ANNALES
POLONICI MATHEMATICI
XLI (1983)

Error estimates for the finite element solutions of variational inequalities

by M. A. Noor and K. I. Noor (Ahwaz, Iran)

Abstract. For piecewise linear approximation of variational inequalities associated with the mildly non-linear elliptic boundary value problems having auxiliary constraint conditions, we prove that the error estimate for $u-u_h$ in the $W^{1,2}$-norm is of order h, i.e., $\|u-u_h\| = O(h)$.

In this paper, we derive the finite element error estimates for the approximate solution of mildly non-linear elliptic boundary value problems having auxiliary constraint conditions. A much used approach with any elliptic is to reformulate it in a weak or variational forms. It has been shown by Noor and Whiteman [7] that in the presence of a constraint, such approach leads to a variational inequality which is the weak formulation. An approximate formulation of the variational inequality is then defined, and the error estimates involving the difference between the solution of the exact and approximate formulation in the W_2^1-norm is obtained, which is in fact of order h. This result is an extension of that obtained by Falk [2] and Mosco and Strang [4] for the constrained linear problems.

For simplicity, we consider the problem of the following type:

$$(1)\qquad \begin{aligned} -\Delta u(x) &= f(x, u), && x \in \Omega, \\ u(x) &= 0, && x \in \partial\Omega, \end{aligned}$$

where Ω is a simply connected open domain in R^n with boundary $\partial\Omega$ and its closure $\bar{\Omega} = \Omega \cup \partial\Omega$. It is assumed that the boundary $\partial\Omega$ and f are smooth enough to ensure the existence and uniqueness of the solution u of (1). We study this problem in the usual Sobolev space $W_2^1(\Omega) = H^1$, the space of functions which together with their generalized derivatives of order one are in $L_2(\Omega)$. The subspace of functions from H^1, which in a generalized sense satisfy the homogeneous boundary conditions on $\partial\Omega$ is $\mathring{W}_2^1(\Omega) = H_0^1$.

It has been shown by Tonti [9] that in its direct variational formulation, (1) is equivalent to finding $u \in H_0^1$ such that

$$I[u] \leqslant I[v] \quad \text{for all } v \in H_0^1,$$

where

$$(2) \qquad I[v] = \int_{\Omega} \left(\frac{\partial v}{\partial x}\right)^2 d\Omega - 2 \int_{\Omega} \int_{0}^{v} f(x, \eta)\, d\eta\, d\Omega \equiv a(v, v) - 2F(v),$$

is the energy functional associated with (1).

We here consider the case, when the solution u of (1) is required to satisfy the condition $u \geqslant \psi$, where ψ is a given function on Ω. In this situation, our problem is to find

$$u \in K \overset{\text{def}}{\equiv} \{v ; v \in H_0^1, v \geqslant \psi \text{ on } \Omega\},$$

a closed convex subset of H_0^1, see Mosco [3], such that u minimizes $I[v]$ on K. It has been shown by Noor and Whiteman [7] that the minimum of $I[v]$ on K can be characterized by a class of variational inequalities

$$(3) \qquad a(u, v-u) \geqslant \langle F'(u), v-u \rangle \quad \text{for all } v \in K,$$

where $F'(u)$ is the Fréchet differential of $F(u)$ and is, in fact,

$$(4) \qquad \langle F'(u), v \rangle = \int_{\Omega} f(u) v\, d\Omega.$$

The finite dimensional form of (3) is to find $u_h \in K_h$ such that

$$(5) \qquad a(u_h, v_h - u_h) \geqslant \langle F'(u_h), v_h - u_h \rangle \quad \text{for all } v_h \in K_h.$$

Here K_h is a finite dimensional convex subset of H_0^1 for the construction of K_h, see Mosco [3]. Let Ω be the convex polygon. We partition it into triangles of side less than h. We consider $S_h \subset H_0^1$, the subspace of continuous piecewise linear functions on the triangulation of Ω, vanishing on its boundary $\partial\Omega$. Let ψ_h be the interpolant of ψ such that ψ_h agrees with ψ at all the vertices of the triangulation. For our purposes, it is enough to choose the finite dimensional convex subset $K_h = S_h\{v_h > \psi_h \text{ on } \Omega\}$. For other choices of convex subsets K_h, see Natterer [5] and Nitsche [6], where they have choosen $K_h = K \cap S_h$.

We also want to know the regularity of the solution $u \in K$ satisfying (3). In this case Brezis and Stampacchia [1] have shown that if ψ lies in both H_0^1 and H^2, then the solution $u \in K$ satisfying (3) also lies in H^2. Its norm can be estimated from the data:

$$\|u\|_2 < e\|\psi\|_2.$$

Moreover, if $\tilde{u}$ is the interpolant of u, which agrees with u at every vertex of $\tilde{\ }$, then $\tilde{u}$ lies in K_h. It is well known from the approximation theory, see Strang and Fix [8] that

$$(6) \qquad \|u - \tilde{u}\| \leqslant Ch\|u\|_2.$$

We also note that in certain cases, the equality holds instead of inequality in (3). This happens, when v, together with $2u-v$ also lies in K. In this case, we get

$$a(u, v-u) = \langle F'(u), v-u \rangle. \tag{7}$$

Finally let C and C_h be the cones composed of non-negative functions on H_0^1 and its subspace S_h. Thus it is clear that

$$U = u-\psi \quad \text{is in } C,$$

$$U_h = u_h-\psi_h \quad \text{is in } C_h.$$

From these relations it follows that

$$u-u_h = U-U_h+\psi-\psi_h. \tag{8}$$

DEFINITION. An operator T on H_0^1 is said to be *quasi-monotone*, if for all $u, v, w, z \in H_0^1$,

$$\langle Tu-Tv, w-z \rangle \geqslant 0. \tag{9}$$

We also need the following result of Mosco and Strang [4].

THEOREM 1. *Suppose that $U > 0$ in the plane polygon Ω and that U lies in both H_0^1 and H^2. Then there exists a V_h in S_h such that*

$$0 \leqslant V_h \leqslant U \quad \text{in } \Omega$$

and

$$\|U-V_h\| \leqslant Ch\|U\|_2. \tag{10}$$

Now we state and prove the main result.

THEOREM 2. *Let $a(u, v)$ be a continuous coercive bilinear form and $F'(u)$ be a quasi-monotone operator on H_0^1. If $V_h \in C_h$ and $2U-V_h \in C$, then*

$$\|u-u_h\| = O(h),$$

where u and u_h are the solutions of (3) *and* (5) *respectively.*

Proof. Since both $v = \psi+V_h$ and $2u-v = \psi+(2U-V_h)$ are in K, we have from (3) and (7) that

$$a(u, V_h-U) = \langle F'(u), V_h-U \rangle. \tag{11}$$

Letting $v_h = \psi_h+V_h$ and $u_h = \psi_h+U_h$ in (5), we have

$$a(u_h, V_h-U_h) \geqslant \langle F'(u_h), V_h-U_h \rangle, \tag{12}$$

and taking $v = \psi+U_h$ in (3), we get

$$a(u, U_h-U) \geqslant \langle F'(u), U_h-U \rangle. \tag{13}$$

From (11) and (13), we obtain

$$a(u, U_h - V_h) \geq \langle F'(u), U_h - V_h \rangle. \tag{14}$$

and from (12) and (14), we get

$$a(u-u_h, U_h - V_h) \geq \langle F'(u) - F'(u_h), U_h - V_h \rangle.$$

Thus using the quasi-monotonicity of $F'(u)$, we have

$$a(u-u_h, U_h - V_h) \geq 0,$$

which can be written as

$$a(u-u_h, U - U_h) \leq a(u-u_h, U - V_h). \tag{15}$$

Now by the coercivity of $a(u, v)$, it follows that there exists a constant $\varrho > 0$ such that

$$\begin{aligned} \varrho \|u-u_h\|^2 &\leq a(u-u_h, u-u_h) \\ &= a(u-u_h, \psi-\psi_h) + a(u-u_h, U-U_h) \qquad \text{from (8)} \\ &\leq a(u-u_h, \psi-\psi_h) - + a(u-u_h, U-V_h) \\ &\leq \mu \|u-u_h\| \{\|\psi-\psi_h\| + \|U-V_h\|\}, \end{aligned}$$

where μ is a continuity constant of the bilinear form $a(u, v)$.

Hence it follows that

$$\begin{aligned} \|u-u_h\| &\leq \frac{\mu}{\varrho}\{\|\psi-\psi_h\| + \|U-V_h\|\} \\ &\leq \frac{\mu}{\varrho} eh\{\|\psi\|_2 + \|U\|_2\}, \qquad \text{by (6) and (10)} \end{aligned}$$

from which the required estimate follows.

Remark. The problem of deriving the L^∞-norm for the mildly non-linear problems having constraint conditions is still open.

Acknowledgement. The authors would like to thank Proof. Dr. J. R. Whiteman for his helpful critism and encouragement.

References

[1] H. Brezis and G. Stampacchia, *Sur la regularite de la solution d'inéquations elliptiques*, Bull. Soc. Math. France 96 (1968), p. 153–180.

[2] R. S. Falk, *Error estimates for the approximation of a class of variational inequalities*, Math. Comp. 28 (1974), p. 963–971.

[3] U. Mosco, *An introduction to the approximate solution of variational inequalities*, In constructive aspects of Functional Analysis, Edizone Cremonese, Roma (1973), p. 499–685.

[4] U. Mosco and G. Strang, *One sided approximation and variational inequalities*, Bull. Amer. Math. Soc. 80 (1974), p. 308–312.

[5] F. Nattere, *Optimal L_2-Konvergenz finiter elemente bei variationsungleichungen*, I. S. N. M., Birkhauser Verlag, Basel. (to appear).

[6] J. Nitsche, *L_∞-convergence of finite element approximations in Mathematical aspects of finite element method*, Rome, December, 1975.

[7] M. A. Noor and J. R. Whiteman, *Finite element error analysis of nonlinear variational inequalities*, Appl. Math. and Opt. (to appear).

[8] G. Strang and G. Fix, *An analysis of the finite element method*, Prentice Englewood Cliff., N. J., 1973.

[9] E. Tonti, *Variational formulation of nonlinear differential equations*, Bull. Acad. Royale de Belgique (1969), p. 137–165 and 262–278.

MATHEMATICS DEPARTMENT, JUNDI SHAPUR UNIVERSITY, AHWAZ, IRAN

Reçu par la Rédaction le 16.6.1978