t)f + \sum_{n=1}^{\infty} S_n(t)f \\ &= S_0(t)f + \int_0^t T(t-\tau)\sum_{n=1}^{\infty} KS_{n-1}(\tau)f\,d\tau \\ &= S_0(t)f + \int_0^t T(t-\tau)KS(\tau)f\,d\tau \\ &= S_0(t)f + \int_0^t e^{-(t-\tau)}T_0(t-\tau)KS(\tau)f\,d\tau. \end{aligned} \tag{3.5}$$

Now for a fixed $f \in D([0, \pi])$ we define

$$g(t) = KS(t)f.$$

Since $S(t)$ is a stochastic semigroup and K is a Markov operator, for each fixed τ the function $g(\tau)$ is a density. From (3.5) it follows that

$$S(t)f = S_0(t)f + \int_0^t e^{-(t-\tau)}T_0(t-\tau)g(\tau)\,d\tau. \tag{3.6}$$

By Proposition 3.1,

$$T_0(t-\tau)g(\tau)(x) = \int_0^\pi \Gamma_0(t-\tau, x, y)g(\tau)(y)\,dy \geqslant \frac{1}{\pi}\Big(1 - 2\sum_{n=1}^{\infty} e^{-n^2c^2(t-\tau)'}\Big),$$

where $(t-\tau)' = \max(t-\tau, t_1)$. Since $S_0(t)f = e^{-t}T_0(t)f \geqslant 0$, from (3.6) we obtain

$$S(t)f \geqslant \int_0^t \frac{1}{\pi} e^{-(t-\tau)}\Big(1-2\sum_{n=1}^{\infty} e^{-n^2c^2(t-\tau)'}\Big)d\tau. \tag{3.7}$$

Now, taking in account that the integrand in (3.7) is nonnegative and $(t-\tau)' \geqslant t-\tau$, we have

$$\begin{aligned} S(t)f &\geqslant \frac{1}{\pi} \int_0^{t-t_1} e^{-(t-\tau)}\Big(1-2\sum_{n=1}^{\infty} e^{-n^2c^2(t-\tau)'}\Big)d\tau \\ &\geqslant \frac{1}{\pi} \int_0^{t-t_1} e^{-(t-\tau)}\Big(1-2\sum_{n=1}^{\infty} e^{-n^2c^2(t-\tau)}\Big)d\tau \qquad \text{for } t \geqslant t_1. \end{aligned}$$

Calculating the last integral and setting

$$\varepsilon(t) = -\frac{1}{\pi} e^{-t}\Big[1-2\sum_{n=1}^{\infty} \frac{1}{n^2c^2+1} e^{-n^2c^2t}\Big],$$

we obtain

$$S(t)f \geqslant \frac{1}{\pi} e^{-t_1}\Big[1-2\sum_{n=1}^{\infty} \frac{1}{n^2c^2+1} e^{-n^2c^2t_1}\Big] + \varepsilon(t).$$

It is evident that $\varepsilon(t) \to 0$ as $t \to \infty$ and that (by definition of t_1)

$$h = \frac{1}{\pi} e^{-t_1}\Big[1-2\sum_{n=1}^{\infty} \frac{1}{n^2c^2+1} e^{-n^2c^2t_1}\Big] > \frac{1}{\pi} e^{-t_1}\Big[1-2\sum_{n=1}^{\infty} e^{-n^2c^2t_1}\Big] = 0.$$

Thus finally

$$S(t)f \geqslant h+\varepsilon(t)$$

which implies by Lasota theorem the asymptotical stability of $\{S(t)\}$. The proof is completed.

4. Stability on the real line. In the case of unbounded interval it is necessary to admit some additional assumption concerning the kernel k. In general these conditions can be written in the form

$$\int_{-\infty}^{\infty} k(x, y)V(x)dx \leqslant \alpha V(x)+\beta,$$

where $V\colon R \to R_+$ is a Lapunov function. In order to simplify the argument we shall restrict ourselves to the case $V(x) = |x|^s$.

THEOREM 4.1. *If $\Delta = (-\infty, \infty)$ and k is a stochastic kernel satisfying the inequality*

$$(4.1)\qquad \int_{-\infty}^{\infty} k(x, y)|x|^s dx \leqslant \alpha |y|^s + \beta \quad \text{for } y \in R$$

with nonnegative constants α, s, β ($\alpha < 1$, $s > 0$), then the semigroup $\{S(t)\}$ generated by equation (0.1) *is asymptotically stable.*

Proof. The semigroup $\{S(t)\}$ is given by formulae (1.6), (1.7) with

$$(4.2)\qquad Kf(x) = \int_{-\infty}^{\infty} k(x, y) f(y) dy, \qquad T(t) f(x) = \int_{-\infty}^{\infty} G(t, x-y) f(y) dy$$

and

$$(4.3)\qquad G(t, x-y) = e^{-t}\Gamma_0(t, x, y) = e^{-t}\frac{1}{2c\sqrt{\pi t}} e^{-(x-y)^2/4c^2t}.$$

We are going to evaluate the expectation of $|x|^s$ with respect to the density $S(t)f$ for $f \in D(R)$. Thus we consider the integral

$$(4.4)\qquad E(t) = \int_{-\infty}^{\infty} |x|^s S(t) f(x) dx$$

which could be written in the form

$$(4.5)\qquad E(t) = \sum_{n=0}^{\infty} E_n(t),$$

where

$$(4.6)\qquad E_n(t) = \int_{-\infty}^{\infty} |x|^s S_n(t) f(x) dx.$$

The function $E_0(t)$ may be evaluated quite easily. Namely, we have

$$E_0(t) = \int_{-\infty}^{\infty} |x|^s T(t) f(x) dx = \int_{-\infty}^{\infty} |x|^s \int_{-\infty}^{\infty} G(t, x-y) f(y) dy dx.$$

Since $s > 0$, there exists positive constants γ and λ such that

$$(4.7)\qquad |x|^s \leqslant \gamma |x-y|^s + \lambda |y|^s.$$

Thus

$$E_0(t) \leqslant \lambda \iint_{-\infty}^{\infty} |y|^s G(t, x-y) f(y) dy dx + \gamma \iint_{-\infty}^{\infty} |x-y|^s G(t, x-y) f(y) dy dx$$

$$= \lambda \int_{-\infty}^{\infty} |y|^s f(y) dy \int_{-\infty}^{\infty} G(t, z) dz + \gamma \int_{-\infty}^{\infty} |z|^s G(t, z) dz \int_{-\infty}^{\infty} f(y) dy.$$

Setting

$$(4.8)\qquad \int_{-\infty}^{\infty} |y|^s f(y)\,dy = m_s(f) \quad \text{and} \quad \int_{-\infty}^{\infty} |z|^s \frac{1}{\sqrt{4c^2\pi}} e^{-z^2/4c^2}\,dz = k_s$$

and using the fact that f is a density, we obtain

$$E_0(t) \leqslant \lambda m_s(f) \int_{-\infty}^{\infty} G(t, z)\,dz + \gamma \int_{-\infty}^{\infty} |z|^s G(t, z)\,dz = \lambda m_s(f) e^{-t} + \gamma k_s e^{-t} t^{\frac{1}{2}s}$$

or, finally,

$$(4.9)\qquad E_0(t) \leqslant C_0 e^{-t}(1 + t^{\frac{1}{2}s}),$$

where

$$C_0 = C_0(f) = \max[\lambda m_s(f), \gamma k_s].$$

Now from (4.6) and (1.7) it follows that

$$E_n(t) = \int_{-\infty}^{\infty} |x|^s \int_0^t T(t-\tau) K S_{n-1}(\tau) f(x)\,d\tau\,dx$$

$$= \int_0^t \iiint_{-\infty}^{\infty} |x|^s G(t-\tau, x-y) k(y, z) S_{n-1}(\tau) f(z)\,dx\,dy\,dz\,d\tau.$$

Again applying (4.7), we obtain

$$E_n(t) \leqslant \gamma \int_0^t \iiint_{-\infty}^{\infty} |x-y|^s G(t-\tau, x-y) k(y, z) S_{n-1}(\tau) f(z)\,dx\,dy\,dz\,d\tau +$$

$$+ \lambda \int_0^t \iiint_{-\infty}^{\infty} |y|^s G(t-\tau, x-y) k(y, z) S_{n-1}(\tau) f(z)\,dx\,dy\,dz\,d\tau$$

$$= \gamma \int_0^t \iiint_{-\infty}^{\infty} |p|^s G(t-\tau, p) k(y, z) S_{n-1}(\tau) f(z)\,dp\,dy\,dz\,d\tau +$$

$$+ \lambda \int_0^t \iiint_{-\infty}^{\infty} |y|^s G(t-\tau, p) k(y, z) S_{n-1}(\tau) f(z)\,dp\,dy\,dz\,d\tau.$$

Using (4.3) and (4.8), it is easy to verify that

$$\int_{-\infty}^{\infty} G(t-\tau, p)\,dp = e^{-(t-\tau)}, \qquad \int_{-\infty}^{\infty} |p|^s G(t-\tau, p)\,dp = k_s e^{-(t-\tau)} (t-\tau)^{\frac{1}{2}s}.$$

From this and the assumptions concerning the kernel it follows that

$$E_n(t) \leqslant \gamma k_s \int_0^t \int_{-\infty}^{\infty} (t-\tau)^{\frac{1}{2}s} e^{-(t-\tau)} S_{n-1}(\tau) f(z)\, dz\, d\tau +$$

$$+\lambda \int_0^t \int_{-\infty}^{\infty} e^{-(t-\tau)} (\alpha |z|^s + \beta) S_{n-1}(\tau) f(z)\, dz\, d\tau.$$

Further, since

$$\int_{-\infty}^{\infty} S_{n-1}(\tau) f(z)\, dz = \frac{\tau^{n-1}}{(n-1)!} e^{-\tau}$$

and

$$\int_{-\infty}^{\infty} |z|^s S_{n-1}(\tau) f(z)\, dz = E_{n-1}(\tau),$$

the last inequality may be rewritten in the form

(4.10)

$$E_n(t) \leqslant \beta\lambda \frac{t^n}{n!} e^{-t} + \gamma k_s \int_0^t (t-\tau)^{\frac{1}{2}s} e^{-(t-\tau)} \frac{\tau^{n-1}}{(n-1)!} e^{-\tau}\, d\tau + \alpha\lambda \int_0^t e^{-(t-\tau)} E_{n-1}(\tau)\, d\tau.$$

From (4.9) and (4.10) by an induction argument, it follows that every function $E_n(t)$ $(n = 0, 1, \ldots)$ is bounded. Now write

$$F_m(t) = \sum_{n=0}^{m} E_n(t). \tag{4.11}$$

Using (4.10) and (4.11), we obtain

$$\begin{aligned} F_m(t) &\leqslant E_0(t) + \beta\lambda + \gamma k_s \int_0^t (t-\tau)^{\frac{1}{2}s} e^{-(t-\tau)}\, d\tau + \alpha\lambda \int_0^t e^{-(t-\tau)} F_{m-1}(\tau)\, d\tau \\ &\leqslant E_0(t) + \varrho + \alpha\lambda \int_0^t e^{-(t-\tau)} F_{m-1}(\tau)\, d\tau, \end{aligned} \tag{4.12}$$

where

$$\varrho = \beta\lambda + \gamma k_s \int_0^{\infty} t_1^{\frac{1}{2}s} e^{-t_1}\, dt_1.$$

Since $\alpha < 1$, we may choose λ in (4.7) in such a way that $\alpha\lambda < 1$. Setting $\alpha\lambda = q$ we may rewrite (4.12) in the form

$$F_m(t) \leqslant E_0(t) + \varrho + q \int_0^t e^{-(t-\tau)} F_m(\tau)\, d\tau. \tag{4.13}$$

Inequality (4.13) allows easily to evaluate $F_m(t)$. In fact, since the kernel $e^{-(t-\tau)}$ is positive the functions F_m are bounded by the solutions of the corresponding differential equations [6]. Thus

$$F_m(t) < \frac{\varrho}{1-q} + \varepsilon_1(t),$$

where

$$\varepsilon_1(t) = E_0(t) + q\int_0^t e^{-(1-q)(t-\tau)} E_0(\tau)\,d\tau.$$

Using (4.5), (4.11) and passing to the limit as $m \to \infty$, we finally obtain

$$E(t) \leqslant \frac{\varrho}{1-q} + \varepsilon_1(t).$$

Since $E_0(t)$ converges to zero as $t \to \infty$, it is evident that $\varepsilon_1(t)$ has the same property. Thus

$$E(t) \leqslant M = 1 + \frac{\varrho}{1-q}$$

for sufficiently large t (say $t \geqslant t_1(f)$). Finally, using (4.4), we have

$$\int_{-\infty}^{\infty} |x|^s S(t) f(x)\,dx \leqslant M \quad \text{for } t \geqslant t_1. \tag{4.14}$$

Having (4.14), we may use the Chebyshev inequality to evaluate the integral of $S(t)f(x)$. Namely, setting $r = 2M$ and $d = r^{1/s}$ we have

$$\int_{-d}^{d} S(t) f(x)\,dx \geqslant 1 - E(t)/r \geqslant \tfrac{1}{2} \quad \text{for } t \geqslant t_1.$$

Now we may easily find a lower function for $\{S(t)\}$. In fact, from (1.6) and (1.7) it follows that

$$S(t+1)f \geqslant T(1)S(t)f(x)$$

and further by (4.2) and (4.3)

$$S(t+1)f \geqslant \frac{e^{-1}}{2c\sqrt{\pi}} \int_{-d}^{d} e^{-(x-y)^2/4c^2} S(t) f(y)\,dy$$

$$\geqslant \frac{1}{2c\sqrt{\pi}} e^{-\left[1+\frac{1}{c^2}x^2+\frac{1}{c^2}d^2\right]} \int_{-d}^{d} S(t) f(y)\,dy$$

$$\geqslant \frac{1}{4c\sqrt{\pi}} e^{-\left[1+\frac{1}{c^2}x^2+\frac{1}{c^2}d^2\right]} \quad \text{for } t \geqslant t_1(f).$$

The last inequality shows that

$$h(x) = \frac{1}{4c\sqrt{\pi}} e^{-\left[1+\frac{1}{c^2}x^2+\frac{1}{c^2}d^2\right]}$$

is a lower function for $\{S(t)\}$ which according Theorem 2.1 completes the proof.

Remark 4.1. Theorem 4.1 is not true for $\alpha = 1$. To construct a counter-example set

$$c^2 = \tfrac{1}{2}, \quad k(x, y) = \frac{1}{\sqrt{2\pi}} e^{-(x-y)^2/2} \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

In this case equation (0.1) has the form

$$(4.15) \qquad u_t(t, x) = \tfrac{1}{2}u_{xx}(t, x) - u(t, x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-y)^2/2} u(t, y)\,dy$$

with the initial condition

$$(4.16) \qquad u(0, x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

We have

$$\int_{-\infty}^{\infty} k(x, y)|x|\,dx \leqslant \int_{-\infty}^{\infty} |x-y|\,k(x, y)\,dx + |y| \int_{-\infty}^{\infty} k(x, y)\,dx = |y| + \sqrt{2/\pi},$$

and thus, condition (4.1) is satisfied for $\alpha = 1$, $s = 1$ and $\beta = \sqrt{2/\pi}$. A simple calculation based on formula (1.7) shows that the function

$$u(t, x) = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} \frac{1}{\sqrt{2\pi(t+n+1)}} e^{-x^2/2(t+n+1)}$$

is a solution of the Cauchy problem (4.15), (4.16) or more precisely that $u(t, x) = S(t)f(x)$. It is evident that

$$u(t, x) \leqslant \frac{1}{\sqrt{2\pi(t+1)}}$$

which shows that $u(t, x)$ converges to zero as $t \to \infty$ uniformally in x. Thus $u(t, x)$ does not converge to any density as $t \to \infty$ and $\{S(t)\}$ is not asymptotically stable.

5. Properties of limiting functions. From Theorems 3.1 and 4.1 it follows that the limiting function

$$(5.1) \qquad u_* = \lim_{t\to\infty} S(t)f \quad \text{for } f \in D(\Delta)$$

exists and is independent on f. The properties of u_* can be found using the following

PROPOSITION 5.1. *If the assumptions of Theorem* 3.1 *or* 4.1 *are satisfied, then the functions* u_* *given by* (5.1) *is the unique solution of*

$$u = Au + Ku$$

belonging to $D(\Delta)$.

Proof. Passing to the limit as $t \to \infty$ in the equality

$$S(t+\tilde{t})f = S(\tilde{t})S(t)f,$$

we obtain

$$u_* = S(\tilde{t})u_*,$$

which shows that u_* is the common fixed point of all transformation $S(t)$. It is well known that the fixed points of a continuous semigroups are precisely the zeros of its infinitesimal operator. Thus

$$(A - I + K)u_* = 0,$$

which completes the proof.

As an application of Proposition 5.1 consider again equation (4.15) and denote by $\{S(t)\}$ the corresponding semigroup. Assume for a while that $S(t)$ is asymptotically stable. Then according to Proposition 5.1 the limiting function u_* satisfies

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-y)^2/2} u(y)\,dy = 0.$$

Consequently the Fourier transformation $\hat{u}_*$ of u_* is the solution of

$$\left[\tfrac{1}{2}(i\omega)^2 - 1 + \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2}\right]\hat{u}_* = 0.$$

Since the expression in brackets is always different from zero, the last equality implies that $\hat{u}_*$ as well as u_* are identically equal to zero. This contradicts to the definition of u_* and shows once again that the semigroup generated by equation (4.15) is not asymptotically stable.

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