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On foliations of differential spaces

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Abstract. Generalized foliations introduced by Waliszewski [6] in the category of R. Sikorski's differential spaces [2] are investigated. It is proved that the set of all strates given by a coregular mapping of a locally connected differential space foliates this space.

1. Introduction. Waliszewski in [6] introduced the concept of a foliation of a differential space, which is a generalization of the concept of foliation to the category of differential spaces. In the present paper the well-known theorem of R. Thom on defining a foliation by a submersion is extended to the category of differential spaces (see [4] and [5]).

For a differential space M of the form (S, C) (see [1]) we will denote the set S of all its points, the differential structure C and the weakest topology τ_C on S such that all functions $\alpha \in C$ are continuous by $\underline{M}$, by $F(M)$ and $\operatorname{top} M$, respectively. For any $A \subset \underline{M}$, the differential subspace (A, C_A) will be denoted by M_A. Then $\underline{M}_A = A$, $F(M_A) = (F(M))_A$ and $\operatorname{top} M_A$ coincides with the topology induced on A by $\operatorname{top} M$ (see [1]).

2. Foliations of differential spaces and coregularity of mappings. We recall that a family $\mathscr{F}$ of differential spaces is locally homogeneous (see [6]) iff, for any $K, L \in \mathscr{F}$ and any $p \in \underline{K}$, $q \in \underline{L}$, there exists a diffeomorphism $h\colon K_U \to L_V$, where $p \in U \in \operatorname{top} K$, $q \in V \in \operatorname{top} L$ and $h(p) = q$. The family $\mathscr{F}$ is said to be a *foliation of a differential space* M iff

(i) $\underline{M} = \bigcup_{L \in \mathscr{F}} \underline{L}$,

(ii) L is connected and regularly lying in M for $L \in \mathscr{F}$ (see [4]),

(iii) for any $p \in \underline{M}$ there exist a neighbourhood V of p in $\mathscr{F}_p$, where p is a point of $\mathscr{F}_p \in \mathscr{F}$, a neighbourhood U of p in M, a differential space H, a diffeomorphism

$$\varphi\colon M_U \to \mathscr{F}_{pV} \times H, \tag{1}$$

such that the set of all connected components of sets $\underline{L} \cap U$, where $L \in \mathscr{F}$, is equal to the set of all sets $\varphi^{-1}[V \times \{b\}]$, where $b \in \underline{H}$.

THEOREM. *If M is a locally connected differential space,*

$$f\colon M \to N \tag{2}$$

is a coregular mapping (see [4]) *and the family of all differential spaces of the form $M_{f^{-1}n}$, where $n \in \underline{N}$, is locally homogeneous, then the family $\mathscr{F}$ of all differential spaces M_C, C being a connected component of $M_{f^{-1}n}$, $n \in \underline{N}$, is a foliation of M with locally connected leaves.*

Proof. (i) is obvious. To prove (ii) and (iii), take any $p \in \underline{M}$. By coregularity of (1) there exist a differential space K and a diffeomorphism

$$\psi\colon M_U \to K \times N_W, \tag{3}$$

where $p \in U \in \operatorname{top} M$, $W \in \operatorname{top} N$, such that $\operatorname{pr}_2 \circ \psi = f|U$, where $\operatorname{pr}_2\colon K \times N_W \to N_W$ is the canonical projection. We set $f(p) = n$ and for any $x \in U$

$$\psi(x) = \big(\psi_1(x), f(x)\big) \tag{4}$$

and

$$\varphi(x) = \big(\psi^{-1}(\psi_1(x), n), f(x)\big). \tag{5}$$

By assumption, M is locally connected. So, we may assume that U is connected in $\operatorname{top} M$. Therefore, by (3), both K and N_W are connected. Let V denote the set $U \cap f^{-1}n$. It is easy to check that

$$V = \psi^{-1}[\underline{K} \times \{n\}]. \tag{6}$$

Denote by C the connected (in $\operatorname{top} M$) component of the set $f^{-1}n$ containing the point p. Then $\mathscr{F}_p = M_C$. The set V is connected in M, hence, by (6), also in $M_{f^{-1}n}$.

Therefore $V \subset C$. Consequently $V \subset U \cap C$. On the other hand, for any $x \in U \cap C$, by (4), (3) and the inclusion $C \subset f^{-1}n$ we get $\psi(x) = (\psi_1(x), n) \in \underline{K} \times \{n\}$. So, $x \in V$. Thus,

$$V = U \cap C. \tag{7}$$

Let us take any $L \in \mathscr{F}$. Then $L = M_B$, where B is a connected component of the set $f^{-1}b$ for some $b \in \underline{N}$. Denote by A any connected component of the set $U \cap \underline{L}$. We have $\underline{L} = B$ and $A \subset B$. So, $A \subset f^{-1}b$. Take $a \in A$. Thus, $a \in U$ and $f(a) = b$, and therefore $b \in W$. Write $A_0 = \psi^{-1}[\underline{K} \times \{b\}]$. We have $\psi(a) = (\psi_1(a), f(a)) = (\psi_1(a), b) \in \underline{K} \times \{b\}$. So, $a \in A_0 \subset f^{-1}b$. The set A_0 is connected. Thus, $A_0 \subset B$. We have found a connected set $A_0 \subset U \cap B$ with $a \in A_0$. Hence $A_0 \subset A$. If there existed a point $c \in A - A_0$, then $c \in U$ and $\psi(c) = (\psi_1(c), f(c)) \neq (\psi_1(c), b)$. Thus, $f(c) \neq b$, i.e., $c \notin f^{-1}b$. On the other hand, $A \subset B \subset f^{-1}b$. Hence $c \in f^{-1}b$. Therefore $A_0 = A$. It follows that

$$A = \psi^{-1}[\underline{K} \times \{b\}]. \tag{8}$$

Now, according to (3) and (4) we have a smooth mapping $\psi_1\colon M_U \to K$. By (6) we get the smooth mapping $\psi^{-1}(\psi_1(\cdot), n)\colon M_U \to M_V$. Because of $\mathscr{F}_{pV} = (M_C)_V = M_V$ we obtain a smooth mapping (1). From (5), (6) and (8) we conclude without difficulty that $\varphi\colon U \to V \times \underline{H}$ is one-to-one,

$$\varphi^{-1}(s, t) = \psi^{-1}(\psi_1(s), t) \quad \text{for } (s, t) \in V \times \underline{H} \tag{9}$$

and

$$A = \varphi^{-1}[V \times \{b\}]. \tag{10}$$

From (9) and (3) it follows that the mapping (1) defined by (5) is a diffeomorphism. Equality (10) gives the shape of any connected component of the set $U \cap \underline{L}$.

We now show that every set A_1 of the form $\varphi^{-1}[V \times \{b\}]$, where $b \in \underline{H}$, is a connected component of a set $U \cap \underline{L}$, where $l \in \mathscr{F}$. The connectedness of V yields that of A_1. We have the inclusion $A_1 \subset f^{-1}b$. Denoting by C the connected component of $f^{-1}b$ such that $A_1 \subset C$ we have $A_1 \subset U \cap C = U \cap \underline{L}$, where $L = M_C$. Let us take the connected component A_2 of the set $U \cap \underline{L}$ which contains A_1. If there existed $a_1 \in A_2 - A_1$, we would have $\varphi(a_1) \notin V \times \{b\}$. Then we have $\big(\psi^{-1}(\psi_1(a_1), n), f(a_1)\big) \notin V \times \{b\}$. But $\psi^{-1}(\psi_1(a_1), n) \in \psi^{-1}[\underline{K} \times \{n\}] = V$, and therefore $a_1 \notin f^{-1}b$. On the other hand, $a_1 \in A_1 \subset A_2 \subset U \cap \underline{L} \subset \underline{L} = C \subset f^{-1}b$ gives a contradiction. Consequently $A_2 = A_1$. Thus, A_1 is a connected component of $U \cap \underline{L}$, where $L \in \mathscr{F}$. This ends the proof of (iii).

To prove (ii) consider the map (1) defined by (4) and (5). In view of (7), $f(x) = n$ for any $x \in V$. This yields $\varphi(x) = (x, n)$. Thus, $x = \varphi^{-1}(x, n)$ for $x \in V$ and $\varphi^{-1}\colon \mathscr{F}_{pV} \times H \to M_U$ is a diffeomorphism. This shows that $\mathrm{id}_V\colon \mathscr{F}_{pV} \to M$ is a regular mapping (cf. [4] and [5]); hence, $\mathscr{F}_p$ is regularly lying in M. Using (6) and (7), we check that all leaves of the foliation $\mathscr{F}$ are locally connected. This ends the proof of the theorem.

Now, let us consider a differential space M and an equivalence relation $\equiv$ in the set $\underline{M}$. For any $x \in \underline{M}$, denote the coset containing x by $\equiv(x)$. In [4], the differential space $M/\equiv$ is defined, having $\underline{M}/\equiv$ for its set of points, with differential structure $F(M/\equiv)$ defined as the largest one on $\underline{M}/\equiv$ such that the mapping

$$(x \mapsto \equiv(x))\colon M \to M/\equiv \tag{11}$$

is smooth. In the same paper we can find a theorem containing a necessary and sufficient condition for the mapping (11) to be coregular. From this and from the theorem just proved we get

COROLLARY. *If M is a locally connected differential space and $\equiv$ is an equivalence relation in $\underline{M}$ such that*

(iv) $\mathrm{pr}_1: (M \times M)_{\equiv} \to M$ *is regular (the relation* $\equiv$ *is here regarded as a subset of* $\underline{M} \times \underline{M}$),

(v) *for any* $p \in \underline{M}$ *there exist* $U \in \mathrm{top}\, M$, $Q \subset U$ *and a coregular mapping* $s: M_U \to M_Q$ *such that for any* $x \in U$ *the intersection of* Q *and the coset* $\equiv (x)$ *consists of a single point* $s(x)$,

(vi) *the family of all subspaces* $M_{\equiv(x)}$, *where* $x \in \underline{M}$, *is locally homogeneous,*

then the family of all sets M_C, *where* C *is a connected component of* $\equiv (x)$, $x \in \underline{M}$, *is a foliation of* M *having locally connected leaves.*

3. Tangent bundle of differential spaces of finite dimension as a foliation. Following Sikorski (see [3]), we say that differential space M is of (finite) dimension m iff, for any $p \in \underline{M}$, there exist a neighbourhood U of p and smooth vector fields $e_1, \ldots, e_m$ on U such that $e_1(q), \ldots, e_m(q)$ is a base of the tangent space $T_q M$ (see also [1]) for $q \in U$. Such a system of vector fields is called a *vector base* of M on U. Then we may assume that there exist functions $e^1, \ldots, e^m$ belonging to $F(M)$ such that for any $q \in U$ we have $e_i(q)(e^j) = \delta_i^j$, $i, j = 1, \ldots, m$ (Kronecker's delta) (see [3]).

Lemma. *If* M *is a Hausdorff differential space of finite dimension, then the projection* $\pi: \underline{TM} \to M$ *of the tangent bundle* TM (cf. [1]) *is a coregular mapping.*

Proof. Let $v \in TM$. We set $\pi(v) = p$. Taking a neighbourhood U of p, a vector base $e_1, \ldots, e_m$ of M on U and functions $e^1, \ldots, e^m \in F(M)$ such that $e_i(q)(e^j) = \delta_i^j$ for $q \in U$, $i, j = 1, \ldots, m$, we define a mapping

$$\eta: (TM)_{U_0} \to M_U \times R^m, \tag{12}$$

where $U_0 = \pi^{-1}[U]$,

$$\eta(w) = \big(\pi(w), \big(w(e^1), \ldots, w(e^m)\big)\big) \quad \text{for } w \in U_0. \tag{13}$$

Here R^m is the natural differential space having the Cartesian space R^m for its set of points. Using the definition (see [1]) of the differential structure of TM, we check that the mapping (12) is a diffeomorphism. Taking the natural projection $\mathrm{pr}_1: M_U \times R^m \to M_U$, we have by (13) $\mathrm{pr}_1 \circ \eta = \pi \mid U_0$, which ends the proof of the lemma.

Theorem and lemma yield the following

Proposition. *If* M *is a locally connected Hausdorff differential space of finite dimension, then the family of all tangent spaces* $T_p M$, *where* $p \in \underline{M}$, *viewed as differential subspaces of* TM, *is a foliation of* TM.

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