ANNALES
POLONICI MATHEMATICI
XLIV (1984)

Functional equations related to non-additive information measures

by A. KAMIŃSKI (Katowice),
P. N. RATHIE and LILIAN T. SHENG (Campinas, Brazil)

Abstract. A simple method of the solving of the following system of functional equations for measurable functions is presented:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i y_j, u_i v_j) = \sum_{i=1}^{n} g(x_i, u_i)+\sum_{j=1}^{m} h(y_j, v_j)+\lambda\Big[\sum_{i=1}^{n} g(x_i, u_i)\Big]\cdot\Big[\sum_{j=1}^{m} h(y_j, v_j)\Big],$$

where $f, g, h\colon [0, 1]\times[0, 1]\to \boldsymbol{R}$, λ is a real constant ($\neq 0$) and $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_m)$, $(u_1, \ldots, u_n)$, $(v_1, \ldots, v_m)$ are systems of non-negative numbers such that $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j = 1$, $\sum_{i=1}^{n} u_i \leqslant 1$, $\sum_{j=1}^{m} v_j \leqslant 1$ and $n, m = 2, 3$. The method consists in reducing the above system of equations to the Pexider equation on $[0, 1]\times[0, 1]$. As a consequence, a generalization of the result given in [3], [9] and [12] and, in application to the information theory, axiomatic characterizations of some non-additive information measures (non-additive entropy, directed divergence, inaccuracy) are obtained.

Introduction. The use of a given measure in the information theory can be mathematically justified by deriving it from an adequate system of postulates (axioms) describing the main properties of the theory. This often leads to functional equations of a special type, the solving of which may be interesting in itself. The equations have been usually solved for various classes of functions and by using various methods. There exists a large bibliography concerning this subject (cf. [2] and [8]).

In this paper, we shall consider functional equations in the domains

$$\Delta_n = \Big\{P = (p_1, \ldots, p_n)\colon\ p_i \geqslant 0 \text{ for } i = 1, \ldots, n,\ \sum_{i=1}^{n} p_i = 1\Big\}$$

and

$$\Delta'_n = \Big\{P = (p_1, \ldots, p_n)\colon\ p_i \geqslant 0 \text{ for } i = 1, \ldots, n,\ \sum_{i=1}^{n} p_i \leqslant 1\Big\},$$

where $n = 2, 3, \ldots$

We shall namely deal with the equations:

$$(1.1)\qquad \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i y_j) = \sum_{i=1}^{n} g(x_i)+\sum_{j=1}^{m} h(y_j)+\lambda\Big[\sum_{i=1}^{n} g(x_i)\Big]\cdot\Big[\sum_{j=1}^{m} h(y_j)\Big]$$

and

$$(1.2)\qquad \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i y_j,\, u_i v_j)$$
$$= \sum_{i=1}^{n} g(x_i, u_i)+\sum_{j=1}^{m} h(y_j, v_j)+\lambda\Big[\sum_{i=1}^{n} g(x_i, u_i)\Big]\cdot\Big[\sum_{j=1}^{m} h(y_j, v_j)\Big],$$

where $X=(x_1, \ldots, x_n)\in\Delta_n$, $Y=(y_1, \ldots, y_m)\in\Delta_m$, $U=(u_1, \ldots, u_n)\in\Delta'_n$, $V=(v_1, \ldots, v_m)\in\Delta'_m$ for $n, m = 2, 3, \ldots$, the functions f, g, h are real valued, defined on $I=[0, 1]$ and $I\times I$, respectively, and λ is a real constant.

The case $\lambda = 0$ has been considered in several papers, first by T. W. Chaundy, J. B. Mc Leod [4] for equations (1.1) with $f=g=h$ and by Pl. Kannappan [7] for equations (1.2) with $f=g=h$.

The case $\lambda \neq 0$ was studied by M. Behara and P. Nath [3], D. P Mittal [9] and B. D. Sharma and I. J. Taneja [12] (there are some inaccuracies in the results given in those papers – see Remark in Section 3). Equations (1.1) were dealt with in [3] and [12] (for $f=g=h$) and in [9] (for arbitrary f, g, h) and equations (1.1) (for $f=g=h$) in [12]. In each of the three papers, the equations are solved in the class of continuous functions and a common method is used: the reduction of (1.1) or (1.2) to the Cauchy or related equations by checking suitable equations for integers and rational numbers and then by applying continuity. It is worth noting that equations (1.1), (1.2) were considered for all $n, m = 2, 3, \ldots$ in [3], [9], [12].

The latter case will be considered also in this paper and therefore equations (1.1), (1.2) will be always meant with a constant $\lambda \neq 0$ throughout the sequel. We shall extend the results in [3], [9], [12] by giving all measurable solutions of (1.1) and (1.2). Moreover, we shall show that systems (1.1) and (1.2) of equations are equivalent to the systems of equations:

$$(1.3)\qquad \sum_{i=1}^{n}\sum_{j=1}^{m} F(x_i y_j) = \Big[\sum_{i=1}^{n} G(x_i)\Big]\cdot\Big[\sum_{j=1}^{m} H(y_j)\Big] \qquad (X\in\Delta_n,\ Y\in\Delta_m)$$

and

$$(1.4)\qquad \sum_{i=1}^{n}\sum_{j=1}^{m} F(x_i y_j,\, u_i v_j) = \Big[\sum_{i=1}^{n} G(x_i, u_i)\Big]\cdot\Big[\sum_{j=1}^{m} H(y_j, v_j)\Big]$$
$$(X\in\Delta_n,\ Y\in\Delta_m,\ U\in\Delta'_n,\ V\in\Delta'_m),$$

respectively, as well as to the Pexider equations:

$$(1.5)\qquad F(xy) = G(x)\cdot H(y) \qquad (x, y\in I)$$

and

(1.6) $$F(xy, uv) = G(x, u) \cdot H(y, v) \quad (x, y, u, v \in I),$$

respectively. Systems (1.1)–(1.4) can be considered for all $n, m = 2, 3, \ldots$ or only for $n, m = 2, 3$ and both the cases are equivalent. So the problem is reduced to the solving of the Pexider equation on I and on $I \times I$, respectively.

The proof of equivalence and the solving of the Pexider equation will be given (in Section 2) first for the two-dimensional case and later, as a consequence, for the other one (see Section 3). The method of the proof consists in using suitable substitutions and a lemma, based on the result of Z. Daroczy and L. Losonczi [6] on additive measurable functions on a closed interval.

In particular, the solutions of (1.1) and (1.2) for $f = g = h$ lead to characterizations of the following non-additive information measures considered in information theory (see Section 4):

(a) the non-additive entropy of degree α,

(b) the directed divergence of type α,

(c) the inaccurancy of type $1+\beta$.

Axiomatic characterizations of non-additive information measures by using other types of functional equations were given by several authors, e.g. by Z. Daroczy [5], P. N. Rathie and Pl. Kannappan [10] and B. D. Sharma [11]. For more references and details see [2] and [8].

2. The main theorem. We will use the notation described in Section 1: Δ_n; Δ'_n; $X = (x_1, \ldots, x_n)$; $Y = (y_1, \ldots, y_m)$ etc.; $I = [0, 1]$. Moreover, let $I^0 = (0, 1)$ and $\boldsymbol{R}$ – the set of all real numbers. By a measurable function f: $I \times I \to \boldsymbol{R}$, we will mean a function which is measurable with respect to each of its variables.

We start with the following lemma:

LEMMA. *Let f: $I \times I \to \boldsymbol{R}$ be a measurable function such that*

(2.1) $$\sum_{i=1}^{n} f(x_i, u_i) = 0$$

for all $X \in \Delta_n$, $U \in \Delta'_n$ and $n = 2, 3$. Then

(2.2) $$f(x, u) = 0 \quad \textit{for } x, u \in I.$$

Proof. By substituting $X = (x, y, 1-x-y)$, $U = (u, v, 1-u-v) \in \Delta_3$ and $X = (x+y, 1-x-y)$, $U = (u+v, 1-u-v) \in \Delta_2$ in (2.1) and subtracting the identities obtained, we get

(2.3) $$f(x+y, u+v) = f(x, u) + f(y, v),$$

where $x, y, u, v, x+y, u+v \in I$. Hence the functions $\varphi_1(x) = f(x, 0)$ and

$\varphi_2(x) = f(0, x)$ are additive on I, i.e.,

(2.4) $$\varphi_i(x+y) = \varphi_i(x)+\varphi_i(y) \quad \text{if } x, y, x+y \in I$$

for $i = 1, 2$. On the other hand, the substituting $y = u = 0$ in (2.3) yields

(2.5) $$f(x, v) = \varphi_1(x)+\varphi_2(v) \quad (x, v \in I).$$

Since (2.4) implies $\varphi_i(x) = x\varphi_i(0)$ for $x \in I$, by Z. Daroczy and L. Losonczi result [6] (see also [2], p. 8), we obtain from (2.5) the equality

(2.6) $$f(x, u) = xf(1, 0)+yf(0, 1) \quad (x, u \in I).$$

Now, put $X = (1, 0)$, $U = (0, 0)$ and $X = (1, 0)$, $U = (0, 1)$ in (2.1):

(2.7) $$f(1, 0)+f(0, 0) = 0 = f(1, 0)+f(1, 0).$$

Since letting $X = (x, 1-x)$, $U = (u, 1-u) \in \Delta_2$ and $X = (x, 1-x, 0)$, $U = (u, 1-u, 0) \in \Delta_3$ in (2.1) gives $f(0, 0) = 0$, identities (2.7) yield $f(1, 0) = f(0, 1) = 0$ and thus (2.6) turns into (2.2). The proof is completed.

In the following theorem, functions involved in equations are expressed by means of the function $\Gamma(x, y)$ on $I \times I$ which is given by one of the formulae (the symbol x^α for $x = 0$ means 0 if $\alpha \neq 0$ and 1 if $\alpha = 0$):

(2.8) $\Gamma(x, u) = x^\alpha u^\beta$ $\quad (x, u \in I)$;

(2.9) $\Gamma(x, u) = x^\alpha$ if $u > 0$; $\Gamma(x, 0) = 0$ $\quad (x, u \in I)$;

(2.10) $\Gamma(x, u) = u^\beta$ if $x > 0$; $\Gamma(0, u) = 0$ $\quad (x, u \in I)$;

(2.11) $\Gamma(x, u) = 1$ if $xu > 0$; $\Gamma(x, u) = 0$ if $xu = 0$ $\quad (x, u \in I)$;

(2.12) $\Gamma(x, u) = 0$ if $u < 1$; $\Gamma(x, 1) = x^\alpha$ $\quad (x, u \in I)$;

(2.13) $\Gamma(x, u) = 0$ if $x < 1$; $\Gamma(1, u) = u^\beta$ $\quad (x, u \in I)$;

(2.14) $\Gamma(x, u) = 0$ if $u < 1$;
$\Gamma(x, 1) = 1$ if $x > 0$; $\Gamma(0, 1) = 0$ $\quad (x, u \in I)$;

(2.15) $\Gamma(x, u) = 0$ if $x < 1$;
$\Gamma(1, u) = 1$ if $u > 0$; $\Gamma(1, 0) = 0$ $\quad (x, u \in I)$;

(2.16) $\Gamma(x, u) = 0$ if $xu < 1$; $\Gamma(1, 1) = 1$ $\quad (x, u \in I)$;

(2.17) $\Gamma(x, u)$ arbitrary measurable $\quad (x, u \in I)$.

THEOREM 1. *Let f, g, h, F, G, H: $I \times I \to \mathbf{R}$ be measurable functions connected by the identities:*

(2.18) $$f(x, u) = \lambda^{-1}[F(x, u)-x], \quad g(x, u) = \lambda^{-1}[G(x, u)-x],$$
$$h(x, u) = \lambda^{-1}[H(x, u)-x] \quad (x, u \in I),$$

where $\lambda \in \mathbf{R}$, $\lambda \neq 0$. The following conditions are equivalent:

(I) *f, g, h satisfy* (1.2) *for* $n, m = 2, 3, \ldots$;

(II) *f, g, h satisfy* (1.2) *for* $n, m = 2, 3$;

(III) *F, G, H satisfy* (1.4) *for* $n, m = 2, 3, \ldots$;

(IV) *F, G, H satisfy* (1.4) *for* $n, m = 2, 3$;

(V) *F, G, H satisfy* (1.6);

(VI) *f, g, h are of the following form on $I \times I$:*

$$(2.19) \quad f(x, u) = \lambda^{-1}[ab\Gamma(x, u) - x], \quad g(x, u) = \lambda^{-1}[a\Gamma(x, u) - x], \quad h(x, u) = \lambda^{-1}[b\Gamma(x, u) - x];$$

(VII) *F, G, H are of the following form on $I \times I$:*

$$(2.20) \quad F(x, u) = ab\Gamma(x, u), \quad G(x, u) = a\Gamma(x, u), \quad H(x, u) = b\Gamma(x, u),$$

where Γ in (2.19) *and* (2.20) *is given by one of formulae* (2.8)–(2.17) *and a, b are arbitrary real numbers; in case* (2.17) *one of constants a, b equals to* 0.

In particular, all the continuous solutions of (1.2) *are of the form* (2.19) *on $I \times I$, where* 1° *$\Gamma(x, u)$ is given by formula* (2.8) *(for $\alpha, \beta \geqslant 0$) and constants $a, b \in \mathbf{R}$ are arbitrary or* 2° *$\Gamma(x, u)$ is arbitrary continuous and one of constants a, b is* 0.

Proof. In view of (2.18), we have (I)$\Leftrightarrow$(III), (II)$\Leftrightarrow$(IV) and (VI)$\Leftrightarrow$(VII). One can easily check that the functions f, g, h given by formulae (2.19); (2.8)–(2.17) satisfy equations (1.2) for all $n, m = 2, 3, \ldots$ or, equivalently, that the functions F, G, H given by (2.20); (2.8)–(2.17) satisfy equations (1.4) for all $n, m = 2, 3, \ldots$ Since, moreover, (I) implies (II) and (III) implies (IV), it remains only to show the two implications: (IV)$\Rightarrow$(V) and (V)$\Rightarrow$(VII).

To prove the first one, define for arbitrarily fixed $X \in \Delta_n$, $U \in \Delta_n'$ ($n = 2, 3$) the function

$$A_{XU}(y, v) = \sum_{i=1}^{n} F(x_i y, u_i v) - \sum_{i=1}^{n} G(x_i, u_i) \cdot H(y, v)$$

for $y, v \in I$. Because of (1.3), we have

$$\sum_{j=1}^{m} A_{XU}(y_j, v_j) = 0$$

for any $Y \in \Delta_m$, $V \in \Delta_m'$ ($m = 2, 3$) and hence

$$(2.21) \qquad A_{XU}(y, v) = 0 \quad (y, v \in I),$$

in view of the lemma. In turn, let

$$B_{yv}(x, u) = F(xy, uv) - G(x, u) \cdot H(y, v) \quad (x, u \in I),$$

where $y, v \in I$. Then (2.21) can be written as

$$\sum_{i=1}^{n} B_{yv}(x_i, u_i) = 0$$

for any $X \in \Delta_n$, $U \in \Delta'_n$ $(n = 2, 3)$. Again by the lemma, we have

$$B_{yv}(x, u) = 0 \quad (x, u \in I),$$

i.e., (1.6) holds, as desired.

Now, we will show the second implication: (V) $\Rightarrow$ (VII). First denote: $F_1(x) = F(x, 1)$, $F_2(x) = F(1, x)$, $G_1(x) = G(x, 1)$, $G_2(x) = G(1, x)$, $H_1(x) = H(x, 1)$ and $H_2(x) = H(1, x)$ for $x \in I$. Owing to (1.6), we have

$$F(x, u) = G(x, u) \cdot H(1, 1) = G(1, 1) \cdot H(x, u) \quad (x, u \in I); \tag{2.22}$$

$$F(x, u) = G_1(x) \cdot H_2(u) \quad (x, u \in I) \tag{2.23}$$

and

$$F_i(xy) = G_i(x) \cdot H_i(y) \quad (x, y \in I) \tag{2.24}$$

for $i = 1, 2$.

It is not difficult to reduce equation (2.24) to the Cauchy equation on $[0, \infty)$ and, consequently, to obtain the following general measurable solutions of (2.24) on I^0 for $i = 1, 2$:

$$F_i(x) = abx^\alpha, \quad G_i(x) = ax^\alpha, \quad H_i(x) = bx^\alpha \quad (x \in I^0), \tag{2.25}$$

where a, b, α are arbitrary real numbers, depending on $i = 1, 2$;

$$F_i(x) = G_i(x) = 0, \quad H_i(x) \quad \text{arbitrary} \quad (x \in I^0); \tag{2.26}$$

$$F_i(x) = H_i(x) = 0, \quad G_i(x) \quad \text{arbitrary} \quad (x \in I^0) \tag{2.27}$$

for $i = 1, 2$ (cf. [1], p. 142–145).

Combining (2.25)–(2.27) with the identities:

$$F_i(0) = G_i(0) \cdot H_i(x) = G_i(x) \cdot H_i(0) \quad (x \in I)$$

and

$$F_i(x) = G_i(1) \cdot H_i(x) = G_i(x) \cdot H_i(1) \quad (x \in I),$$

following by (2.24), one can easily find all the measurable solutions of (2.24) on I for $i = 1, 2$:

$$F_i(x) = ab(x), \quad G_i(x) = a\Gamma(x), \quad H_i(x) = b\Gamma(x) \quad (x \in I), \tag{2.28}$$

where

$$\Gamma(x) = x^\alpha \quad (x \in I); \tag{2.29}$$

$$\Gamma(x) = 1 \quad \text{if } x > 0; \quad \Gamma(0) = 0 \quad (x \in I); \tag{2.30}$$

$$(2.31) \qquad \Gamma(x) = 0 \quad \text{if } x < 1; \quad \Gamma(1) = 1 \quad (x \in I);$$

$$(2.32) \qquad \Gamma(x) \quad \text{arbitrary measurable} \quad (x \in I);$$

$\Gamma(x)$ and constants a, b, α depend on $i = 1, 2$; for (2.29)–(2.31) constants a, b are arbitrary, for (2.32) one of them is 0.

Substituting in (2.23) functions $G_1(x)$, $H_2(u)$ of the form (2.25); (2.29)–(2.31), we see that each non-trivial $F(x, u)$ is of the form:

$$F(x, u) = ab\Gamma(x, u) \qquad (x, u \in I),$$

where $a, b \neq 0$ and $\Gamma(x, u)$ is a function given by one of formulae (2.8)–(2.16).

Now, relations (2.22) allow us to determine the functions $G(x, u)$ and $H(x, u)$ as in formulae (2.20); (2.8)–(2.17) and the proof of implication (V) $\Rightarrow$ (VII), as well as of the theorem, is finished.

Remark. Formulae (2.8)–(2.16) can be expressed by one formula, if the following notation is adopted: $\tau_1 f = f$,

$$(\tau_0 f)(x) = \begin{cases} 0 & \text{if } x \in [0, 1), \\ f(1) & \text{if } x = 1 \end{cases}$$

and

$$\sigma_i^\alpha(x) = \begin{cases} x^\alpha & \text{if } x \in (0, 1],\ \alpha \in \mathbf{R}, \\ 0 & \text{if } x = 0,\ \alpha \neq 0, \\ i & \text{if } x = 0,\ \alpha = 0 \end{cases}$$

for $i = 0, 1$. Then the formula

$$\Gamma(x, u) = \tau_i \sigma_j^\alpha(x) \cdot \tau_k \sigma_l^\beta(u) \qquad (x, u \in I)$$

for $i, j, k, l = 0, 1$ and arbitrary $\alpha, \beta \in \mathbf{R}$ covers all cases (2.8)–(2.16).

3. Corollaries. From Theorem 1, we will deduce its one-dimensional analogue:

THEOREM 2. *Let $f, g, h, F, G, H\colon I \to \mathbf{R}$ be measurable functions connected by the identities:*

$$(3.1) \quad f(x) = \frac{F(x)-x}{\lambda}, \qquad g(x) = \frac{G(x)-x}{\lambda}, \qquad h(x) = \frac{H(x)-x}{\lambda} \qquad (x \in I),$$

where $\lambda \in \mathbf{R}$, $\lambda \neq 0$. The following conditions are equivalent:

(i) *f, g, h satisfy equations* (1.1) *for $n, m = 2, 3, \ldots$;*

(ii) *f, g, h satisfy equations* (1.1) *for $n, m = 2, 3$;*

(iii) *F, G, H satisfy equations* (1.3) *for $n, m = 2, 3, \ldots$;*

(iv) *F, G, H satisfy equations* (1.3) *for $n, m = 2, 3$;*

(v) *F, G, H satisfy equation* (1.5);

(vi) *f, g, h are of the following form on I:*

(3.2) $$f(x) = \frac{ab\Gamma(x) - x}{\lambda}, \quad g(x) = \frac{a\Gamma(x) - x}{\lambda}, \quad h(x) = \frac{b\Gamma(x) - x}{\lambda};$$

(vii) *F, G, H are of the following form on I:*

(3.3) $$F(x) = ab\Gamma(x), \quad G(x) = a\Gamma(x), \quad H(x) = b\Gamma(x),$$

where $\Gamma(x)$ in (3.2) *and* (3.3) *is given by one of formulae* (2.29)–(2.32) *and a, b are arbitrary real numbers; in case* (2.32) *one of them equals to* 0.

In particular, all continuous solutions of (1.1) *are of the form* (3.2) *on I, where* 1° *$\Gamma(x)$ is given by* (2.29) *(for $\alpha \geq 0$) and constants $a, b \in \mathbf{R}$ are arbitrary or* 2° *$\Gamma(x)$ is arbitrary continuous and one of constants a, b is* 0 (cf. [9]).

Proof. The functions $\bar{f}$, $\bar{g}$, $\bar{h}$, $\bar{F}$, $\bar{G}$, $\bar{H}$: $I \times I \to \mathbf{R}$ defined by the formulae:

(3.4) $$\begin{matrix} \bar{f}(x, u) = f(x), & \bar{g}(x, u) = g(x), & \bar{h}(x, u) = h(x) \\ \bar{F}(x, u) = F(x), & \bar{G}(x, u) = G(x), & \bar{H}(x, u) = H(x) \end{matrix} \quad (x, u \in I)$$

are measurable and if any of conditions (i)–(vii) is satisfied by f, g, h or F, G, H, respectively, then the suitable condition from among (I)–(VII) holds for $\bar{f}$, $\bar{g}$, $\bar{h}$ or $\bar{F}$, $\bar{G}$, $\bar{H}$, respectively. The converse relation holds, due to (3.4) and the following substitutions $u_i = v_j$ ($i = 1, \ldots, n$; $j = 1, \ldots, m$) in (1.2) and (1.4); $u = v = 0$ in (1.6); $u = 1$ in (2.19) and (2.20).

Therefore the assertion follows, by virtue of Theorem 1.

Immediately from Theorems 1 and 2, we obtain

Corollary 1. *All measurable solutions of* (1.2) *with $f = g = h$ (for $n = m = 2, 3$ or, equivalently, for $n, m = 2, 3, \ldots$) are of the form*

(3.5) $$f(x, u) = \frac{\Gamma(x, u) - x}{\lambda} \quad (x, u \in I),$$

where $\Gamma(x, u)$ is given by one of formulae (2.8)–(2.16) *or by*

(3.6) $$\Gamma(x, u) = 0 \quad (x, u \in I).$$

In particular, all continuous solutions of (1.2) *with $f = g = h$ are of the form* (3.5) *where $\Gamma(x, u)$ is given by* (2.8) *(for $\alpha, \beta \geq 0$) or by* (3.6) (cf. [12]).

Corollary 2. *All measurable solutions of* (1.1) *with $f = g = h$ (for $n, m = 2, 3$ or, equivalently, for $n, m = 2, 3, \ldots$) are of the form*

(3.7) $$f(x) = \frac{\Gamma(x) - x}{\lambda} \quad (x \in I),$$

where $\Gamma(x)$ is given by one of formulae (2.29)–(2.31) *or by*

(3.8) $$\Gamma(x) = 0 \quad (x \in I).$$

In particular, all continuous solutions of (1.1) *with* $f = g = h$ *are of the form* (3.7), *where* $\Gamma(x)$ *is given by* (2.29) (*for* $\alpha \geq 0$) *or by* (3.8) (cf. [3] and [12]).

Remark. Let us compare the results obtained for continuous solutions with those given in [3], [9] and [12].

In [3], the general continuous solutions of (1.1) for $f = g = h$ are discussed occasionally (see remarks after the proof of Theorem 3.1 in [3]). The list of solutions is not complete: the trivial solution, i.e., of the form (3.7) with $\Gamma(t) \equiv 0$, is omitted. Moreover, there is not clearly written that in the non-trivial solution (i.e., with $\Gamma(t)$ of the form (2.29)) the parameter α is non-negative.

In [12], the range of parameters, describing non-trivial continuous solutions of (1.1) for $f = g = h$ and of (1.2) for $f = g = h$, is not given properly. Namely, the assumption that the parameters α, β in formulae (2.29) and (2.8) are non-negative is omitted; on the other hand, the assumptions $\alpha \neq 1$ in (2.29) and $\alpha \neq 1$, $\beta \neq 1$ in (2.8) are superfluous and eliminate some of solutions (see [12], Theorems 1 and 2).

In [9], the continuous solutions of (1.1) are the same as in the second part of Theorem 2. However, it is not explained (for $0 \leq \beta \leq 1$) what values at the point 0 are taken by the solutions given in Theorem 2 in [9], p. 33:

$$f(x) = \frac{x}{\lambda}(abx^{\beta-1} - 1), \quad g(x) = \frac{x}{\lambda}(ax^{\beta-1} - 1), \quad h(x) = \frac{x}{\lambda}(bx^{\beta-1} - 1).$$

4. Applications to information theory. Now, we will derive from Corollaries 1 and 2 some characterizations of the following non-additive information measures:

(a) the non-additive entropy of degree β with $\beta \neq 1$ (see [5]; [2], p. 184):

$$H_n(P) = (2^{1-\beta} - 1)^{-1}\left(\sum_{i=1}^{n} p_i^{\beta} - 1\right); \tag{4.1}$$

(b) the directed divergence of type α with $\alpha \neq 1$:

$$D_n(P, Q) = (2^{\alpha-1} - 1)^{-1}\left(\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - 1\right); \tag{4.2}$$

(c) the inaccuracy of type $1+\beta$ with $\beta > 0$:

$$I_n(P, Q) = (2^{-\beta} - 1)^{-1}\left(\sum_{i=1}^{n} p_i q_i^{\beta} - 1\right), \tag{4.3}$$

where $P = (p_1, \ldots, p_n)$ is a complete probability distribution and $Q = (q_1, \ldots, q_n)$ is a probability distribution, possibly incomplete (i.e., $P \in \Delta_n$, $Q \in \Delta_n'$) for $n = 2, 3, \ldots$ The convention $0^{\alpha} = 0$ for $\alpha < 0$ and $0^0 = 1$, adopted in Section 2, is used here.

For probability distributions

$$P = (p_1, \ldots, p_n) \in \Delta'_n \quad \text{and} \quad Q = (q_1, \ldots, q_m) \in \Delta'_m,$$

let

$$P * Q = (p_1 q_1, \ldots, p_1 q_m, \ldots, p_n q_1, \ldots, p_n q_m).$$

Of course, if $P \in \Delta'_n$, $Q \in \Delta'_m$, then $P * Q \in \Delta'_{n \cdot m}$ and if $P \in \Delta_n$, $Q \in \Delta_m$, then $P * Q \in \Delta_{n \cdot m}$ $(n, m = 2, 3, \ldots)$.

THEOREM 3. *If functions $H_n(P)$ defined for $P \in \Delta_n$ satisfy the postulates:*

(i) *The Non-additivity Postulate:*

$$H_{nm}(P * Q) = H_n(P) + H_m(Q) + (2^{1-\beta} - 1) H_n(P) \cdot H_m(Q) \tag{4.4}$$

for $P \in \Delta_n$, $Q \in \Delta_m$ $(n, m = 2, 3)$, where β is a constant different from 0 *and* 1;

(ii) *The Sum Postulate:*

$$H_n(P) = \sum_{i=1}^{n} f(p_i) \tag{4.5}$$

for $P \in \Delta_n$ $(n = 2, 3, \ldots)$, where f is a measurable function such that:

(iii) *The Normalizing Postulate:*

$$f(\tfrac{1}{2}) = \tfrac{1}{2}, \tag{4.6}$$

then $H_n(P)$ represent the non-additive entropy of degree β.

Conversely, the functions $H_n(P)$ of the form (4.1) *satisfy postulates* (i)–(iii) *for all $n = 2, 3, \ldots$*

Proof. From (4.4) and (4.5), we obtain system (1.1) of functional equations for $X \in \Delta_n$, $Y \in \Delta_m$ $(n, m = 2, 3)$ and $\lambda = 2^{1-\beta} - 1 \neq 0$.

By Corollary 2, one of measurable solutions of (1.1) is of the form (3.7); (2.29):

$$f(x) = \frac{x^\alpha - x}{\lambda} \qquad (x \in I),$$

where α is an arbitrary real constant; from (4.6), it follows that $\alpha = \beta$. The other solutions ((3.7); (2.30)–(2.31) and (3.8)) are eliminated by Normalizing Postulate.

Thus postulates (i)–(iii) imply (4.1). One can easily check that $H_n(P)$ given by (4.1) fulfil postulates (i)–(iii) for all $n, m = 2, 3, \ldots$

THEOREM 4. *If functions $D_n(P, Q)$ defined for $P \in \Delta_n$, $Q \in \Delta'_n$ satisfy the postulates:*

(I) *The Non-additive Postulate:*

$$\begin{aligned} &D_{nm}(P_1 * P_2, Q_1 * Q_2) \\ &\quad = D_n(P_1, Q_1) + D_m(P_2, Q_2) + (2^{\alpha-1} - 1) D_n(P_1, Q_1) \cdot D_m(P_2, Q_2) \end{aligned} \tag{4.7}$$

for $P_1 \in \Delta_n$, $P_2 \in \Delta_m$, $Q_1 \in \Delta'_n$, $Q_2 \in \Delta'_m$ $(n, m = 2, 3)$, *where* α *is a constant different from* 0 *and* 1;

(II) *The Sum Postulate*:

$$D_n(P, Q) = \sum_{i=1}^{n} f(p_i, q_i) \tag{4.8}$$

for $P \in \Delta_n$, $Q \in \Delta'_n$ $(n = 2, 3, \ldots)$, *where* f *is a measurable function such that*:

(III) *The Normalizing Postulate*:

$$f(1, \tfrac{1}{2}) = 1, \quad f(\tfrac{1}{2}, \tfrac{1}{2}) = 0, \tag{4.9}$$

then $D_n(P, Q)$ *represent the directed divergence of type* α *for* $n = 2, 3, \ldots$

Conversely, the functions $D_n(P, Q)$ *of the form* (4.2) *satisfy postulates* (I)–(III) *for all* $n = 2, 3, \ldots$

THEOREM 5. *If functions* $I_n(P, Q)$ *defined for* $P \in \Delta_n$, $Q \in \Delta'_n$ *satisfy the postulates*:

(I′) *The Non-additive Postulate*:

$$\begin{aligned} &I_{nm}(P_1 * P_2, Q_1 * Q_2) \\ &\quad = I_n(P_1, Q_1) + I_m(P_2, Q_2) + (2^{-\beta} - 1) I_n(P_1, Q_1) \cdot I_m(P_2, Q_2) \end{aligned} \tag{4.10}$$

for $P_1 \in \Delta_n$, $P_2 \in \Delta_m$, $Q_1 \in \Delta'_n$, $Q_2 \in \Delta'_m$ $(n, m = 2, 3)$, *where* β *is a positive constant*;

(II′) *The Sum Postulate*:

$$I_n(P, Q) = \sum_{i=1}^{n} f(p_i, q_i) \tag{4.11}$$

for $P \in \Delta_n$, $Q \in \Delta'_n$ $(n = 2, 3, \ldots)$, *where* f *is a measurable function such that*:

(III′) *The Normalizing Postulate*:

$$f(1, \tfrac{1}{2}) = 1, \quad f(\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{2}, \tag{4.12}$$

then $I_n(P, Q)$ *represent the inaccuracy of type* $1+\beta$ *for* $n = 2, 3, \ldots$

Conversely, the functions $I_n(P, Q)$ *of the form* (4.3) *satisfy postulates* (I′)–(III′) *for all* $n = 2, 3, \ldots$

Proof of Theorems 4 and 5. From postulates (I), (II) and (I′), (II′), we obtain systems (1.2) of functional equations for $X \in \Delta_n$, $Y \in \Delta_m$, $U \in \Delta'_n$, $V \in \Delta'_m$ $(n, m = 2, 3)$ with $\lambda = 2^{\alpha-1} - 1$ and $\lambda = 2^{-\beta} - 1$, respectively.

By Corollary 1, measurable solutions can be given by formulae (3.5); (2.8), i.e.,

$$f(x, y) = \frac{x^{\bar{\alpha}} y^{\bar{\beta}} - x}{2^{\alpha-1} - 1} \quad (x, y \in I) \tag{4.13}$$

and

$$f(x, y) = \frac{x^{\bar{\alpha}} y^{\bar{\beta}} - x}{2^{-\beta} - 1} \quad (x, y \in I), \tag{4.14}$$

respectively.

The remain solutions of (1.2), given by (3.5); (2.9)–(2.16) and (3.6), do not fulfil Normalizing Postulates (III), (III').

By using those postulates in (4.13) and (4.14), we get $\bar{\alpha} = \alpha$, $\bar{\beta} = 1-\alpha$ in (4.13) and $\bar{\alpha} = 1$, $\bar{\beta} = \beta$ in (4.14). In view of the Sum Postulates, the functions $D_n(P, Q)$ and $I_n(P, Q)$ of the form (4.2) and (4.3), are obtained as a result of postulates (I)–(III) and (I')–(III'), respectively.

On the other hand, the functions of the form (4.2) and (4.3) satisfy postulates (I)–(III) and (I')–(III'), respectively, for all $n = 2, 3, \ldots$

Acknowledgement. The authors would like to thank the referee for some useful suggestions.

References

[1] J. Aczél, *Lectures on functional equations and their applications*, Academic Press, 1966.
[2] — and Z. Daroczy, *On measures of information and their applications*, Academic Press, 1975.
[3] M. Behara and P. Nath, *Additive and non-additive entropies of finite measurable partitions*, Lecture Notes in Mathematics 296, Springer, 1973, 102–138.
[4] T. W. Chaundy, J. B. Mc Leod, *On a functional equations*, Proc. Edin. Math. 12, Edin. Math. Notes 43 (1960), 7–8.
[5] Z. Daroczy, *Generalized information functions*, Information and Control 16 (1970), 36–51.
[6] — and L. Losonczi, *Über die Erweiterung der auf einer Punktmenge additive Funktionen*, Publ. Math. Debrecen 14 (1967), 239–245.
[7] Pl. Kannappan, *On Shannon's entropy, directed divergence and inaccuracy*, Z. Wahrs. Verw. Geb. 22 (1972), 95–100.
[8] A. M. Mathai, P. N. Rathie, *Basic concepts in information theory and statistics: axiomatic foundations and applications*, Wiley, 1975.
[9] D. P. Mittal, *On continuous solution of a functional equation*, Metrika 22 (1976), 31–40.
[10] P. N. Rathie and Pl. Kannappan, *A directed-divergence function of type β*, Information and Control 20 (1972), 38–45.
[11] B. D. Sharma and I. J. Taneja, *Entropy of type (α, β) and other generalized measures in information theory*, Metrika 22 (1975), 205–212.

INSTITUTE OF MATHEMATICS, THE POLISH ACADEMY OF SCIENCES
40-013 KATOWICE, WIECZORKA 8, POLAND
and
INSTITUTE OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE
UNIVERSITY OF CAMPINAS,
13100 CAMPINAS – SP, BRAZIL

Reçu par la Rédaction le 20. 11. 1979