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L_1 contains every two-dimensional normed space*

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Abstract. We exhibit a real Banach space which contains (a linear isometric copy of) every two-dimensional normed space, but does not contain every three-dimensional normed space. This answers a question of Rolewicz. The example is $L_1(0, 1)$. Our proofs are short and completely elementary, requiring only the definition of $L_1(0, 1)$ and the fundamental theorem of calculus.

The following question was asked quite recently [7], Problem 9.9.5. If a Banach space is universal for all two-dimensional normed spaces, is it then universal for all finite-dimensional normed spaces? We answer this question in the negative. Specifically, we show that $L_1(0, 1)$ contains every two-dimensional normed space, but does not contain (isometrically) the three-dimensional space $l_\infty(3)$.

The fact that $L_1(0, 1)$ contains every two-dimensional space is already known. Dor [3], Proposition 1.3, Ferguson [4], Section 2, Herz [5], Theorem 2, and Lindenstrauss [6], p. 493, each obtained this as a by-product of deeper work concerning potential transforms, multivariate distributions, negative definite functions, and extensions of operators, respectively. Our proof requires no knowledge of these topics.

The fact that $L_1(0, 1)$ does not contain a linear isometric copy of $l_\infty(3)$ is also known. Ferguson [4] and Herz [5] used negative definite functions to establish this. It also follows from the fact that an octahedron, the unit ball of $l_\infty(3)^*$, is not a zonoid [1]. Dor [3] gave a better result, showing that $l_p(3)$ does not embed in $L_1(0, 1)$ for any $p > 2$. We give a completely elementary proof that there is not even a non-linear isometry from $l_\infty(3)$ into $L_1(0, 1)$. Our argument shows that this is also true for $l_p(3)$, for any $p \geqslant 3.2$.

These two facts about $L_1(0, 1)$ do not seem to be very well known. The discovery of simple direct proofs should be good reason to publicize them.

THEOREM 1. *Every two-dimensional normed space embeds linearly and isometrically in* $L_1(0, 1)$.

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Proof. Let E be any two-dimensional normed space. We actually show that E^* embeds in $L_1(0, \lambda)$ for some positive λ. This clearly establishes the result.

Denote by K the boundary of the unit ball of E. By convexity, K is a rectifiable curve. Choose some x on K, and let C be an arc of K joining x to $-x$. Let $\psi\colon [0, \lambda] \to C$ be a parametrization of C, where λ is the length of C. That is, ψ is an isometry with $\psi(0) = x$ and $\psi(\lambda) = -x$.

Being absolutely continuous, ψ is differentiable almost everywhere and satisfies the fundamental theorem of calculus [2], p. 180. Define $T\colon E^* \to L_1(0, \lambda)$ by $Tf = \frac{1}{2} f \circ \psi'$. It is obvious that T is a linear map.

To show T is an isometry, fix f in E^*, and choose y in K so that $f(y) = \|f\|$. Then $y \in C$ or $-y \in C$; without loss of generality we assume the former. Then $y = \psi(t)$ for some t in $[0, \lambda]$. Note that $f \circ \psi$ is an increasing function on the interval $(0, t)$ and a decreasing function on the interval (t, λ). Hence

$$\|Tf\| = \int_0^\lambda |Tf| = \tfrac{1}{2}\int_0^\lambda |(f\circ\psi)'| = \tfrac{1}{2}\int_0^t (f\circ\psi)' - \tfrac{1}{2}\int_t^\lambda (f\circ\psi)'$$

$$= \tfrac{1}{2}[f\circ\psi]_0^t - \tfrac{1}{2}[f\circ\psi]_t^\lambda = f(y) - \tfrac{1}{2}f(x) - \tfrac{1}{2}f(-x) = \|f\|.$$

Being linear and norm-preserving, T is an isometry.

THEOREM 2. *The three-dimensional space $l_\infty(3)$ is not isometric to any subset of $L_1(0, 1)$.*

Proof. It is straightforward, and not too time consuming, to show that

$$|a-b|+|b-c|+|c-a|+|x| \leqslant |x-a|+|x-b|+|x-c|+|a|+|b|+|c|$$

for all real numbers a, b, c, x. Since the integral is linear and monotone, it follows that

$$\|a-b\|+\|b-c\|+\|c-a\|+\|x\| \leqslant \|x-a\|+\|x-b\|+\|x-c\|+\|a\|+\|b\|+\|c\|$$

for all functions a, b, c, x in $L_1(0, 1)$.

Now consider, in $l_\infty(3)$, the points $a = (1, 1, 1)$, $b = (1, 1, -1)$, $c = (1, -1, 1)$ and $x = (2, 0, 0)$. These points do not satisfy the inequality above. So the five element set $\{0, a, b, c, x\}$, and hence also $l_\infty(3)$, cannot be isometric to any subset of $L_1(0, 1)$.

The same argument shows that $l_p(3)$ is not isometric to any subset of $L_1(0, 1)$, at least for $p \geqslant 16/5$.

We remark that it is necessary to consider an inequality involving five points in the preceding proof. It is easy to show that every four element metric space embeds isometrically in $l_1(2)$, and hence $L_1(0, 1)$.

Smiley and Smiley [8] studied the quadrilateral inequality,

$$\|x+y\|+\|x+z\|+\|y+z\| \leqslant \|x\|+\|y\|+\|z\|+\|x+y+z\|$$

which, being valid in $\boldsymbol{R}$, is also valid in $L_1(0, 1)$. By considering the points $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$, they showed that the quadrilateral inequality is violated in $l_p(3)$, whenever $p > \log 3/\log 1.5 \simeq 2.7$. Thus $l_p(3)$ (for such p) is not linearly isometric to any subspace of $L_1(0, 1)$, but this argument does not preclude the possibility of a non-linear embedding.

Rolewicz [7], Problem 9.9.4, also asked if a Banach space universal for finite-dimensional normed spaces is automatically universal for all separable normed spaces. Szankowski [9] has answered this in the negative. Szankowski constructed a separable reflexive Banach space which contains a linear isometric (even norm-one complemented) copy of every finite-dimensional normed space.

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