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# On the maximum term and maximum modulus of analytic functions represented by Dirichlet series

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**Abstract.** In the present paper, the growth of functions represented by Dirichlet series and analytic in a half-plane has been studied in terms of their maximum modulus and maximum term. Thus, let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

be an analytic function represented by Dirichlet series in the half-plane $\sigma < A$ $(-\infty < A < \infty)$, where $s = \sigma + it$, $0 < \lambda_n < \lambda_{n+1}$, $\lambda_n \to \infty$ and $\liminf\limits_{n\to\infty}(\lambda_n - \lambda_{n-1}) = \beta > 0$. Set

$$M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |f(\sigma+it)|, \quad m(\sigma) = \max_{n\geq 1}\{|a_n|e^{\sigma\lambda_n}\},$$

$$N(\sigma) = \max\{n \mid m(\sigma) = |a_n|e^{\sigma\lambda_n}\},$$

then it has been shown that for all $\sigma$ sufficiently closed to $A$ and $\gamma < \beta$

$$M(\sigma) < m(\sigma)\left[1 + \frac{1+\gamma}{\gamma} N\left\{\sigma + \frac{1-e^{\sigma-A}}{N(\sigma)}\right\}\right](1-e^{\sigma-A})^{-1}.$$

To study the growth of $f(s)$ precisely the concepts of order and lower order have been introduced. Thus, $f(s)$ is of order $\varrho$ and lower order $\lambda$ in the halfplane $\sigma < A$, if and only if,

$$\lim_{\sigma\to A} \frac{\sup}{\inf} \frac{\log\log M(\sigma)}{-\log(1-e^{\sigma-A})} = \frac{\varrho}{\lambda}.$$

It has been shown that order $\varrho$ and lower order $\lambda$ of $f(s)$ are also given by

$$\frac{\varrho}{\lambda} = \lim_{\sigma\to A} \frac{\sup}{\inf} \frac{\log\log m(\sigma)}{-\log(1-e^{\sigma-A})} = -1 + \lim_{\sigma\to A} \frac{\sup}{\inf} \frac{\log \lambda_{N(\sigma)}}{-\log(1-e^{\sigma-A})}.$$

Finally it has been shown that

$$1 + \lambda \leqslant (1+\varrho) \liminf_{n\to\infty} \frac{\log \lambda_n}{\log \lambda_{n+1}}.$$

**1.** Consider the Dirichlet series

$$\sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \tag{1.1}$$

where $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geqslant 0$, $\lim_{n\to\infty} \lambda_n = \infty$, $s = \sigma + it$ ($\sigma$, $t$ real variables) and

$$\limsup_{n\to\infty} \frac{n}{\lambda_n} = D < \infty. \tag{1.2}$$

If $C$, $A$ are the abscissa of convergence and the abscissa of absolute convergence of the series (1.1) respectively, then, since (1.2) is satisfied, we have ([2], p. 166)

$$C = A = -\limsup_{n\to\infty} \frac{\log|a_n|}{\lambda_n}. \tag{1.3}$$

It is well known that the series (1.1) represents a holomorphic function $(s)$ in the half-plane $\sigma < A$ $(-\infty < A < \infty)$ and that if [1]

$$M(\sigma) \equiv M(\sigma, f) = \operatorname*{l.u.b.}_{-\infty<t<\infty} |f(\sigma+it)|,$$

then $\log M(\sigma)$ is a convex function of $\sigma$ for $\sigma < A$.

To determine the growth of $M(\sigma, f)$ precisely, the concept of order has been introduced. Thus $f(s)$ is said to be of order $\varrho$ $(0 \leqslant \varrho \leqslant \infty)$ if

$$\limsup_{\sigma\to A} \frac{\log\log M(\sigma, f)}{-\log(1-e^{\sigma-A})} = \varrho. \tag{1.5}$$

Further, it has been shown [3] that

$$\frac{\varrho}{1+\varrho} = \limsup_{n\to\infty} \frac{\log\{A\lambda_n + \log|a_n|\}}{\log \lambda_n}. \tag{1.6}$$

In a similar fashion, the concept of lower order can be introduced. Thus $f(s)$ is of lower order $\lambda$ if

$$\liminf_{\sigma\to A} \frac{\log\log M(\sigma, f)}{-\log(1-e^{\sigma-A})} = \lambda \quad (0 \leqslant \lambda \leqslant \infty). \tag{1.7}$$

$f(s)$ will be said to be of regular growth if $\varrho = \lambda$.

Since the series (1.1) converges absolutely for every $\sigma < A$, the sequence

$$|a_1|e^{\sigma\lambda_1},\ |a_2|e^{\sigma\lambda_2},\ \ldots,\ |a_n|e^{\sigma\lambda_n},\ \ldots$$

tends to zero for all values of $\sigma < A$. For every $\sigma < A$, there is therefore one term of this sequence which is greater than or equal to all the rest.

This term will be called the maximum term for the given value of $\sigma$ and we shall denote its value by $m(\sigma)$. Thus

$$m(\sigma) \equiv m(\sigma, f) = \max_{n \geq 1} \{|a_n| \exp(\sigma\lambda_n)\}.$$

If there are more than one maximum term for some $\sigma < A$, we shall consider the term with the highest $\lambda_n$ as the maximum term. Thus, if

$$\lambda_{N(\sigma)} \equiv \lambda_{N(\sigma, f)} = \max\{\lambda_n \mid m(\sigma) = |a_n| e^{\sigma\lambda_n}\}$$

we have

$$m(\sigma) = |a_{N(\sigma)}| e^{\sigma\lambda_{N(\sigma)}}. \tag{1.8}$$

The purpose of the present paper is to study the growth of the functions $m(\sigma, f)$ and $\lambda_{N(\sigma, f)}$ in relation to the growth of the function $M(\sigma, f)$. We first obtain certain relations which show how the growth of one influences the growth of the other. Further, it is shown that (1.5) and (1.7) still hold good with $M(\sigma, f)$ replaced by $m(\sigma, f)$. We also obtain formulae which give the growth constants $\varrho$ and $\lambda$ in terms of the function $\lambda_{N(\sigma, f)}$. Finally we give a decomposition theorem for functions of irregular growth. We shall suppose throughout that the series (1.1) converges absolutely to $f(s)$ for all $\sigma < A$ and that condition (1.2) is satisfied.

**2.** We first give a lemma.

LEMMA. *If the Dirichlet series* $\sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *converges absolutely to* $f(s)$ *for all* $\sigma < A$, *then*

$$\log m(\sigma) = \log m(\sigma_1) + \int_{\sigma_1}^{\sigma} \lambda_{N(u)}\, du, \qquad -\infty < \sigma_1 < \sigma < A. \tag{2.1}$$

Proof. Taking the axes of coordinate $OX$ and $OY$, if we plot the points $A_n$ with the coordinates $(\lambda_n, g_n)$, where $g_n = -\log|a_n|$, we can construct a Newton polygon $\Pi(f)$ having some of the points $A_n$ as its vertices while the remainder lie either on it or on one side of it. If $G_n$ is the ordinate of the point of the abscissa $\lambda_n$ on the curve $\Pi(f)$, then $G_n \leq g_n$. So it can easily be seen that the series $\sum_{n=1}^{\infty} |a_n| \exp(\sigma\lambda_n)$ will be dominated by $W(\sigma) = \sum_{n=1}^{\infty} \exp(\sigma\lambda_n - G_n)$ with the rectifying ratio $R_n = \exp\left(\frac{G_n - G_{n-1}}{\lambda_n - \lambda_{n-1}}\right)$. Both the series have the same maximum terms and rank for all $\sigma < A$. $R_n$ is a non-decreasing function of $n$ and tends to $\exp(A)$ as $n \to \infty$. The rank of the maximum term is equal to $n$ for all $\sigma$ satisfying $\log R_n \leq \sigma < \log R_{n+1}$. Now the lemma can easily be proved on lines similar to those given by G. Valiron [4]. Hence we omit the proof.

THEOREM 1. *If* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *represents a holomorphic function for* $\sigma < A$ *and*

(2.2) $$\liminf_{n\to\infty}(\lambda_n - \lambda_{n-1}) = \beta > 0,$$

*then for every* $\gamma < \beta$

(2.3) $$M(\sigma) < m(\sigma)\left[1+\frac{\gamma+1}{\gamma}N\left\{\sigma+\frac{1-e^{\sigma-A}}{N(\sigma)}\right\}\right](1-e^{\sigma-A})^{-1}.$$

Proof. We have

$$M(\sigma) \leqslant \sum_{n=1}^{\infty}|a_n|\exp(\sigma\lambda_n) < W(\sigma) = \sum_{n=1}^{\infty}\exp(\sigma\lambda_n - G_n)$$
$$\leqslant (1+n)m(\sigma)+\sum_{k=1}^{\infty}\exp(-G_{n+k}+\sigma\lambda_{n+k}), \qquad n \geqslant N(\sigma) > n_0(\gamma).$$

Now, for $k \geqslant 1$,

$$\begin{aligned}\sigma\lambda_{n+k}-G_{n+k} &= (\lambda_{n+k}-\lambda_{n+k-1})(A-\log R_{n+k})+\\ &\quad+(\lambda_{n+k-1}-\lambda_{n+k-2})(A-\log R_{n+k-1})+\ldots+\\ &\quad+(\lambda_{n+1}-\lambda_n)(A-\log R_{n+1})+\log m(\sigma)+(\sigma-A)(\lambda_{n+k}-\lambda_n)\\ &\leqslant (\lambda_{n+k}-\lambda_n)(\sigma-\log R_{n+1})+\log m(\sigma)\\ &< \gamma k(\sigma-\log R_{n+1})+\log m(\sigma), \qquad \text{since } n > n_0(\gamma).\end{aligned}$$

So,

$$\sum_{k=1}^{\infty}\exp\{\sigma\lambda_{n+k}-G_{n+k}\} < m(\sigma)\exp(\gamma\sigma)\{R_{n+1}^{\gamma}-\exp(\gamma\sigma)\}^{-1}.$$

Hence,

$$M(\sigma) < m(\sigma)\left\{1+n+\frac{\exp(\gamma\sigma)}{R_{n+1}^{\gamma}-\exp(\gamma\sigma)}\right\}.$$

Let $n = N(\bar{\sigma})$ be a number such that $\sigma < \bar{\sigma} < A$ and $\bar{\sigma} < \log R_{n+1}$; then

$$M(\sigma) < m(\sigma)\left\{1+N(\bar{\sigma})+\frac{\exp(\gamma\sigma)}{\exp(\gamma\bar{\sigma})-\exp(\gamma\sigma)}\right\}$$
$$< m(\sigma)\left[1+N(\bar{\sigma})+\frac{1}{(\bar{\sigma}-\sigma)\gamma}\right].$$

If $\bar{\sigma} = \sigma + \dfrac{1-e^{\sigma-A}}{N(\sigma)}$, then

$$M(\sigma) < \left[1+\frac{\gamma+1}{\gamma}N\left\{\sigma+\frac{1-e^{\sigma-A}}{N(\sigma)}\right\}\right]m(\sigma)(1-e^{\sigma-A})^{-1},$$

which completes the proof.

THEOREM 2. *If* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *is an analytic function in* $\sigma < A$, *having order* $\varrho$ *and lower order* $\lambda$ $(0 < \varrho < \infty)$ *and* (2.2) *is satisfied, then*

$$(2.4)\qquad \lim_{\sigma\to A} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m(\sigma)}{-\log(1-e^{\sigma-A})} = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

Proof. The proof of the theorem follows in a straightforward manner by using relations (2.1), (2.3) and the fact that $m(\sigma) \leqslant M(\sigma)$.

THEOREM 3. *If* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *is an analytic function in* $\sigma < A$ *with order* $\varrho$ $(0 < \varrho)$ *and lower order* $\lambda$ *and* (2.2) *is satisfied, then*

$$(2.5)\qquad 1+\varrho = \limsup_{\sigma\to A} \frac{\log \lambda_{N(\sigma)}}{-\log(1-e^{\sigma-A})}.$$

Proof. Using (1.6), we can easily obtain

$$(2.6)\qquad \log m(\sigma) < \lambda_{N(\sigma)}^{\frac{\varrho}{1+\varrho}+\varepsilon_2} \quad \text{for} \quad \sigma > \sigma_0(\varepsilon_2);$$

or

$$\frac{\log\log m(\sigma)}{-\log(1-e^{\sigma-A})} < \left(\frac{\varrho}{1+\varrho}+\varepsilon_2\right)\frac{\log \lambda_{N(\sigma)}}{-\log(1-e^{\sigma-A})}.$$

Taking limits, we have

$$(2.7)\qquad 1+\varrho \leqslant \limsup_{\sigma\to A} \frac{\log \lambda_{N(\sigma)}}{-\log(1-e^{\sigma-A})}.$$

Further, it can easily be derived that for $\sigma > \sigma_1(\varepsilon)$

$$(2.8)\qquad C\lambda_{N(\sigma)}(1-e^{\sigma-A}) \leqslant \int_{\sigma}^{\sigma+\frac{1}{2}(1-e^{\sigma-A})} \lambda_{N(x)}\,dx$$
$$< \log m\{\sigma+\tfrac{1}{2}(1-e^{\sigma-A})\} < B(1-e^{\sigma-A})^{-(\varrho+\varepsilon)},$$

where $B$ and $C$ are constants.

(2.8) easily leads to

$$(2.9)\qquad 1+\varrho \geqslant \limsup_{\sigma\to A} \frac{\log \lambda_{N(\sigma)}}{-\log(1-e^{\sigma-A})}.$$

Combining (2.7) and (2.9) we get (2.5).

THEOREM 4. *If* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *is an analytic function in* $\sigma < A$ *with order* $\varrho$ $(0 < \varrho < \infty)$ *and lower order* $\lambda$ *and* (2.2) *is satisfied, then*

$$(2.10)\qquad 1+\lambda = \liminf_{\sigma\to A} \frac{\log \lambda_{N(\sigma)}}{-\log(1-e^{\sigma-A})}.$$

To prove the theorem we need the following

LEMMA. *Let $\varphi(\sigma)$ be a positive, non-decreasing function on $(-\infty, A)$ such that*

$$\liminf_{\sigma\to A} \frac{\log\varphi(\sigma)}{-\log(1-e^{\sigma-A})} = a.$$

*Let $a$, $\beta$ be real numbers for which $\beta > a$ and $a/\beta < \gamma < 1$. If $\sigma'$ is a number for which*

$$1-2^{-\frac{1}{1-\gamma}} \leqslant \exp(\sigma'-A) < 1,$$

*and also*

$$\frac{\log\varphi(\sigma')}{-\log(1-e^{\sigma'-A})} \leqslant \beta\gamma,$$

*hen for $2\sigma'-A \leqslant \sigma \leqslant \sigma'$,*

$$\varphi(\sigma) \leqslant \{1-\exp(\sigma-A)\}^{-\beta}.$$

Proof. We see that

$$\log\varphi(\sigma) \leqslant \log\varphi(\sigma') \leqslant -\beta\gamma\log(1-e^{\sigma'-A}). \tag{2.11}$$

Since

$$\exp(\sigma'-A) \geqslant 1-\exp\left\{-\frac{\log 2}{1-\gamma}\right\},$$

we have

$$\{1-\exp(\sigma'-A)\}^{-\gamma} \leqslant \{1-\exp(\sigma-A)\}^{-1} \quad \text{for} \quad 2\sigma'-A \leqslant \sigma. \tag{2.12}$$

Using (2.11) and (2.12), we get

$$\varphi(\sigma) \leqslant \{1-\exp(\sigma-A)\}^{-\beta} \quad \text{for} \quad 2\sigma'-A \leqslant \sigma \leqslant \sigma'.$$

Hence the lemma.

Proof of Theorem 4. From (2.8) we have for $\sigma > \sigma_0$,

$$C\lambda_{N(\sigma)}(1-e^{\sigma-A}) < \log m\{\sigma+\tfrac{1}{2}(1-e^{\sigma-A})\}$$

or

$$\frac{\log\lambda_{N(\sigma)}}{-\log(1-e^{\sigma-A})} < 1+\frac{\log\log m\{\sigma+\frac{1}{2}(1-e^{\sigma-A})\}}{-\log(1-e^{\sigma-A})}+o(1).$$

Taking limits and making use of (2.4), we obtain

$$a \equiv \liminf_{\sigma\to A} \frac{\log\lambda_{N(\sigma)}}{-\log(1-e^{\sigma-A})} \leqslant 1+\lambda. \tag{2.13}$$

Further, from (2.6) we can easily conclude that

$$\lambda \leqslant \frac{a\varrho}{1+\varrho}. \tag{2.14}$$

Again from (2.6) we get, for $\sigma > \sigma_0$,

$$\log m(\sigma) < \lambda_{N(\sigma)}. \tag{2.15}$$

Let $\varepsilon$ be a small positive number for which $\alpha+\varepsilon \neq 1$, and let $S_1$ be the set of all maximal (in length) intervals of the form $[\xi_i, \xi_i']$ for which

$$\log m(\sigma) \leqslant \{1-\exp(\xi-A)\}^{-(\lambda+\varepsilon/2)}, \qquad \xi_i \leqslant \xi \leqslant \xi_i'. \tag{2.16}$$

Thus, every $\xi$ in $(-\infty, A)$ for which (2.16) holds is in precisely one interval in the set $S_1$. From the above lemma it is clear that each interval $[\xi_i, \xi_i']$ in $S_1$ contains the associated interval $[2\xi_i'-A \leqslant \xi \leqslant \xi_i']$ provided $\xi_i' > \sigma_1$.

Now let $S_2$ be the set of all maximal (in length) intervals of the form $[\sigma_i, \sigma_i']$ for which

$$\lambda_{N(\sigma)} \leqslant \{1-\exp(\sigma-A)\}^{-(\alpha+\varepsilon)} \quad \text{for} \quad \sigma_i \leqslant \sigma < \sigma_i'.$$

Since (2.15) holds for $\sigma > \sigma_0$, we have

$$\log m(\sigma) < \{1-\exp(\sigma-A)\}^{-(\alpha+\varepsilon)} \quad \text{for} \quad \sigma_0 < \sigma_i \leqslant \sigma < \sigma_i'.$$

But from (2.14) we obtain

$$\alpha+\varepsilon > \lambda+\varepsilon/2.$$

Hence each $[\xi_i, \xi_i']$ with $\xi_i > \sigma_0$ is contained in some $[\sigma_j, \sigma_j']$. Therefore for $[\xi_i, \xi_i']$ with $\xi_i \geqslant \max[\sigma_0, \sigma_1]$ we find

$$\begin{aligned} \log m(\xi_i') - \log m(2\xi_i'-A) &= \int_{2\xi_i'-A}^{\xi_i'} \lambda_{N(x)}\,dx \leqslant \int_{2\xi_i'-A}^{\xi_i'} \{1-\exp(x-A)\}^{-(\alpha+\varepsilon)}\,dx \\ &\leqslant \frac{\{1-\exp(\xi_i'-A)\}^{-(\alpha+\varepsilon-1)}}{(\alpha+\varepsilon-1)\exp\{2(\xi_i'-A)\}}\,[1-\{1+\exp(\xi_i'-A)\}^{-(\alpha+\varepsilon-1)}]. \end{aligned} \tag{2.17}$$

However, the properties of $S_1$, and in particular (2.16), give

$$\begin{aligned} \log m(\xi_i') - \log m(2\xi_i'-A) &\geqslant \{1-\exp(\xi_i'-A)\}^{-(\lambda+\varepsilon/2)} - \{1-\exp(2\xi_i'-2A)\}^{-(\lambda+\varepsilon/2)} \\ &= \{1-\exp(\xi_i'-A)\}^{-(\lambda+\varepsilon/2)}[1-\{1+\exp(\xi_i'-A)\}^{-(\lambda+\varepsilon/2)}]. \end{aligned} \tag{2.18}$$

Combining (2.17) and (2.18), we see for $\xi_i' > \max(\sigma_0, \sigma_1)$

$$\begin{aligned} &\{1-\exp(\xi_i'-A)\}^{-(\lambda+\varepsilon/2)}[1-\{1+\exp(\xi_i'-A)\}^{-(\lambda+\varepsilon/2)}] \\ &\qquad \leqslant \frac{\{1-\exp(\xi_i'-A)\}^{-(\alpha+\varepsilon-1)}}{(\alpha+\varepsilon)\exp\{2(\xi_i'-A)\}}\,[1-\{1+\exp(\xi_i'-A)\}^{-(\alpha+\varepsilon-1)}] \end{aligned}$$

or

$$[1-\exp(\xi_i'-A)]^{-(\lambda+\varepsilon/2)}K \leqslant (\alpha+\varepsilon-1)^{-1}\{1-\exp(\xi_i'-A)\}^{-(\alpha+\varepsilon-1)}, \tag{2.19}$$

where

$$K = \frac{[1-\{1+\exp(\xi_i'-A)\}^{-(\lambda+\varepsilon)/2}]\exp\{2(\xi_i'-A)\}}{1-\{1+\exp(\xi_i'-A)\}^{-(\alpha+\varepsilon-1)}}.$$

We at once note that $a \geqslant 1$, since $\lambda \geqslant 0$, $\varepsilon$ can be arbitrarily small, and there must be $\xi_i'$ arbitrarily close to $A$ for which (2.19) holds. Again using the fact that there must be $\xi_i'$ arbitrarily close to $A$ for which (2.19) holds, we conclude that

$$\lambda+\varepsilon/2 \leqslant a+\varepsilon-1.$$

Since $\varepsilon$ is arbitrarily small, we have

$$\lambda+1 \leqslant a. \tag{2.20}$$

Combining (2.13) and (2.20), we obtain (2.10).

**3.** In this section we obtain some results concerning functions of irregular growth. First we have

THEOREM 5. *If* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *is an analytic function for* $\sigma < A$ *with order* $\varrho$ $(0<\varrho)$ *and lower order* $\lambda$ *and* (2.2) *is satisfied, then*

$$1+\lambda \leqslant (1+\varrho)\liminf_{n\to\infty} \frac{\log\lambda_n}{\log\lambda_{n+1}}. \tag{3.1}$$

Proof. Let

$$a = \liminf_{n\to\infty} \frac{\log\lambda_n}{\log\lambda_{n+1}};$$

then for every $\varepsilon>0$ there exists a sequence $\{p(t)\}$ such that

$$\lambda_{n_{p(t)}} < \lambda^{\beta}_{n_{p(t)+1}}, \qquad \beta = a+\varepsilon.$$

Let $\sigma_t$ be a value of $\sigma$ at which $N(\sigma)$ jumps from a value less than or equal to $n_{p(t)}$ to a value greater than or equal to $n_{p(t)}+1$; i.e.,

$$N(\sigma_t-0) \leqslant n_{p(t)} < n_{p(t)+1} \leqslant N(\sigma_t+0);$$

then

$$\lambda_{N(\sigma_t-0)} \leqslant \lambda_{n_{p(t)}} < \lambda^{\beta}_{n_{p(t)+1}} \leqslant \lambda^{\beta}_{N(\sigma_t+0)}$$

or

$$\log\lambda_{N(\sigma_t-0)} < \beta\log\lambda_{N(\sigma_t-0)}$$

or

$$\frac{\log\lambda_{N(\sigma_t-0)}}{-\log\{1-\exp(\sigma_t-A)\}} < \beta\frac{\log\lambda_{N(\sigma_t+0)}}{-\log\{1-\exp(\sigma_t-A)\}}.$$

Taking limits, we have

$$1+\lambda \leqslant \limsup_{t\to\infty} \frac{\log\lambda_{N(\sigma_t-0)}}{-\log\{1-\exp(\sigma_t-A)\}}$$

$$\leqslant \beta\limsup_{t\to\infty} \frac{\log\lambda_{N(\sigma_t+0)}}{-\log\{1-\exp(\sigma_t-A)\}} \leqslant \beta(1+\varrho)$$

or

$$1+\lambda \leqslant (1+\varrho)\liminf_{n\to\infty}\frac{\log\lambda_n}{\log\lambda_{n+1}}.$$

Hence the theorem.

THEOREM 6. *If* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *is an analytic function in* $\sigma < A$ *with order* $\varrho$ $(> 0)$ *and lower order* $\lambda$ *such that* $\lambda < \mu < \varrho$, *then*

$$f(s) = g(s)+h(s),$$

*where* $g(s)$ *has order less than or equal to* $\mu$ *and*

$$h(s) = \sum_{k=1}^{\infty} b_k \exp(s\lambda_{m_k})$$

*(*$\{m_k\}$ *the integers for which* $b_k \neq 0$*) satisfies*

$$1+\lambda \geqslant (1+\mu)\liminf_{k\to\infty}\frac{\log\lambda_{m_k}}{\log\lambda_{m_{k+1}}}. \tag{3.2}$$

Proof. Let

$$g(s) = \sum_{k=1}^{\infty} C_k \exp(s\lambda_k),$$

where $C_k = a_k$ if $\log|a_k| \leqslant \lambda_k^{\mu/(1+\mu)} - A\lambda_k$, otherwise $C_k = 0$. The order of $g(s)$ is less than or equal to $\mu$. Set

$$h(s) = f(s)-g(s) = \sum_{k=1}^{\infty} b_{m_k}\exp(s\lambda_{m_k}),$$

where $\{m_k\}$ is the sequence of integers for which $b_{m_k} \neq 0$,

$$B_{m_k} = |b_{m_k}| \quad \text{and} \quad \log B_{m_k} > \lambda_{m_k}^{\mu/(1+\mu)} - A\lambda_{m_k}.$$

Let

$$\sigma_k - A = \log(1-e^{-1}\lambda_{m_k}^{-1/(1+\mu)}).$$

Then, for $\sigma_k \leqslant \sigma < \sigma_{k+1}$, we have

$$M(\sigma) \geqslant B_{m_k}\exp(\sigma\lambda_{m_k}) \geqslant B_{m_k}\exp(\sigma_k\lambda_{m_k})$$

or

$$\begin{aligned}\log M(\sigma) &\geqslant \log B_{m_k} + \sigma_k\lambda_{m_k} > \lambda_{m_k}^{\mu/(1+\mu)} + (\sigma_k - A)\lambda_{m_k}\\ &= \lambda_{m_k}\{\lambda_{m_k}^{-1/(1+\mu)} + \log(1-e^{-1}\lambda_{m_k}^{-1/(1+\mu)})\}\end{aligned}$$

or

$$\log\log M(\sigma) \geqslant \log\lambda_{m_k} + \log\{\lambda_{m_k}^{-1/(1+\mu)} + \log(1-e^{-1}\lambda_{m_k}^{-1/(1+\mu)})\}.$$

But

$$-\log\{1-\exp(\sigma-A)\} < \frac{1+\mu+\log\lambda_{m_{k+1}}}{1+\mu}$$

and

$$\lambda_{m_k}^{-1/(1+\mu)}+\log(1-e^{-1}\lambda_{m_k}^{-1/(1+\mu)}) > \left(1-\frac{3}{2e}\right)\lambda_{m_{k+1}}^{-1/(1+\mu)}.$$

So

$$\begin{aligned}\frac{\log\log M(\sigma)}{-\log(1-e^{\sigma-A})} &\geqslant \frac{(1+\mu)\log\lambda_{m_k}}{1+\mu+\log\lambda_{m_{k+1}}}+(1+\mu)\frac{\log\{\lambda_{m_k}^{-1/(1+\mu)}+\log(1-e^{-1}\lambda_{m_k}^{-1/(1+\mu)})\}}{1+\mu+\log\lambda_{m_{k+1}}}\\ &> -1+o(1)+(1+\mu)\frac{\log\lambda_{m_k}}{\log\lambda_{m_{k+1}}}.\end{aligned}$$

Passing to limits, we have

$$1+\lambda \geqslant (1+\mu)\liminf_{k\to\infty}\frac{\log\lambda_{m_k}}{\log\lambda_{m_{k+1}}}$$

which completes the proof.

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