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On the Hardy class of certain subclass of Bazilevič function

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Abstract. The Hardy classes for a certain subfamily of Bazilevič functions and their derivative are obtained.

Introduction. Let $B_a(m)$ be the subclass of normalized Bazilevič functions f [1] defined in the unit disc $E = \{z: |z| < 1\}$ $(f(0) = 0, f'(0) = 1)$, given by

$$f(z) = \Big\{m \int_0^z [g(s)]^m p(s) s^{-1} ds\Big\}^{1/m}, \tag{1}$$

where $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a starlike function of order a, $P(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ is analytic function in E, $\operatorname{Re} P(z) > 0$ and $m > 0$.

Definition. Let $K(\beta, \gamma)$ denote the subclass of the class of close-to-convex function analytic in E, such that for $F \in K(\beta, \gamma)$ there exists a starlike function g of order a such that

$$\left|\arg \frac{zF'(z)}{g(z)}\right| < \frac{\beta\pi}{2}. \tag{2}$$

It is well known that $K(0, 0)$ is the class of convex function, $K(1, 0)$ is the class of close-to-convex functions, and $K(\beta, 0)$ is the subclass of close-to-convex functions of order β.

The object of this note is to obtain the Hardy classes for f and f', where $f \in B_a(m)$.

It is clear that, if $F \in K(\beta, \gamma)$, then

$$F'(z) = \frac{g(z)}{z} \{p(z)\}^{\beta}, \tag{3}$$

where $\operatorname{Re}\{p(z)\} > 0$. Denote by $F_\gamma(z)$ the function which is obtained from (3) by letting $g(z)$ be the Koebe function

$$g_\gamma(z) = az(1 - ze^{i\tau})^{2\gamma-2} \quad \text{for some complex } a \text{ and real } \tau.$$

We need in our proof the following

Lemma. *If $F \in K(\beta, \gamma)$ and F is not a rotation of F_γ, then there exists $\varepsilon = \varepsilon(F) > 0$ such that*

$$F' \in H^{1/[2(1-\gamma)+\beta]+\varepsilon}.$$

Proof. Since $F \in K(\beta, \gamma)$, then from (3) we have

$$F'(z) = \frac{g(z)}{z}[p(z)]^\beta,$$

where $g(z)$ is starlike of order γ and $\operatorname{Re} P(z) > 0$. Then from [7], p. 64, it follows that $P(z) \in H^\lambda$, $\forall\lambda$, $\lambda < 1$ and since $g(z) \neq g_\gamma(z)$, then from [2], Theorem 5, it follows that

$$\frac{g(z)}{z} \in H^{1/2(1-\gamma)+\delta} \quad \text{for some } \delta = \delta(g) > 0.$$

Application of Hölder inequalities to $F(z)$ with conjugate indices

$$p = 1 + \frac{\beta[1+2\delta(1-\gamma)]}{2(1-\gamma)} \quad \text{and} \quad q = 1 + \frac{2(1-\gamma)}{\beta[1+2\delta(1-\gamma)]}$$

yields for $z = re^{i\theta}$, $0 < r < 1$,

$$\int_{-\pi}^{\pi} |F'(z)|^{1+2\delta(1-\gamma)/[2(1-\gamma)+\beta+2\delta(1-\gamma)\beta]}$$

$$\leqslant \left(\int_{-\pi}^{\pi} \left|\frac{g(z)}{z}\right|^{p[1+2\delta(1-\gamma)]/[2(1-\gamma)+\beta+2\delta(1-\gamma)\beta]}\right)^{1/p} \left(\int_{-\pi}^{\pi} |P(z)|^{\frac{\beta q[1+2\delta(1-\gamma)]}{[2(1-\gamma)+2\delta(1-\gamma)\beta]}}\right)^{1/q}.$$

If δ is sufficiently small, each of the integrals on the right-hand side of the above inequality remains bounded as r tends to 1. Hence there exists $\varepsilon = \varepsilon(F) > 0$ such that

$$F' \in H^{1/[2(1-\gamma)+\beta]+\varepsilon}.$$

Corollary 1. *For $0 \leqslant \gamma \leqslant 1/2$, from [4], Theorem 33, it follows that there exists $\varepsilon' = \varepsilon'(F) > 0$ such that*

$$F \in H^{1/[1-2\gamma+\beta]+\varepsilon'}.$$

Corollary 2. *For $\gamma = 0$ we can get the result of Theorem* 3 [3].

Corollary 3. *For $\gamma = 0$, $\beta = 1$ we can get the result of Theorem* 6 [2].

Denote by $f_a(z)$ the function which is obtain from (1) by letting $g(z) = g_a(z)$.

Theorem. *If $f \in B_a(m)$ and $f(z)$ is not a rotation of $f\ (z)$, then we have*

(i) *if $0 < m < 1$, there exists $\varepsilon_2 = \varepsilon_2(f) > 0$ such that*

$$f' \in H^{1/[3-2am]+\varepsilon_1},$$

(ii) *if* $m \geqslant 1$, *there exists* $\varepsilon_3 = \varepsilon_3(f) > 0$ *such that*

$$f' \in H^{1/[1+2m(1-a)]+\varepsilon_3}.$$

Proof. Since $f \in B_a(m)$, from (1) we have

$$f(z) = \left\{ m \int_0^z \left[\frac{g(s)}{s} \right]^m P(s) s^{m-1} ds \right\}^{1/m}. \tag{4}$$

Define

$$F'(z) = P^{1/m}(z) \left(\frac{g(z)}{z} \right) \tag{5}$$

then from (4) and (5) we have

$$f(z) = \left\{ m \int_0^z [F'(z)]^m s^{m-1} ds \right\}^{1/m}$$

and

$$f'(z) = z^{m-1} [f(z)]^{1-m} [F'(z)]^m. \tag{6}$$

Case 1. $0 < m < 1$.

Since f is univalent, then follows from [5], p. 214, that $f(z) \in H^\lambda$, $\forall \lambda$, $\lambda < \frac{1}{2}$ and since $g(z)$ is starlike of order a and is not rotation of $g_a(z)$, it follows from the lemma that $F'(z) \in H^{m/[2(1-a)m+1]+\varepsilon}$ for some $\varepsilon = \varepsilon(F) > 0$. Application of Hölder inequalities to $f(z)$ with conjugate indices

$$p = 1 + \frac{m[1+2m(1-a)]}{2(1-m)\{m+\varepsilon[1+2m(1-a)]\}},$$

$$q = 1 + \frac{2(1-m)\{m+\varepsilon[1+2m(1-a)]\}}{m[1+2m(1-a)]},$$

yields for $z = re^{i\theta}$ $(0 < r < 1)$ that

$$\int_{-\pi}^{\pi} \left| \frac{f'(z)}{z^{m-1}} \right|^{\{m+\varepsilon[1+2m(1-a)]\}/\{m[3-2ma]+2\varepsilon(1-m)[1+2m(1-a)]\}}$$

$$\leqslant \left(\int_{-\pi}^{\pi} |f(z)|^{(1-m)p\{[m+\varepsilon[1+2m(1-a)]]\}/[m[3-2ma]+2\varepsilon(1-m)[1+2m(1-a)]]\}} \right)^{1/p} \times$$

$$\times \left(\int_{-\pi}^{\pi} |F'(z)|^{aq\{[m+\varepsilon[1+2m(1-a)]]/[m[3-2ma]+2\varepsilon(1-m)[1+2m(1-a)]]\}} \right)^{1/q}.$$

If ε is sufficiently small, each of the integrals on the right-hand side of the above inequality remains bounded as r tends to 1, since $[3-2am] > 1$

for $m < 1$. Hence there exists $\varepsilon_1 = \varepsilon_1(f) > 0$ such that

$$f'(z) \in H^{1/(3-2am)+\varepsilon_1}.$$

Case 2. $m \geqslant 1$.

It is well known that $|f(re^{i\theta})| \geqslant r/(1+r)^2$ and so from (6), since $m \geqslant 1$ we get

$$|f'(z)| \leqslant \left(\frac{1}{1+r}\right)^{2(1-m)} |F'(z)|^m$$

and from the lemma, it follows that there exists $\varepsilon_2 = \varepsilon_2(f) > 0$ such that

$$f'(z) \in H^{1/[1+2m(1-a)]+\varepsilon_2}.$$

Corollary 4. *From* [4], *Theorem* 33, *under the same conditions of the above theorem, we get, there exists* $\varepsilon_2 = \varepsilon_3(f) > 0$ *such that*

$$f(z) \in H^{1/2(1-am)+\varepsilon_3}.$$

Corollary 5. *From the theorem, for* $a = 0$, *we get the result of Theorem* 3 [4].

Corollary 6. *From Corollary* 4 *and for* $a = 0$ *we get the result of Theorem* 1 [6].

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