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On certain oscillatory properties for solutions of an equation with a biharmonic leading part

by Zdzisław Frydrych (Kraków)

Abstract. Using the expansion of the spherical mean value for the analytic functions, the oscillatory properties for the non-trivial solutions of the equation

$$\Delta^2 u(x,y,z)+\Delta u(x,y,z)+u(x,y,z) = 0$$

are investigated. Moreover, a theorem of the Harnack type is proved.

1. In this paper we shall give a theorem concerning the oscillatory behaviour for non-trivial solutions of the equation

$$\Delta^2 u+\Delta u+u = 0, \qquad u = u(x,y,z), \qquad \Delta^2 u = \Delta\Delta u. \tag{1}$$

In the proof we shall use the mean value theorem for the solutions of equation (1) and some lemmas.

2. The mean value theorem [1].

If the function $u(x,y,z)$ is analytic in the sphere $K[R, X_0]$ with radius R and centre X_0, then

$$M[R, X_0, u] = \frac{1}{4\pi R^2}\iint_{\partial K} u(Y)\,dS = \sum_{i=0}^{\infty}\frac{R^{2i}}{(2i+1)!}\,\Delta^i u(X_0). \tag{2}$$

Applying formula (2), we shall prove

LEMMA 1. *If $u(x,y,z)$ is an analytic solution of* (1) *in* $K[R, X_0]$, *then*

$$M[R, X_0, u] = \frac{1}{R}\,u(X_0)\cdot f(R)+\frac{1}{R}\,\Delta u(X_0)\cdot g(R), \tag{3}$$

$$f(R) = R-\frac{R^5}{5!}+\frac{R^7}{7!}-\frac{R^{11}}{11!}+\frac{R^{13}}{13!}-\frac{R^{17}}{17!}+\frac{R^{19}}{19!}-\ldots, \tag{4}$$

$$g(R) = \frac{R^3}{3!}-\frac{R^5}{5!}+\frac{R^9}{9!}-\frac{R^{11}}{11!}+\frac{R^{15}}{15!}-\frac{R^{17}}{17!}+\frac{R^{19}}{19!}-\ldots \tag{5}$$

Proof. From (1) it follows that

$$\begin{gathered}\Delta^2 u = -\Delta u-u, \qquad \Delta^3 u = u, \qquad \Delta^4 u = \Delta u,\\ \Delta^5 u = -\Delta u-u, \qquad \ldots;\end{gathered} \tag{6}$$

using formulae (4), (5) and (2) we get (3).

Both f and g satisfy $y^{(6)}-y = 0$. This implies, together with their initial data,

$$f(R) = \frac{\sqrt{3}}{3} e^{\frac{1}{2}R}\cdot\cos\frac{\sqrt{3}}{2}R+\frac{\sqrt{3}}{3}e^{-\frac{1}{2}R}\cdot\cos\frac{\sqrt{3}}{2}R,$$

$$g(R) = \frac{1}{2}e^{\frac{1}{2}R}\cdot\cos\frac{\sqrt{3}}{2}R+\frac{\sqrt{3}}{6}e^{\frac{1}{2}R}\cdot\cos\frac{\sqrt{3}}{2}R+\frac{1}{2}e^{-\frac{1}{2}R}\cdot\sin\frac{\sqrt{3}}{2}R+$$
$$+\frac{\sqrt{3}}{6}e^{-\frac{1}{2}R}\cdot\cos\frac{\sqrt{3}}{2}R.$$

THEOREM 1. *If $u(x, y, z)$ is an analytic non-trivial solution of equation* (1) *in E_3, then $u(x, y, z)$ changes the sign exterior to every sphere.*

Proof. Let $u(x, y, z)$ be a non-trivial analytic solution of (1) in E_3 and X_0 a point such that $u(X_0) \neq 0$. Let

$$A = \frac{\sqrt{3}}{3}u(X_0)+\frac{\sqrt{3}}{6}\Delta u(X_0), \qquad B = \Delta u(X_0).$$

Then $A^2+B^2 \neq 0$. If $A \neq 0$, then

$$R\cdot M[R, X_0, u] = e^{\frac{1}{2}R}\cdot\left(A\cos\frac{\sqrt{3}}{2}R+B\sin\frac{\sqrt{3}}{2}R\right)+h(R)$$
$$= e^{\frac{1}{2}R}\cdot C\cdot\sin\left(\frac{\sqrt{3}}{2}R+\varphi\right)+h(R), \qquad \text{where } h(R)\to 0 \text{ if } R\to\infty.$$

For the sequence of points

$$R_n = \frac{2n\pi-2\varphi}{\sqrt{3}}, \qquad n = 1, 2, \ldots,$$

the function

$$C\cdot e^{\frac{1}{2}R}\cdot\sin\left(\frac{\sqrt{3}}{2}R+\varphi\right) \quad \text{changes the sign.}$$

For the sequence

$$R_n^{(1)} = \frac{(4n+1)\pi-2\varphi}{\sqrt{3}}, \qquad n = 1, 2, \ldots,$$

$$C\cdot e^{\frac{1}{2}R^1_{(n)}}\cdot\sin\left(\frac{\sqrt{3}}{2}R_n^{(1)}+\varphi\right)+h(R_n^{(1)})\to\infty,$$

and for

$$R_n^{(2)} = \frac{(4n-1)\pi - 2\varphi}{\sqrt{3}}, \quad n = 1, 2, \ldots,$$

$$C \cdot e^{\frac{1}{2}R_n^{(2)}} \cdot \sin\left(\frac{\sqrt{3}}{2} R_n^{(2)} + \varphi\right) + h(R) \to -\infty.$$

From the last conditions it follows that $R \cdot M[R, X_0, u]$ change the sign exterior to every sphere.

In the case $B \neq 0$ the proof is analogous.

Now we shall prove

THEOREM 2. *If the function $u(x, y, z)$ is of class C^6 in $K[R, X_0]$ and satisfies condition* (3), *then $u(x, y, z)$ satisfies* (1) *at the point X_0.*

Proof. The function $M[R, X_0, u]$ satisfies the condition

$$(7) \qquad RM[R, X_0, u] = R \cdot u(X_0) + \frac{R^3}{3!} \Delta u(X_0) + \frac{R^5}{5!} \Delta^2 u(X_0) + \frac{R^7}{7!} \varphi(R),$$

$\varphi(R)$ being of class C^6 on $K[R, X_0]$, [2].

Using (3) and (7) we get

$$(8) \qquad R \cdot u(X_0) + \frac{R^3}{3!} \Delta u(X_0) + \frac{R^5}{5!} \Delta^2 u(X_0) + \frac{R^7}{7!} \varphi(R) = u(X_0) \cdot f(R) + \Delta u(X_0) \cdot g(R).$$

Differentiating the last equation five times with respect to R, we obtain for $R = 0$

$$\Delta^2 u(X_0) = -\Delta u(X_0) - u(X_0).$$

3. We shall apply Theorem 2 to the proof of Theorem 3 of the Harnack type.

THEOREM 3. *If the sequence $u_n(x, y, z)$ satisfies the conditions:*

1° *$u_n(x, y, z)$ are of class C^6 on D,*

2° *$u_n(x, y, z)$ satisfies equation* (1) *on D,*

3° *$u_n(x, y, z)$ is uniformly convergent on D,*

4° *the sequence $\Delta u_n(x, y, z)$ is convergent on D, then the function $u(x, y, z) = \lim u_n(x, y, z)$ satisfies* (1) *on D.*

Proof. Using condition (3) we get

$$(9) \qquad \lim M[R, X_0, u_n(X_0)] = \frac{1}{R} \lim u_n(X_0) \cdot f(R) + \frac{1}{R} \Delta u_n(X_0) \cdot g(R)$$

at every point of D. From (9) and Theorem 2 it follows that the function $u(x, y, z)$ satisfies equation (1).

Bibliography

[1] R. Courant and D. Hilbert, *Metody matematiczeskoj fizyki*, Tom II, 1951.
[2] W. Walter, *Mittelwertsatze und ihre Verwendung zur Losung der Randwertaufgaben*, Jahresbericht der Deutscher Mathematiker Vereinigung 1958.

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