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Existence problem of the osculating planes of a curve in R_n

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The linear elements of figures, particularly of curves, can be defined in two different ways: the first we can briefly name "the limiting processes", the second one is based on investigating the order of contact of the linear elements with respect to the given figure.

This paper deals with a certain problem connected with the second method.

The result of this work concerns a theorem of Z. Moszner [1].

Let us take a curve C given by the vectorial equation

$$(1) \qquad \boldsymbol{r} = \boldsymbol{r}(\tau), \quad \tau \in I \ (^1)$$

in an n-dimensional Euclidean space R_n.

In a neighbourhood of the point $M(\tau_0)$ (2) we choose a point $M(\tau)$ on the curve C.

We shall denote by L_j any j-dimensional plane through the point $M(\tau_0)$. By $\boldsymbol{r}^*(\tau)$ we denote the radius-vector of the normal projection $M^*(\tau)$ of the point $M(\tau)$ on the plane L_j.

DEFINITION 1. We shall say that a plane L_j has *the order of contact with the curve C at the point $M(\tau_0)$ equal to the number $\mu > 0$* if there exists a finite and positive limit

$$(2) \qquad \lim_{\tau\to\tau_0} \frac{|\boldsymbol{r}-\boldsymbol{r}^*|}{|\boldsymbol{r}-\boldsymbol{r}_0|^{\mu}} = \gamma ,$$

where $\boldsymbol{r} = \boldsymbol{r}(\tau)$, $\boldsymbol{r}_0 = \boldsymbol{r}(\tau_0)$ and $\boldsymbol{r}^* = \boldsymbol{r}^*(\tau)$.

Now we shall give the following definition of the osculating plane (of the order j of a curve C at the point $M(\tau_0)$) (3).

DEFINITION 2. The j-dimensional plane L_j^0 through the point $M(\tau_0)$ will be named *the osculating plane* of the order j of the curve C at the point $M(\tau_0)$ if

(1) I denotes an interval (finite or infinite).

(2) $M(\tau_0)$ is the point of curve C which corresponds to the value τ_0 of parameter τ.

(3) This definition is a slight modification of that given by S. Gołąb.

a) either the plane L_j^0 has the order of contact at the point $M(\tau_0)$ equal to μ_0, and every other plane L_j through the point $M(\tau_0)$ has a definite order of contact at the point $M(\tau_0)$ equal to μ, $\mu < \mu_0$;

b) or L_j^0 is the unique plane which has no order of contact at the point $M(\tau_0)$.

Now we can put the following problem: what are weakest conditions of regularity of the curve C under which the curve C will have an osculating plane L_j^0.

A partial solution of the problem gives the following

THEOREM. *Let j be a natural number satisfying the inequality $1 \leqslant j \leqslant n-1$. Let a curve C, defined by vectorial equation* (1), *satisfy the following conditions*:

1. *the function $\boldsymbol{r}(\tau)$ has continuous derivatives*

$$\boldsymbol{r}^{(1)}, \boldsymbol{r}^{(2)}, \ldots, \boldsymbol{r}^{(j-1)}, \quad \tau \in I, \quad \textit{where } \boldsymbol{r}^{(i)} \textit{ denotes } \frac{d^i\boldsymbol{r}(\tau)}{d\tau^i};$$

2. *the derivative $\boldsymbol{r}^{(j)}$ exists at the point τ_0*;

3. *the vectors $\{\boldsymbol{r}_0^{(i)}\}$ are linearly independent* $(i = 1, \ldots, j)$; *$\boldsymbol{r}_0^{(i)}$ denotes $\boldsymbol{r}^{(i)}(\tau_0)$*;

4. *there exists at most one j-dimensional plane through the point $M(\tau_0)$ which has infinitely many points in common with C, and these points have $M(\tau_0)$ as an accumulation point.* (In the following the plane described above will be named the exceptional plane.)

Then there exists an osculating plane L_j^0 of order j of the curve C at the point $M(\tau_0)$ (in the sence of definition 2).

Remark 1. The last condition 4 of the above theorem is the necessary condition for the existence of the positive limit in (2). Otherwise there would exist at least two exceptional planes, and the limit in (2) would be equal to 0 for every $\mu > 0$, which contradicts assumption a) in definition 2.

Proof of the theorem. For any plane L_j we can find the representation

$$\boldsymbol{r} = \boldsymbol{r}_0 + \sum_{\nu=1}^{j} \lambda_\nu \boldsymbol{a}_\nu, \tag{3}$$

where the vectors

$$\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_j \tag{4}$$

are constant and linearly independent (λ_ν denote the real parameters).

Denote by p the largest number i for which every vector among $\boldsymbol{r}_0^{(1)}, \boldsymbol{r}_0^{(2)}, \ldots, \boldsymbol{r}_0^{(i)}$, $i \leqslant j$ is linearly dependent on the vectors in (4). If $\boldsymbol{r}_0^{(1)}$ is linearly independent on the vectors in (4), then we put $p = 0$. We have $0 \leqslant i \leqslant p \leqslant j$.

We denote by H_j the j-dimensional plane through the point $M(\tau_0)$ with the vectorial base $r_0^{(1)}, r_0^{(2)}, \ldots, r_0^{(j)}$. Its equation is

$$r = r_0 + \sum_{\varrho=1}^{j} \lambda_\varrho r_0^{(\varrho)} . \tag{5}$$

Of course the necessary and sufficient condition for the identity $L_j = H_j$ is that p be equal to j.

Now we shall prove that

I. Each of the plane $L_j \neq H_j$ has the order of contact with the curve C at the point $M(\tau_0)$ equal to $p+1$.

II. The order of contact, if it exists, for the plane H_j is larger than j.

We shall prove is this way that the plane H_j will satisfy all the conditions required in order that it should be an osculating plane of the order j.

Let us number the vectors in (4) in such a way that the following equality holds:

$$j = \operatorname{ord}(a_1, \ldots, a_j) = \operatorname{ord}(r_0^{(1)}, \ldots, r_0^{(p)}, a_{p+1}, \ldots, a_j) . \tag{6}$$

To get (6) let us take into consideration the following facts:

1) The plane H_p with the vectorial base $r_0^{(1)}, \ldots, r_0^{(p)}$, $p \leqslant j$ is the linear subspace of the linear space L_j.

2) The vector $r_0^{(p+1)}$ does not belong to L_j, thus it belongs to the complement of the space L_j (to the whole space R_n).

It is possible (in different ways) to number the vectors in (4) in such a way that the vectors $a_{p+1}, \ldots, a_j$ do not belong to H_p. Then the set of the vectors

$$r_0^{(1)}, \ldots, r_0^{(p)}, a_{p+1}, \ldots, a_j \tag{4'}$$

is also the vectorial base for the plane L_j. It yields the condition (6). (In the case $p = j$ the set of the vectors $a_{p+1}, a_{p+2}, \ldots, a_j$ is empty.)

Hence by (6) the equation in (3) of the plane L_j can be written in the form

$$r = r_0 + \sum_{\varrho=1}^{p} \lambda_\varrho r_0^{(\varrho)} + \sum_{\nu=p+1}^{j} \lambda_\nu a_\nu . \tag{3'}$$

For a given system of linearly independent vectors in (4) we can find the set of linearly independent vectors

$$a_{j+1}, \ldots, a_n \tag{7}$$

which satisfy the conditions

$$a_k \cdot a_l = 0 \quad (k = 1, \ldots, n\,,\; l = j+1, \ldots, n) \tag{8}$$

for $k \neq l$.

When instead of (3) we take (3′), conditions (8) will be transformed into the form

$$\tag{8'}\begin{aligned} r_0^{(\varrho)}\cdot a_l &= 0 \quad (\varrho = 1, \ldots, p\,,\ l = j+1, \ldots, n)\,,\\ a_\nu\cdot a_l &= 0 \quad (\nu = p+1, \ldots, j\,,\ l = j+1, \ldots, n)\,,\\ a_k\cdot a_l &= 0 \quad (k \neq l\,,\ k, l = j+1, \ldots, n)\,. \end{aligned}$$

Let us denote by H_{n-j} the $(n-j)$-dimensional plane through the point $M(\tau)$ of the curve C whose vectorial base is given by (7). Its equation is

$$\hat{r} = r + \sum_{k=j+1}^{n} \lambda_k a_k\,.$$

It is obvious that the only point of the planes L_j and H_{n-j} is the point $M^*(\tau)$.

Now we shall find the radius-vector $r^*(\tau)$ of the point $M^*(\tau)$. We put

$$\tag{9} r + \sum_{k=j+1}^{n} \lambda_k a_k = r_0 + \sum_{\varrho=1}^{p} \lambda_\varrho r_0^{(\varrho)} + \sum_{\nu=p+1}^{j} \lambda_\nu a_\nu\,,$$

where the parameters $\lambda_{j+1}, \lambda_{j+2}, \ldots, \lambda_n$ must be computed.

Multiplying (9) scalarly by a_l and using (8′) we obtain the equalities

$$\lambda_k = -(r - r_0)a_k \quad (k = j+1, \ldots, n)\,.$$

Hence

$$r^* = r - \sum_{k=j+1}^{n} [(r-r_0)a_k]a_k\,.$$

Therefore

$$\tag{2'} \frac{|r-r^*|}{|r-r_0|^\mu} = \frac{\left|\sum_{k=j+1}^{n}[(r-r_0)a_k]a_k\right|}{|r-r_0|^\mu}\,.$$

According to the conditions in 2 and 3, which have been assumed for the function $r(\tau)$ we can apply to the difference $r - r_0$ in the numerator of the fraction in (2′) the formula of Peano up to the order j. To the difference $r - r_0$ in the denominator of the fraction in (2′) we apply the same formula with order 1 of the derivative.

Writing $h = \tau - \tau_0$ we have

$$\tag{2''} \frac{|r-r^*|}{|r-r_0|^\mu} = \frac{\left|\sum_{k=j+1}^{n}\left[\left(\sum_{\varrho=1}^{j}\frac{h^\varrho}{\varrho!}r_0^{(\varrho)} + \frac{h^j}{j!}e\right)a_k\right]a_k\right|}{|h|^\mu|r_0^{(1)} + e_1|^\mu} \equiv f(h)\,.$$

It may easily be verified that (using (8′)) we get in the case $p = j$ and $0 < \mu \leqslant j$

$$f(h) = \frac{|h|^j \left|\sum_{k=j+1}^{n} (ea_k)a_k\right|}{j!\,|h|^\mu |r_0^{(1)} + e_1|^\mu} \xrightarrow[h\to 0]{} 0\,.$$

This means that the plane H_j which occurs in this case has the order of contact larger than j, if it exists.

Now in the case $0 \leqslant p \leqslant j-1$ we have (using (8′))

$$f(h) = \frac{|h|^{p+1}\left|\sum_{k=j+1}^{n}\left[\sum_{\varrho=p+1}^{j} \frac{h^{\varrho-p-1}}{\varrho!}(r_0^{(\varrho)}a_k) + \frac{h^{j-p-1}}{j!}(ea_k)\right]a_k\right|}{|h|^\mu |r_0^{(1)} + e_1|^\mu}$$

which for $\mu = p+1$ has the limit, by $h\to 0$, equal to

$$\gamma \stackrel{\text{def}}{=} \frac{\left|\sum_{k=j+1}^{n} (r_0^{(p+1)}a_k)a_k\right|}{(p+1)!\,|r_0^{(1)}|^{p+1}}\,.$$

Remark that (because of the linear independence of the vectors $a_{j+1}, a_{j+2}, \ldots, a_n$) the necessary and sufficient condition for the equality $\gamma = 0$ is

$$(10) \qquad r_0^{(p+1)}a_k = 0 \quad (k = j+1, \ldots, n)\,.$$

But the vector $r_0^{(p+1)}$ is not equal to zero because of the assumption 3. On the other hand it does not belong to the space L_j, so not all the relations in (10) are satisfied.

That means that the number γ is larger than zero, which implies that for $p = 0, 1, \ldots, j-1$ there exists an order of contact of the plane L_j and it is equal to $p+1$.

The theorem is proved.

Equation (5) is the equation of the osculating plane L_j^0 of the order j of the curve C at the point $M(\tau_0)$.

Remark 2. If osculating plane L_j^0 of the curve C (satisfying the conditions 1-4) exists at the point $M(\tau_0)$ then all the osculating planes $L_1^0, L_2^0, \ldots, L_{j-1}^0$ exist at that point.

This fact follows immediately from the proof of the theorem.

Remark 3. If there exists a plane L_q of q dimensions ($q = 1, \ldots, n-1$) for which there is a neighbourhood of the point $M(\tau_0)$ on the curve C contained in L_q, and if we write $q_0 = \min q$ [4], then there may only exist osculating planes L_m^0 (in the sence of definition 2) at the point $M(\tau_0)$ of the order m equal to q_0 at most.

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[4] Where min is taken over all the planes L_q having the property in the question.

References

[1] Z. Moszner, *Sur quelques théorèmes concernant les hyperplans osculateurs*, Rocznik Naukowo-Dydaktyczny Wyższej Szkoły Pedagogicznej w Krakowie 13 (1962), pp. 17-37.

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