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Extremal problems in some class of regular functions defined in the unit circle and applications

by PAWEŁ WIATROWSKI (Łódź)

1. Let H be the family of non-decreasing functions $\mu(t)$ defined in the interval $[0, 2\pi]$ which satisfy the condition

$$\int_0^{2\pi} d\mu(t) = 1,$$

let m, $0 \leqslant m \leqslant 1$, be an arbitrary fixed number and let $\wp_m$ be a family of functions represented by the Herglotz–Stielties integral

$$(1.1) \qquad p(z) = \int_0^{2\pi} \frac{1+e^{it}z}{1-me^{it}z}\, d\mu(t), \qquad |z| < 1,\ \mu(t) \in H.$$

Denote by $\wp$ the family of regular functions $p(z)$, $p(0) = 1$, defined in the circle $K = \{z: |z| < 1\}$ which satisfy the condition $\operatorname{re} p(z) > 0$ for $z \in K$.

Obviously, $\wp_m \subset \wp$ and $\wp_1 \equiv \wp$. It can easily be proved that $\wp_m$ is a compact and connected family for every $0 \leqslant m \leqslant 1$.

For a fixed ζ, $\zeta \in K$, we assign to every function $p(z) \in \wp_m$ a number

$$(1.2) \qquad \mathscr{F}(p) = \mathscr{F}[p(\zeta), \zeta p'(\zeta)],$$

where $\mathscr{F}(u, v)$ is an analytical function defined in the semiplane $\operatorname{re} u > 0$ and in the plane v which satisfies the condition $|\mathscr{F}'_u|^2 + |\mathscr{F}'_v|^2 > 0$ at every point (u, v). The set of all points $F(p)$ of the plane with $p(z) \in \wp_m$ will be denoted by $D(\zeta)$ and called the *set of values of functional* (1.2). Because of the compactness and connectedness of the family $\wp_m$ and of the continuity of functional (1.2) $D(\zeta)$ is a closed set.

Let $\Gamma(\zeta)$ denote the boundary of the region $D(\zeta)$ and $p_0(z)$ and arbitrary boundary function with respect to functional (1.2), i.e. a function such that $\mathscr{F}(p_0) \in \Gamma(\zeta)$.

Basing ourselves on variation formulas in the family $\wp_m$, we have investigated in this paper some properties of the boundary functions of this family with respect to functional (1.2), we have obtained sharp upper and lower estimates of the functional $\operatorname{re}\frac{zp'(z)}{p(z)}$, $|z| = r < 1$, $p(z) \in \wp_m$ and found the radius of convexity of some classes of star-like functions generated by functions of the family $\wp_m$.

2. THEOREM 1. *Every boundary function* $p_0(z)$ *with respect to functional* (1.2) *is of the form*

$$p_0(z) = \lambda_1 \frac{1+\varepsilon_1 z}{1-m\varepsilon_1 z} + \lambda_2 \frac{1+\varepsilon_2 z}{1-m\varepsilon_2 z}, \tag{2.1}$$

where $|\varepsilon_k| = 1$, $\lambda_k \geqslant 0$ *for* $k = 1, 2$, $\lambda_1 + \lambda_2 = 1$.

Proof. 1° Applying the theorem on variation in the classes of functions represented by structure formulas [1] to an arbitrary boundary function $p_0(z)$ at the point $z = \zeta$, we obtain the following two variation formulas:

$$p_\varepsilon(\zeta) = p_0(\zeta) + \varepsilon \int_{t_1}^{t_2} \frac{i(1+m)e^{it}\zeta}{(1-me^{it}\zeta)^2} |\mu_0(t) - d|\, dt, \qquad \mu_0(t) \in H, \tag{2.2}$$

where $-1 \leqslant \varepsilon \leqslant 1$, $t_1, t_2, t_1 < t_2$, are arbitrary points of the interval $[0, 2\pi]$, and d is a constant, and

$$p_\varepsilon(\zeta) = p_0(\zeta) + \varepsilon \left[\frac{1+e^{it_1}\zeta}{1-me^{it_1}\zeta} - \frac{1+e^{it_2}\zeta}{1-me^{it_2}\zeta} \right], \tag{2.3}$$

where $\varepsilon > 0$ is a sufficiently small number and t_1, t_2, where $t_1 < t_2$, $t_1, t_2 \in [0, 2\pi]$ are the discontinuity points of the function $\mu_0(t)$. Assume

$$\Delta\mathscr{F} = \mathscr{F}(p_\varepsilon) - \mathscr{F}(p_0), \tag{2.4}$$

$$T(z) = \int_{t_1}^{t_2} \frac{i(1+m)e^{it}z}{(1-me^{it}z)^2} |\mu_0(t) - d|\, dt, \tag{2.5}$$

$$\Delta p_0^{(k)} = p_\varepsilon^{(k)} - p_0^{(k)}, \quad p_\varepsilon^{(k)} = p_\varepsilon^{(k)}(\zeta), \quad p_0^{(k)} = p_0^{(k)}(\zeta), \tag{2.6}$$

for $k = 0, 1$, $p_\varepsilon^{(0)} \equiv p_\varepsilon$, $p_0^{(0)} \equiv p_0$ and

$$L(p_0) = \sum_{k=0}^{1} q_k T_\zeta^{(k)}(\zeta), \qquad T_\zeta^{(0)}(\zeta) \equiv T(\zeta), \tag{2.7}$$

where

$$q_0 = \mathscr{F}'_u(p_0), \qquad q_1 = \mathscr{F}'_v(zp'_0).$$

Let $\mathring{\mathscr{F}}$ be an arbitrary exterior point of the region $D(\zeta)$. Then there exists a point $\mathscr{F}_0$ of the boundary Γ of the region $D(\zeta)$ such that

$$|\mathscr{F}_0 - \mathring{\mathscr{F}}| \leqslant |\mathscr{F} - \mathring{\mathscr{F}}| \tag{2.8}$$

for every point $\mathscr{F}$ of the region $D(\zeta)$ which belongs to a sufficiently small neighbourhood of the point $\mathscr{F}_0$. Putting

$$\mathscr{F}_0 = \mathscr{F}(p_0), \quad \mathscr{F} = \mathscr{F}(p_\varepsilon)$$

and denoting

$$\beta = \exp[i\arg\overline{\mathscr{F}_0 - \mathring{\mathscr{F}}}],$$

by (2.4) we may write condition (2.8) in an equivalent form

$$\Delta^2\mathscr{F} + 2|\mathscr{F}_0 - \mathring{\mathscr{F}}|\operatorname{re}(\beta\Delta\mathscr{F}) \geqslant 0. \tag{2.9}$$

Functional (1.2) may be expanded in a neighbourhood of the point $\mathscr{F}(p_0)$ in a power series, and thus because of (2.2), (2.5)-(2.7), we obtain

$$\Delta\mathscr{F} = \varepsilon L(p_0) + o(\varepsilon). \tag{2.10}$$

Substituting $\Delta\mathscr{F}$ from (2.10) into (2.9), we obtain the inequality

$$2\varepsilon|\mathscr{F}_0 - \mathring{\mathscr{F}}|\operatorname{re}[\beta L(p_0)] + o(\varepsilon) \geqslant 0;$$

thus since ε is an arbitrary number, $-1 \leqslant \varepsilon \leqslant 1$, we have the equality

$$\operatorname{re}[\beta L(p_0)] = 0. \tag{2.11}$$

Replacing ζ by z and putting $a_k = \beta q_k$, $k = 0, 1$, we obtain from (2.11) by (2.5) and (2.7) the equation

$$\int_{t_1}^{t_2} G(t)\,|\mu_0(t) - d| = 0, \tag{2.12}$$

where

$$G(t) = \operatorname{re}\frac{i(1+m)e^{it}}{(1-me^{it})^3}[a_0 z(1-me^{it}z) + a_1(1+me^{it}z)].$$

By hypothesis we have $|q_0|^2 + |q_1|^2 > 0$, and thus also $|a_0|^2 + |a_1|^2 > 0$.

If the equation $G(t) = 0$ has no roots in the interval (t_1, t_2), then by (2.12) we have in this interval $\mu_0(t) - d = 0$, and thus

$$\mu_0(t) = \text{const}.$$

If the equation $G(t) = 0$ has roots in this interval, then equation (2.12) is equivalent to the equation

$$G(t) = 0,$$

i.e. to an equation of the fourth degree with respect to e^{it}. Thus this equation has no more than four roots e^{it}. Hence it follows that the function $\mu_0(t)$ is interval-wise constant, and thus it may have at most four discontinuity points (of the first kind).

2° We shall prove that the function $\mu_0(t)$ has at most two discontinuity points.

Supposing for the proof that the contrary is the case, denote by t_1, t_2, t_3, t_4, $t_1 < t_2 < t_3 < t_4$ the discontinuity points of $\mu_0(t)$ and apply the second variation formula (2.3) to the function $p_0(z)$ in the interval (t_k, t_{k+1}), $k = 1, 2, 3$.

Proceeding in the same way as in the first proof, we obtain the equation

$$\operatorname{re}[\beta \hat{L}(p_0)] = 0,$$

where

$$\hat{L}(p_0) = \sum_{k=0}^{1} q_k U_z^{(k)}(z), \qquad U_z^{(0)} = U(z),$$

$$U(z) = \operatorname{re}\left[\frac{1+e^{it_k}z}{1-me^{it_k}z} - \frac{1+e^{it_{k+1}}z}{1-me^{it_{k+1}}z}\right].$$

Assume

$$H(t) = \operatorname{re}\left[a_0 \frac{1+e^{it}z}{1-me^{it}z} + a_1 \frac{(1+m)e^{it}}{(1-me^{it}z)^2}\right]$$

and observe that $H'(t) = G(t)$ and $H(t_k) = H(t_{k+1})$. Thus there exists a point t_k^0, $t_k^0 \in (t_k, t_{k+1})$ such that

$$H_t'(t_k^0) = 0, \qquad k = 1, 2, 3.$$

The equation $G(t) = 0$ would thus have seven roots, which is impossible.

Supposing that the function $\mu_0(t)$ has three points of discontinuity, we also obtain a contradiction. Since the function $\mu_0(t)$ has at most two discontinuity points, by structure formulas (1.1) the function corresponding to it $p_0(z)$ is of the form (2.1), which was to be proved. If functional (1.2) is reduced to the form $\mathscr{F}(p) = \mathscr{F}[p(\zeta)]$, we obtain, by an analogous argument as in the proof of Theorem 1, the following

THEOREM 1′. *The boundary function $p_0(z)$ with respect to the functional*

$$\mathscr{F}(p) = \mathscr{F}[p(\zeta)]$$

is of the form

$$p_0(z) = \frac{1+\varepsilon z}{1-m\varepsilon z}, \qquad |\varepsilon| = 1.$$

COROLLARY. *The region of values of the functional*

$$\mathscr{F}(p) = p(\zeta)$$

is the circle $K(c, \varrho)$ with the centre c and the radius ϱ, where

$$c = c(r) = \frac{1+mr^2}{1-m^2r^2}, \quad \varrho = \varrho(r) = \frac{(1+m)r}{1-m^2r^2}, \quad r = |\zeta|. \tag{2.13}$$

Denote by $\wp_{m,n}$ a subclass of functions of the family $\wp_m$ of the form

$$p(z) = \sum_{k=1}^{n} \lambda_k \frac{1+\varepsilon_k z}{1-m\varepsilon_k z}, \quad \lambda_k \geqslant 0, \quad \sum_{k=1}^{n} \lambda_k = 1, \quad |\varepsilon_k| = 1.$$

Analogously as in paper [5] we prove the following theorem:

THEOREM 2. *If $p(z) \in \wp_{m,2}$, then for $z = re^{i\varphi}$ we have*

$$p(z) = c + \varkappa\gamma, \tag{2.14}$$

where c and ϱ are defined by formulas (2.13) *and* $0 \leqslant \varkappa \leqslant \varrho$, $|\gamma| = 1$.

Preserving the notation adopted above, we shall prove the following

THEOREM 3. *Let*

$$\mathscr{F}(u, v) = A(u) + B(u)v, \tag{2.15}$$

where $A(u)$ and $B(u)$ are analytical functions defined in the semiplane $\mathrm{re}\, u > 0$ *and v a finite-valued complex variable. If*

$$u = \lambda_1 p_1(z) + \lambda_2 p_2(z), \quad v = \lambda_1 z p_1'(z) + \lambda_2 z p_2'(z),$$
$$p_k(z) = \frac{1+\varepsilon_k z}{1-m\varepsilon_k z}, \quad k = 1, 2, \tag{2.16}$$

where z is an arbitrary point of the circumference with the centre at the origin and the radius r, $0 < r < 1$, then the value of the function $\mathscr{F}(u, v)$ for u and v given by (2.16) *may be written in the form*

$$\mathscr{F}(u,v) = A(u) + \frac{B(u)}{1+m}[mu^2 + (1-m)u - 1 + m(\varkappa^2 - \varrho^2)\eta], \tag{2.17}$$

where $|\eta| = 1$.

Proof. Observe first that

$$zp_k'(z) = \frac{1}{1+m}[mp_k^2(z) + (1-m)p_k(z) - 1], \quad k = 1, 2,$$

by which

$$v = \frac{1}{1+m}\left[m\big(\lambda_1 p_1^2(z) + \lambda_2 p_2^2(z)\big) + (1-m)\big(\lambda_1 p_1(z) + \lambda_2 p_2(z)\big) - 1\right].$$

Since

$$u^2 = \lambda_1 p_1^2(z) + \lambda_2 p_2^2(z) - \lambda_1\lambda_2[p_1(z) - p_2(z)]^2,$$

we have

$$v = \frac{1}{1+m}\left[mu^2 + (1-m)u + m\lambda_1\lambda_2\big(p_1(z) - p_2(z)\big)^2 - 1\right]. \tag{2.18}$$

Since, for $z = re^{i\varphi}$,

$$p_k = c + \varrho\eta_k, \quad k = 1, 2, \tag{2.19}$$

where

$$\eta_k = \varepsilon_k e^{i\varphi}\frac{1-mr\bar{\varepsilon}_k e^{-i\varphi}}{1-mr\varepsilon_k e^{i\varphi}}, \tag{2.20}$$

we have

$$p_1(z)-p_2(z) = \varrho(\eta_1-\eta_2).$$

Thus

$$\lambda_1\lambda_2[p_1(z)-p_2(z)]^2 = \lambda_1\lambda_2\varrho^2(\eta_1^2+\eta_2^2-2\eta_1\eta_2). \tag{2.21}$$

Assuming

$$\eta = \eta_1\eta_2 \tag{2.22}$$

we may write (2.21) in the form

$$\lambda_1\lambda_2[p_1(z)-p_2(z)]^2 = \lambda_1\lambda_2\varrho^2\eta[2\cos(\gamma_1-\gamma_2)-2]$$

or finally in the form:

$$\lambda_1\lambda_2[p_1(z)-p_2(z)]^2 = -4\lambda_1\lambda_2\varrho^2\sin^2\frac{\gamma_1-\gamma_2}{2}. \tag{2.23}$$

Because of formula (2.19) we have for $z = re^{i\varphi}$

$$\lambda_1 p_1(z)+\lambda_2 p_2(z) = c+\varrho(\lambda_1\eta_1+\lambda_2\eta_2);$$

thus by Theorem 2

$$\varkappa = \varrho(\lambda_1\eta_1+\lambda_2\eta_2)$$

holds.

Thus

$$\varkappa^2 = \varrho^2-4\lambda_1\lambda_2\varrho^2\sin^2\frac{\gamma_1-\gamma_2}{2}. \tag{2.24}$$

Hence by (2.23) and (2.24) we obtain

$$\lambda_1\lambda_2[p_1(z)-p_2(z)]^2 = (\varkappa^2-\varrho^2)\eta; \tag{2.25}$$

(2.18) and (2.25) imply (2.17).

Assuming in (2.15)

$$u = se^{it},$$

by (2.17) we may write (2.15) in the form

$$\mathscr{F}[se^{it}, v(se^{it})] = C(se^{it}),$$

where

$$C(se^{it}) = A(se^{it})+\frac{B(se^{it})}{1+m}[ms^2e^{2it}+(1-m)se^{it}-1-(\varrho^2-s^2-c^2+ +2cs\cos t)m\eta].$$

Because of (2.14) the variables s and t vary in the intervals:

$$c-\varrho\leqslant s\leqslant c+\varrho,\quad -\psi\leqslant t\leqslant\psi,\quad \text{where } \cos\psi = \frac{s^2+c^2-\varrho^2}{2cs},$$

respectively.

Theorem 2 of [3] immediately implies the following

THEOREM 4. *Let*

$$\Phi(p) = \operatorname{re}\frac{zp'(z)}{p(z)}, \quad |z| = r < 1.$$

Then for every function $p(z) \in \wp_m$ *we have*

$$U(r, m) \leqslant \Phi(p) \leqslant V(r, m), \tag{2.26}$$

where

$$U(r, m) = -\frac{(1-m)r}{(1-r)(1+mr)}, \quad V(r, m) = \frac{(1+m)r}{(1+r)(1-mr)}.$$

Estimate (2.26) is sharp, the equalities being realized by the function

$$p(z) = \frac{1+\varepsilon z}{1-m\varepsilon z}, \quad |\varepsilon| = 1.$$

3. Denote by S the family of regular and univalent functions $w = f(z)$, $f(0) = 0$, $f'(0) = 1$, defined in the circle K, by S^* its subclass of starlike functions with respect to the point $w = 0$ and by S_m^*, $0 \leqslant m \leqslant 1$, the family of those functions of the class S for which

$$\frac{zf'(z)}{f(z)} \in \wp_m \qquad \text{for } z \in K.$$

Obviously $S_m^* \subset S^*$ and $S_1^* \equiv S^*$.

Theorem 2 of [2] immediately implies the following

THEOREM 5. *The radius of convexity of the family* S_m^*, $0 \leqslant m \leqslant 1$, *is given by the formula*

$$\text{r. c. } S_m^* = \frac{3+m-\sqrt{m^2+6m+5}}{2}.$$

4. Denote by Σ the family of regular and univalent functions $F(z)$ defined in the ring $P = \{z\colon 0 < |z| < 1\}$ which have the single pole at the point $z = 0$ and which in a neighbourhood of that point may be expanded in a power series of the form

$$w = F(z) = \frac{1}{z} + a_0 + a_1 z + \ldots$$

Let Σ^* be the subclass of starlike functions of the family Σ, i.e. the subclass of functions which map the ring P onto a set whose complement to the plane is a starlike set with respect to the point $w = 0$. Thus, if $F(z) \in \Sigma^*$, then

$$\frac{-zF'(z)}{F(z)} = p(z), \quad z \in K$$

and respectively $p(z) \in \wp$.

Moreover, denote by $\wp(M)$, $M \geqslant 1$, the family of regular functions $p(z)$, $p(0) = 1$, defined in the circle K and such that

$$|p(z)-M| < M \quad \text{for } z \in K$$

(comp. [2]). $\wp(M)$ is a subclass of the family $\wp$ and $\wp(\infty) \equiv \wp$. It can easily be proved that $\wp_m \subset \wp(M)$ for $M = 1/(1-m)$, $m \neq 1$. Finally, let $\Sigma^*(M)$ and Σ_m^* denote the families of functions of the class Σ for which

$$\frac{-zF'(z)}{F(z)} = p(z), \quad p(z) \in \wp(M), \; z \in K$$

and respectively

$$\frac{-zF'(z)}{F(z)} = p(z), \quad p(z) \in \wp_m, \; z \in K.$$

Obviously, $\Sigma_m^* \subset \Sigma^*(M) \subset \Sigma^*$ for $m = 1-1/M$ and $\Sigma_1^* \equiv \Sigma^*(\infty) \equiv \Sigma^*$.

The radius of convexity of the family $\Sigma^*(M)$ has been found in paper [4]. It is given by the formula

$$\text{r.c. } \Sigma^*(M) = \sqrt{\frac{2\sqrt{2}-\sqrt{1+m}}{2\sqrt{2}+\sqrt{1+m}}}, \quad m = 1-1/M.$$

We shall prove the following

THEOREM 6. *The radius of convexity* r.c.Σ_m^* *of the family* Σ_m^* *is given by the formula*

$$\text{r.c. } \Sigma_m^* = \begin{cases} \sqrt{\dfrac{2}{1-m+\sqrt{2(1+m)}}} & \text{for } 0 \leqslant m \leqslant m_0, \\ \sqrt{\dfrac{3}{m(3m+2+2\sqrt{3m^2+1})}} & \text{for } m_0 \leqslant m \leqslant 1, \end{cases}$$

where m_0, $0.52 < m_0 < 0.53$ *is the smallest positive root of the equation*

$$3\sqrt{2(1+m)}-(6m^2+7m-3)-4m\sqrt{3m^2+1} = 0.$$

Proof. If $F(z) \in \Sigma_m^*$, then

$$-\frac{zF'(z)}{F(z)} = p(z), \quad z \in K \tag{4.1}$$

for some function $p(z) \in \wp_m$. A minor calculation yields with (4.1)

$$-\left[1+\frac{zF''(z)}{F'(z)}\right] = p(z)-\frac{zp'(z)}{p(z)}.$$

Thus the problem of determining the radius of convexity of the family Σ_m^* is reduced to finding the smallest root r_0, $0 < r_0 < 1$ of the equation $\Omega(r) = 0$, where

$$\Omega(r) = \min_{\substack{|z|=r \\ p(z)\in\wp_m}} \operatorname{re}\left[p(z) - \frac{zp'(z)}{p(z)}\right].$$

Since the functional $\mathscr{F}(p) = p(z) - zp'(z)/p(z)$, $|z| = r$ is continuous and the family Σ_m^* is compact, this functional attains its upper and lower bounds, and according to Theorem 1 it attains them in the subclass functions $\wp_{m,2}$.

Preserving the notation adopted earlier, and basing ourselves on Theorem 3, we obtain for $|z| = r$:

$$(4.2) \quad p(z) - \frac{zp'(z)}{p(z)}$$
$$= se^{it} - \frac{1}{1+m}\left[mse^{it} + (1-m) - \frac{1}{s}e^{-it} - \frac{m}{s}e^{-it}(\varrho^2 - c^2 - s^2 + 2cs\cos t)\eta\right].$$

Thus

$$(4.3) \qquad \operatorname{re}\left[p(z) - \frac{zp'(z)}{p(z)}\right] \geqslant \Phi(s, t),$$

where

$$(4.4) \quad \Phi(s, t)$$
$$= \frac{1}{1+m}\left[\left(s + \frac{1}{s} - 2cm\right)\cos t - (1-m) - \frac{m}{s}(\varrho^2 - c^2) + ms\right].$$

The function $\Phi(s, t)$ has been defined in the region $D = \{(s, t): c - \varrho < s < c + \varrho,\ -\psi < t < \psi\}$ and on its boundary ∂D, with

$$(4.5) \qquad \psi = \arccos\frac{c^2 + s^2 - \varrho^2}{2cs}, \qquad 0 < \arccos\frac{c^2 + s^2 - \varrho^2}{2cs} < \frac{\pi}{2}.$$

Since

$$\Phi'_s(s, t) = \frac{2}{1+m}\left[\left(1 - \frac{1}{s^2}\right)\cos t + m - \frac{m}{s^2}\frac{1-r^2}{1-m^2r^2}\right],$$

$$\Phi'_t(s, t) = \frac{-2}{1+m}\left(s + \frac{1}{s} - 2cm\right)\sin t, \qquad \Phi''_{tt}(s, t) = \frac{-2}{1+m}\left(s + \frac{1}{s} - 2cm\right)\cos t,$$

$$\Phi''_{ss}(s, t) = \frac{4}{1+m}\left(\frac{\cos t}{s^3} + \frac{m}{s^3}\frac{1-r^2}{1-m^2r^2}\right), \qquad \Phi''_{st}(s, t) = \frac{-2}{1+m}\left(1 - \frac{1}{s^2}\right)\sin t,$$

the function $\Phi(s, t)$ attains at the point $(s, 0) \in D$, where

$$(4.6) \qquad s = \sqrt{\frac{1 - mr^2}{1 - m^2r^2}},$$

its local minimum, if

$$s^2 - 2cms + 1 < 0. \tag{4.7}$$

Then

$$\min_{(s,t)\in D} \operatorname{loc} \Phi(s,t) = s - \frac{2cm+1-m}{1+m} + \frac{1+m(c^2-\varrho^2)}{(1+m)s}.$$

We find

$$\min_{(s,t)\in D} \operatorname{loc} \Phi(s,t) = 0,$$

if

$$2s = \frac{1+m^2r^2}{1-m^2r^2}. \tag{4.8}$$

Eliminating s from (4.6) and (4.8), we obtain an equation with the unknown r:

$$m^3(m-4)r^4 + 2m(2+3m)r^2 - 3 = 0. \tag{4.9}$$

If condition (4.7) is satisfied for s given by formula (4.6), then $m > \frac{1}{2}$. In this case we obtain from (4.9)

$$r = r_1 = \sqrt{\frac{3}{m(3m+2+2\sqrt{3m^2+1})}}. \tag{4.10}$$

When $m = 1$, we obtain $r = 1/\sqrt{3}$ (comp. [3]).

Supposing now that the function $\Phi(s,t)$ attains its local minimum equal to zero at the point $\left(\sqrt{\dfrac{1-mr^2}{1-m^2r^2}}, 0\right)$, with r given by formula (4.10), we obtain by (2.13) and (4.8) respectively

$$c = \frac{3m+5+2\delta}{2(1+\delta)}, \quad s = \frac{3m+1+\delta}{2(1+\delta)}, \quad \text{where } \delta = \sqrt{3m^2+1};$$

thus

$$s^2 - 2cms + 1$$
$$= \frac{1}{4(1+\delta)^2}[(3m+1+\delta)(3m+1+\delta-6m^2-10m-4m\delta)+4(1+\delta)^2].$$

Consequently inequality (4.7) is satisfied if

$$h(m) \equiv 2m^4 + 4m^3 - 33m^2 + 6m + 5 < 0. \tag{4.11}$$

We have $h''(m) < 0$ for $0 \leqslant m \leqslant 1$; thus $h'(m)$ decreases in this interval; $h'(0) > 0$ and $h'(1) < 0$, by which there exists a number m_1, $0 < m_1 < 1$ such that the function $h(m)$ increases if $0 < m \leqslant m_1$, while $h(m)$ decreases if $m_1 < m < 1$. Hence by $h(0) > 0$ and $h(1) < 0$ it follows that there exists a number m', $m_1 < m' < 1$ such that $h(m) > 0$ for $0 < m < m'$ and $h(m) < 0$ for $m' < m < 1$.

Thus inequality (4.11) is satisfied with $0 \leqslant m \leqslant 1$ if and only if $m' < m < 1$. It can easily be verified that $0.51 < m' < 0.52$.

Simultaneously $h(m) = 0$ has two negative and two positive roots; the number m' is the smaller positive root of this equation.

The function $\Phi(s, t)$ attains its local minimum on the boundary of the region D if $t = \pm\psi$ and then

$$\Delta(s) = \Phi(s, \pm\psi) = \frac{1}{1+m}\left[\left(s+\frac{1}{s}\right)\frac{c^2+s^2-\varrho^2}{2cs} - (1-m)\right].$$

Since

$$\Delta'(s) = \frac{s^4+\varrho^2-c^2}{(1+m)cs^4},$$

the function $\Delta(s)$ attains at the point $s = \sqrt[4]{c^2-\varrho^2}$ of the boundary of the region D its local minimum equal to

$$\Delta(s) = \frac{1}{2(1+m)c}[s^4+2s^2+1-2(1-m)c],$$

where

$$s = \sqrt[4]{c^2-\varrho^2}. \tag{4.12}$$

We get

$$\min_{(s,t)\in\partial D}\operatorname{loc}\Delta(s) = 0,$$

if

$$s^2 = \frac{1}{2}\frac{(1+2m-m^2)r^2-2m}{1-m^2r^2}. \tag{4.13}$$

Eliminating s from (4.12) and (4.13) we obtain an equation with the unknown r

$$(m^2-4m-1)r^4-4(1-m)r^2+4 = 0;$$

hence

$$r = r_2 = \sqrt{\frac{2}{1-m+\sqrt{2}(1+m)}}. \tag{4.14}$$

Let r.c. Σ_m^* denote the radius of convexity of the family Σ_m^*. It follows from the above argument that

$$\text{r.c. } \Sigma_m^* \geqslant \begin{cases} r_2 & \text{for } 0 \leqslant m \leqslant m', \\ \min(r_1, r_2) & \text{for } m' < m \leqslant 1. \end{cases} \tag{4.15}$$

The inequality $r_1 < r_2$ is equivalent to the inequality

$$\varphi(m) - \psi(m) < 0,$$

where

$$\varphi(m) = 3\sqrt{2(1+m)}, \quad \psi(m) = 6m^2+7m-3+4m\sqrt{3m^2+1}.$$

We easily find that $\varphi(m)-\psi(m)$ is a decreasing function with $\varphi(0)-\psi(0) > 0$ and $\varphi(1)-\psi(1) < 0$. Thus there exists a point m_0 such that $\varphi(m)-\psi(m) < 0$, for $m > m_0$ and $\varphi(m)-\psi(m) > 0$, for $m < m_0$.

The number m_0, $0.52 < m_0 < 0.53$ is the only root of the equation

$$\varphi(m)-\psi(m) = 0.$$

Thus ultimately

$$\min(r_1, r_2) = \begin{cases} r_2 & \text{for } m' \leqslant m \leqslant m_0, \\ r_1 & \text{for } m_0 \leqslant m \leqslant 1. \end{cases} \tag{4.16}$$

For $m = m_0$ we have $r_1 = r_2$. Thus by (4.15) and (4.16) we obtain

$$\text{r.c. } \Sigma_m^* \geqslant r_0$$

with

$$r_0 = \begin{cases} r_2 & \text{for } 0 \leqslant m \leqslant m_0, \\ r_1 & \text{for } m_0 \leqslant m \leqslant 1, \end{cases}$$

where r_1 and r_2 are given by formulas (4.10) and (4.14) respectively.

We shall prove that

$$\text{r.c. } \Sigma_m^* \leqslant r_0.$$

Denote by $p_0(z)$ a function of the family $\wp_{m,2}$, $m_0 \leqslant m \leqslant 1$, for which

$$p_0(r_1) = s_0, \tag{4.16'}$$

where

$$s_0 = \frac{1}{2}\frac{1+m^2r_1^2}{1-m^2r_1^2} \tag{4.17}$$

[comp. (4.8)]. Then $\eta = -1$ [comp. (4.2)-(4.4)].

Thus because of (2.22) we obtain $\eta_2 = -\bar{\eta}_1$. Consequently

$$p_0(r_1) = \lambda_1 p_1(r_1) + \lambda_2 p_2(r_2), \tag{4.18}$$

where

$$p_1(r_1) = c_1 + \varrho_1\eta_1, \quad p_2(r_1) = c_1 - \varrho_1\bar{\eta}_1, \tag{4.19}$$

with

$$c_1 = c(r_1), \quad \varrho_1 = \varrho(r_1), \quad \eta_k = \varepsilon_k\frac{1-mr_1\bar{\varepsilon}_k}{1-mr_1\varepsilon_k}, \quad k = 1, 2$$

[comp. (2.13), (2.19) and (2.20)]. By (4.16′) and (4.19) equality (4.18) may be written in the form

$$s_0 = c_1 + \varrho_1(\lambda_1 - \lambda_2)x + i\varrho_1 y, \tag{4.20}$$

where $x = \operatorname{re}\eta_1$ and $y = \operatorname{im}\eta_1$.

It immediately follows from (4.20) that $y = 0$, because s_0 is positive. Consequently we may assume $\eta_1 = 1$. Hence $\eta_2 = -1$ and we obtain $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$.

Now we shall determine λ_1 and λ_2. By (4.18) and (4.19) we have

$$p_0(r_1) = c_1 + (\lambda_1 - \lambda_2)\varrho_1.$$

Hence we find

$$\lambda_1 - \lambda_2 = \frac{p_0(r_1) - c_1}{\varrho_1}$$

and thus because of (4.17) we shall have

$$\lambda_1 - \lambda_2 = -\frac{1 + m(2-m)r_1^2}{2(1+m)r_1}.$$

Hence

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left[1 - \frac{4+\sqrt{3m^2+1}}{3(1+m)} \cdot \sqrt{\frac{2+3m-2\sqrt{3m^2+1}}{4-m}}\right], \\ \lambda_2 &= \frac{1}{2}\left[1 + \frac{4+\sqrt{3m^2+1}}{3(1+m)} \sqrt{\frac{2+3m-2\sqrt{3m^2+1}}{4-m}}\right]. \end{aligned} \tag{4.21}$$

Thus for $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ we have

$$p_0(z) = \lambda_1 \frac{1+z}{1-mz} + \lambda_2 \frac{1-z}{1+mz}, \tag{4.22}$$

where λ_1 and λ_2 are given by (4.21).

Denote by $F_0(z)$ a function of the family Σ for which

$$-\frac{zF_0'(z)}{F_0(z)} = p_0(z), \quad z \in K, \tag{4.23}$$

where $p_0(z)$ is given by formula (4.22). Since $p_0(z) \in \wp_m$, we have $F_0(z) \in \Sigma_m^*$.

Integrating equation (4.23) from 0 to z, we obtain

$$\log zF_0(z) = \frac{1+m}{m}[\lambda_1 \log(1-mz) + \lambda_2 \log(1+mz)]; \tag{4.24}$$

by $\log zF_0(z)$, $\log(1-mz)$, $\log(1+mz)$, denote respectively the single-valued branches $l_1(z)$, $l_2(z)$, $l_3(z)$ of multi-valued functions $L_1(z) = \log zF_0(z)$, $L_2(z) = \log(1-mz)$, $L_3(z) = \log(1+mz)$, for which $l_1(0) = 0$, $l_2(0) = 0$ and $l_3(0) = 0$.

From (4.24) we obtain

$$(4.25)\qquad F_0(z) = \frac{[(1-mz)^{\lambda_1}(1+mz)^{\lambda_2}]^{(1+m)/m}}{z} \qquad \text{for } m_0 \leqslant m \leqslant 1.$$

For $m = 1$, we have $\lambda_1 = \frac{1}{2}(1-1/\sqrt{3}), \lambda_2 = \frac{1}{2}(1+1/\sqrt{3})$, and thus

$$F_0(z) = \frac{(1-z)^{1-1/\sqrt{3}}(1+z)^{1+1/\sqrt{3}}}{z}$$

(comp. [3]).

We shall prove that the function $F_0(z)$ defined by formula (4.25) is not convex in the circle $|z| < r$, for $r > r_1$. In fact, by (4.22), we have

$$p_0(z) = \frac{mz^2-(1+m)(\lambda_2-\lambda_1)z+1}{1-m^2z^2}.$$

Differentiating $p_0(z)$ with respect to z, we obtain after performing some calculations

$$(4.26)\qquad p_0(z) - \frac{zp_0'(z)}{p_0(z)} = \frac{m^3(4-m)z^4-2m(2+3m)z^2+3}{4(1-m^2z^2)[mz^2-(1+m)(\lambda_2-\lambda_1)z+1]}.$$

Thus

$$\operatorname{re}\left[1+\frac{zF_0''(z)}{F_0'(z)}\right] = 0$$

for $z = r_1$ [comp. (4.10)]. It also immediately follows from (4.26) that the function $F_0(z)$ is not convex in the circle $|z| < r$ for $r > r_1$. Thus r.c. $\Sigma_m^* \leqslant r_1$, whence by r.c. $\Sigma_m^* \geqslant r_1$, we obtain r.c. $\Sigma_m^* = r_1$, for $m_0 \leqslant m \leqslant 1$. Thus the theorem is proved if $m_0 \leqslant m \leqslant 1$.

Denote by $\tilde{p}_0(z)$ a function of the family $\wp_{m,2}$, $0 \leqslant m < m_0$, for which

$$\tilde{p}_0(r_2) = \tilde{s}_0 e^{i\tilde{t}_0},$$

where

$$(4.27)\qquad \tilde{s}_0 = \left[\frac{(1+2m-m^2)r_2^2-2m}{2(1-m^2r_2^2)}\right]^{1/2},$$

$$(4.28)\qquad \cos\tilde{t}_0 = \frac{\tilde{s}_0^2+c_2^2-\varrho_2^2}{2c_2\tilde{s}_0},$$

with $c_2 = c(r_2)$ and $\varrho_2 = \varrho(r_2)$ [comp. (2.13), (4.5) and (4.13)]. By (4.28) the point $\tilde{p}_0(r_2)$ lies on the circumference with the centre c_2 and the radius ϱ_2; consequently $\tilde{p}_0(z) \in \wp_{m,1}$. Thus, let

$$(4.29)\qquad \tilde{p}_0(z) = \frac{1+\varepsilon z}{1-m\varepsilon z}, \qquad |\varepsilon| = 1.$$

To find ε we first express the product $\tilde{s}_0 \cos \tilde{t}_0$ in terms of m.

Substituting r_2 from (4.14) to (4.27) and (4.28), after some calculations we obtain

$$\tilde{s}_0 = \frac{1}{\sqrt{1+\sqrt{2}x}}, \qquad \cos \tilde{t}_0 = \frac{1}{\tilde{s}_0}\left(1-\frac{x}{\sqrt{2}}\right),$$

where $x = \sqrt{1+m}$, by which we get

$$\tilde{s}_0 \cos \tilde{t}_0 = 1-\frac{x}{\sqrt{2}}. \tag{4.30}$$

On the other hand, assuming $\varepsilon = e^{ia}$ (a — real number), we get by (4.29)

$$\operatorname{re}\tilde{p}_0(r_2) = \frac{1-mr_2^2+(1-m)r_2\cos a}{1+m^2r_2^2-2mr_2\cos a}. \tag{4.31}$$

Equating the right-hand sides of formulas (4.30) and (4.31), we obtain ultimately

$$\cos a = -\frac{\sqrt{2}(1-m)+x}{2\sqrt{2}} r_2$$

and

$$\sin a = \pm \frac{x}{2\sqrt{2}} \sqrt{4-r_2^2}.$$

Denote by $\tilde{F}_0(z)$ a function of the family Σ for which

$$-\frac{z\tilde{F}_0'(z)}{\tilde{F}_0(z)} = \tilde{p}_0(z), \qquad z \in K. \tag{4.32}$$

Since $\tilde{p}_0(z) \in \wp_m$, we have $\tilde{F}_0(z) \in \Sigma_m^*$. Integrating equation (4.32) from 0 to z we get

$$\tilde{F}_0(z) = \begin{cases} \dfrac{(1-m\varepsilon z)^{\frac{1+m}{m}\varepsilon}}{z} & \text{for } 0 < m \leqslant m_0, \\[2ex] \dfrac{1}{z} e^{-\varepsilon z} & \text{for } m = 0. \end{cases}$$

We shall prove that the function $\tilde{F}_0(z)$ is not convex in the circle $|z| < r$ for $r > r_2$.

In fact, by (4.29) we have first

$$\tilde{p}_0(z) - \frac{z\tilde{p}_0'(z)}{\tilde{p}_0(z)} = \frac{\varepsilon^2 z^2 + (1-m)\varepsilon z + 1}{(1-m\varepsilon z)(1+\varepsilon z)};$$

thus at the point $z = r$, $0 < r < 1$ we have

$$\text{(4.33)} \qquad \operatorname{re}\left\{-\left[1 + \frac{z\tilde{F}_0''(z)}{\tilde{F}_0'(z)}\right]\right\} \equiv K(r) = \frac{g(r)}{|1-m\varepsilon r|^2|1+\varepsilon r|^2},$$

where

$$g(r) = a_0 r^4 + a_1 r^3 + a_2 r^2 + a_3 r + a_4,$$

with

$$a_0 = -m, \quad a_1 = (1-m)^2\cos a, \quad a_2 = (1-m)(2\cos^2 a - m),$$
$$a_3 = 2(1-m)\cos a, \quad a_4 = 1.$$

Since $K(r_2) = 0$, the polynomial $g(r)$ may be decomposed into factors

$$\text{(4.34)} \qquad g(r) = \frac{r_2 - r}{2r_2} \cdot h(r),$$

where

$$h(r) = b_0 r^3 + b_1 r^2 + b_2 r + b_3,$$

with

$b_0 = 2mr_2$, $b_1 = 2r_2^2[a(1-m)^2 + m]$, $b_2 = r_2[\sqrt{2}mx - (1-m)(1-2m)]$, $b_3 = 2$

and

$$a = \frac{(1-m)\sqrt{2} + x}{2\sqrt{2}}.$$

If $b_2 > 0$, then

$$\text{(4.35)} \qquad h(r) > 0 \quad \text{for } r \geqslant 0.$$

If $b_2 < 0$, then

$$b_2 r + b_3 > -(1-m)(1-2m)r_2 r + b_3 > b_3 - 1 = 1.$$

Thus also in this case condition (4.35) is satisfied.

Hence by (4.34) and (4.33) we conclude that the function $\tilde{F}_0(z)$ is not convex in the circle $|z| < r$ for $r > r_2$, which was to be proved.

Consequently r.c. $\Sigma_m^* \leqslant r_2$; hence by r.c. $\Sigma_m^* \geqslant r_2$, we find r.c. $\Sigma_m^* = r_2$ for $0 \leqslant m \leqslant m_0$, which ends the proof of Theorem 6.

References

[1] Г. М. Голузин, *Геометрическая теория функций комплексного переменного*, Москва 1966.
[2] W. Janowski, *Extremum problems for a family of functions with positive real part and for some related families*, Ann. Polon. Math. 23 (1970), p. 167-185.
[3] M. S. Robertson, *Extremal problems for analytic functions with positive real part and applications*, Trans. Amer. Math. Soc. (1963), p. 236-253.
[4] P. Wiatrowski, *On the radius of convexity of some family of functions regular in the ring* $0 < |z| < 1$, Ann. Polon. Math. 25 (1971), p. 85-98.
[5] В. А. Заморович, *О границах выпуклости звездных функций порядка* a *в круге* $|z| < 1$ *и круговой области* $0 < |z| < 1$, Мат. Сборник 68 (110), № 4 (1965), p. 518-526.

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