ANNALES
POLONICI MATHEMATICI
LIV.1 (1991)

An integral formula for the derivatives of solutions of certain elliptic systems

by H. RENELT (Halle an der Saale)

Abstract. As was shown by Polozhii and Shabat, the solutions f of elliptic systems $f_{\bar z} = \nu f_z + \mu \overline{f_z}$ satisfy a generalized Cauchy integral formula. Here we will show that the derivatives f_z, too, satisfy an integral formula. This formula, announced already in [6], rests upon the notion of generalized (-1)th powers and represents an astonishing analogue to the classical Cauchy integral formula for the derivative of an analytic function. The proof rests on an integral relation for generalized powers, which is of independent interest and which likewise represents a generalization of a classical relation.

1. Introduction and main results. The Cauchy integral formula for analytic functions has a counterpart for (ν, μ)-solutions which was shown in [4] and [7], also cf. [5], § V.1. Here a (ν, μ)-*solution* means a function continuous in a domain G, possessing generalized derivatives f_z and $f_{\bar z}$ in $L_{2,\mathrm{loc}}(G)$ and satisfying the elliptic system

$$f_{\bar z} = \nu f_z + \mu \overline{f_z} \quad \text{a.e. in } G.$$

Such a system is called here a (ν, μ)-*system*, with ν, μ in $L_\infty(G)$ and $\||\nu| + |\mu|\|_{L_\infty(G)} < 1$.

As communicated in [6] without proof, the Cauchy integral formula for the derivative which may be written in the form

$$f'(z) = \frac{-1}{2\pi i} \oint_C f(\zeta)\, d\left(\frac{1}{\zeta - z}\right)$$

has also its analogue for (ν, μ)-solutions. This points out once more a deep and unexpected similarity between analytic functions and (ν, μ)-solutions.

The proposed integral formula rests upon the notion of generalized powers of (ν, μ)-solutions. Effectively, only the (-1)th power occurs in this formula. However, considering only generalized (-1)th powers would not be very natural. Moreover, possible further developments in this direction, as generalized Taylor and Laurent expansions and a certain kind of generalized higher derivatives, require nth powers with n an arbitrary integer.

1985 *Mathematics Subject Classification*: 30G20, 35C15, 35C20.

On account of the Bers–Nirenberg representation theorem, the notions of zero and pole of nth order of a (ν, μ)-solution at a point z_0 are well defined, also with $z_0 = \infty$ (cf. e.g. [5], III.1). For brevity, we shall call z_0 a *point of order n* of a (ν, μ)-solution f in $G\backslash\{z_0\}$, $z_0 \in G$, G a domain, if either n is a positive integer and z_0 is a zero of order n of f, or n is a negative integer and z_0 is a pole of order $-n$ of f; the (trivial) case $n = 0$ is not considered.

In defining generalized powers, asymptotic expansions of (ν, μ)-solutions at points of order n play a crucial role, and that again requires additional assumptions on the coefficients ν and μ.

DEFINITION 1. Let D be any subset of the finite plane $\mathbf{C}$ and p any real number $\geqslant 1$. By $HL_p(D)$ we denote the set of all functions f defined and (Lebesgue-) measurable in $\mathbf{C}$, satisfying

$$\text{(i)}\qquad \frac{f(z)-f(z_0)}{z-z_0} \in L_p \quad \text{(as a function of } z\text{) for every } z_0 \in D$$

and

$$\text{(ii)}\qquad \|f\|_{HL_p(D)} = \|f\|_{L_\infty} + \sup\left\{\left\|\frac{f(z)-f(z_0)}{z-z_0}\right\|_{L_p} \ z_0 \in D\right\} < \infty .$$

Here L_p always means $L_p(\mathbf{C})$, also with $p = \infty$. The condition (i) seems to appear for the first time in [2].

We say that $f \in HL_p(\{\infty\})$ if f is defined and finite on $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and $f(1/z) - f(\infty) \in HL_p(\{0\})$ (the latter is the same as $f(1/z) \in HL_p(\{0\})$). $\|f\|_{HL_p(\{\infty\})}$ means $\|g\|_{HL_p(\{0\})}$ with $g(z) = f(1/z)$.

Until further notice we suppose that

$$\text{(1)}\qquad \begin{gathered} |\nu(z)| + |\mu(z)| \leqslant \text{const} = k < 1 \quad \forall z \in \bar{\mathbf{C}}, \quad \text{and} \\ \nu, \mu \in HL_p(\{z_0\}) \cap HL_p(\{\infty\}), \end{gathered}$$

where z_0 is a fixed point in $\mathbf{C}$, and $p > 2$.

Note that the condition for ν, μ to belong to $HL_p(\{z_0\})$ is weaker than Hölder continuity at z_0.

Concerning asymptotic expansions we have the following.

THEOREM 1 ([5], Sect. III). (I) *Let $z_0 \neq \infty$ be a point of order n of the (ν, μ)-solution $w(z)$. Then $w(z)$ admits the asymptotic expansion*

$$\text{(i)}\qquad w(z) = c\big(z - z_0 + \mathfrak{b}\overline{(z-z_0)}\big)^n - b\bar{c}\big(\overline{z-z_0} + \bar{\mathfrak{b}}(z-z_0)\big)^n + O(|z-z_0|^{n+\alpha})$$

at z_0 with some constants $c \neq 0$, $\alpha > 0$ and

$$\text{(ii)}\qquad \begin{gathered} b = \omega(\varkappa), \qquad \mathfrak{b} = \omega(\sigma), \\ \varkappa = -\mu(z_0)/\big(1 + |\mu(z_0)|^2 - |\nu(z_0)|^2\big), \\ \sigma = \nu(z_0)/\big(1 + |\nu(z_0)|^2 - |\mu(z_0)|^2\big), \end{gathered}$$

where $\omega(x) = 2x/(1 + \sqrt{1-4x^2})$, $|b| \leqslant k$, $|\mathfrak{b}| \leqslant k$, $\forall z_0 \in \bar{\mathbf{C}}$, with k defined in (1).

(II) *Let ∞ be a point of order n of the (ν, μ)-solution $w(z)$. Then $w(z)$ has the asymptotic expansion*

(iii) $$w(z) = c(z+\mathfrak{b}\bar{z})^{-n} - b\bar{c}(\bar{z}+\bar{\mathfrak{b}}z)^{-n} + O(|z|^{-n-\alpha})$$

at ∞ with some constants $c \neq 0$, $\alpha > 0$, and b, $\mathfrak{b}$ as in (ii) *with z_0 replaced by ∞.*

For generalized powers we have

THEOREM 2 [6]. *Let $n \neq 0$ be any integer and $c \neq 0$ any complex number. Under the assumption* (1) *there exists a unique (ν, μ)-solution $w(z)$ in $\mathbf{C}\backslash\{z_0\}$ with the properties*:

(i) *$w(z)$ has the asymptotic expansion*

$$w(z) = c(z-z_0+\mathfrak{b}(\overline{z-z_0}))^n - b\bar{c}(\overline{z-z_0}+\bar{\mathfrak{b}}(z-z_0))^n + O(|z-z_0|^{n+\alpha}),$$

at z_0 with b, $\mathfrak{b}$, α as in Theorem 1, *and*

(ii) *∞ is a point of order $-n$ of $w(z)$.*

This unique $w(z)$ is called a *generalized n-th power* and denoted by $[c(z-z_0)^n]_{(\nu,\mu)}$ or simply by $[c(z-z_0)^n]$ if no misunderstanding is possible.

Note that (i) means that z_0 is a point of order n of $w(z)$. In the special case $\nu \equiv 0$, μ Hölder continuous, such generalized powers have been considered in [1].

Now we are able to formulate our main result (announced in [6]).

THEOREM 3. *Let G be a simply connected domain $\subseteq \mathbf{C}$, $\nu, \mu \in HL_p(G) \cap HL_p(\{\infty\})$, $p > 2$, $|\nu(z)|+|\mu(z)| \leqslant \text{const} = k < 1$, $\forall z \in \bar{\mathbf{C}}$, and let f be a (ν, μ)-solution in G. With the constants b, $\mathfrak{b}$ corresponding to z_0, ν, μ as in Theorem* 1(ii), *we have*

$$\operatorname{Re}\left\{-\frac{1}{2\pi i}\oint_C f(z)\,d[c(z-z_{\bar{\mu}})^{-1}]_{(\nu,\bar{\mu})}\right\} = \frac{1-|b|^2}{1-|b\mathfrak{b}|^2}\operatorname{Re}\{f_z(z_0)(c+\bar{c}\bar{b}\mathfrak{b})\}$$

for any positively oriented rectifiable Jordan curve C within G and any z_0 inside C.

N.B. The formula given in [6], Theorem 3.2, contains two misprints.

By putting $c+\bar{c}\bar{b}\mathfrak{b} = 1$ and $= i$, respectively, we obtain from Theorem 3

$$f_z(z_0) = \operatorname{Re}\frac{-1}{2\pi i}\oint_C f(z)\,d[c_1(z-z_0)^{-1}]_{(\nu,\bar{\mu})} + i\operatorname{Im}\frac{-1}{2\pi i}\oint_C f(z)\,d[c_2(z-z_0)^{-1}]_{(\nu,-\bar{\mu})}$$

with $c_1 = (1-\bar{b}\mathfrak{b})/(1-|b|^2)$, $c_2 = (1+\bar{b}\mathfrak{b})/(1-|b|^2)$.

The following theorem could be seen as a deeper reason for Theorem 3. In any case, it represents a surprising result on generalized powers, of independent interest. Moreover, it might lead to a reasonable theory of generalized Taylor and Laurent expansions as well as to a residue theory for (ν, μ)-solutions.

THEOREM 4. *Let ν, μ satisfy the assumptions of Theorem* 3, *and let n, j be any nonzero integers. Then*

$$\operatorname{Re}\frac{1}{2\pi i}\oint_C [a(z-z_0)^n]_{(\nu,\mu)}\, d[c(z-z_0)^j]_{(\nu,\bar{\mu})} = (1-|b|^2) j \delta_{n,-j} \operatorname{Re}(ac)$$

with C as in Theorem 3 *and b as in Theorem* 1 (ii); *$\delta_{n,m}$ denotes the Kronecker symbol.*

(Of course, Theorem 4 also remains valid if either n or j or both are zero, the 0-th power being understood as a constant.)

Obviously, Theorem 4 is a generalization of the classical relation

$$\frac{1}{2\pi i}\oint_C \frac{dz}{(z-z_0)^m} = \delta_{1,m}.$$

The proofs of Theorems 3 and 4 (given in Sect. 4 below) require asymptotic expansions for the derivatives of (ν, μ)-solutions (Sect. 2) and some approximation considerations (Sect. 3).

2. Asymptotic expansions for the derivatives of (ν, μ)-solutions

THEOREM 5. *In addition to* (1), *let ν, μ possess Hölder continuous partial derivatives with respect to z and $\bar{z}$ in a whole neighbourhood $U(z_0)$ of z_0, and let $f(z)$ be a (ν, μ)-solution in $U(z_0)\backslash\{z_0\}$ having an asymptotic expansion*

$$f(z) = c\big(z-z_0+\mathfrak{b}(\overline{z-z_0})\big)^n - b\bar{c}\big(\overline{z-z_0}+\bar{\mathfrak{b}}(z-z_0)\big)^n + O(|z-z_0|^{n+\alpha})$$

at z_0, with c, $\mathfrak{b}$, b, n, α as in Theorem 1. *Then $f_z(z)$ admits the asymptotic expansion*

$$f_z(z) = nc\big(z-z_0+\mathfrak{b}(\overline{z-z_0})\big)^{n-1} - nb\bar{c}\bar{\mathfrak{b}}\big(\overline{z-z_0}+\bar{\mathfrak{b}}(z-z_0)\big)^{n-1} + O(|z-z_0|^{n-1+\alpha'})$$

at z_0 with $\alpha' > 0$.

Proof. By means of affine transformations (in the z- and f-planes, cf. [5], §II.3) we can reduce the assertion to the case when $z_0 = 0$, $\nu(0) = \mu(0) = 0$, $\nu, \mu \in HL_p(\{\infty\})$, ν, μ and their partial derivatives with respect to z and $\bar{z}$ are Hölder continuous in a neighbourhood $U(0)$,

$$f(z) = cz^n + O(|z|^{n+\alpha})$$

and

$$f_z(z) = ncz^{n-1} + O(|z|^{n-1+\alpha'}), \qquad \alpha' > 0.$$

In view of well-known smoothness results, our conditions on ν, μ imply that every (ν, μ)-solution f in $U(0)\backslash\{0\}$ is twice continuously differentiable there, and the derivative $f_z(z) = h(z)$ satisfies the elliptic system

(2) $$h_{\bar{z}} = \frac{\nu}{1-|\mu|^2} h_z + \frac{\bar{\nu}\mu}{1-|\mu|^2}\overline{h_z} + \frac{\nu_z + \mu\overline{\mu_{\bar z}}}{1-|\mu|^2} h + \frac{\mu_z + \mu\overline{\nu_{\bar z}}}{1-|\mu|^2}\bar{h}$$

in $U(0)\backslash\{0\}$. Due to the representation theorem we then have

(3) $$f_z(z) = h(z) = e^{s(z)} F \circ \chi(z) \quad \text{in } U(0)\setminus\{0\}$$

where $s(z)$ is bounded and Hölder continuous in $\mathbf{C}$, and $\chi(z)$ can be chosen to be a schlicht solution in $\mathbf{C}$ of

$$\chi_{\bar{z}} = \lambda \cdot \chi_z$$

with

$$\lambda = \frac{1}{1-|\mu|^2}\left(\nu + \mu\bar{\nu}\frac{\overline{h_z}}{h_z}\right) \text{ a.e. in } U(0), \quad \lambda = 0 \text{ otherwise in } \mathbf{C}.$$

Moreover, $\chi(z)$ is conformal in a neighbourhood of infinity, with a Laurent expansion

$$\chi(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

there, and F is analytic in $\chi(U(0)\setminus\{0\})$, in particular it is analytic in some punctured disk $\{0 < |\chi - \chi(0)| < R\}$. $\lambda(z)$ is equivalent to a function from $HL_p(\{0\})$, for

$$|\lambda(z)| \leqslant \frac{1}{1-|\mu|^2}(|\nu| + |\nu\mu|) = \frac{|\nu|}{1-|\mu|} \leqslant |\nu| + |\mu| \quad \text{a.e. in } \mathbf{C}.$$

Hence, by Theorem 1, $\chi(z)$ admits an expansion

(4) $$\chi(z) = \chi(0) + dz + O(|z|^{1+\alpha^*}) \quad \text{at } z = 0,$$

with constants $d \neq 0$, $\alpha^* > 0$.

Next we have to show that $\chi(0)$ is not an essential singularity of F. It is clear that F is even analytic at $\chi(0)$ if n is positive. Now let n be negative. Since $s(z)$ is bounded in $\mathbf{C}$, there exist positive constants K_1, K_2 such that

(5) $$K_1|f_z(z)| \leqslant |F \circ \chi(z)| \leqslant K_2|f_z(z)| \quad \forall z \in U(0)\setminus\{0\}.$$

Consider the annulus

$$G' = \{r < |z| < r_0\} \Subset U(0)\setminus\{0\}, \quad r_0 \geqslant 3r.$$

Then Lemma I.3.22 from [5] gives

(6) $$\max\{|F \circ \chi(z)|: |z| = 2r\} < Kr^{-q/(2-q)}\|F \circ \chi\|_{L_q(G')},$$

where q is a constant satisfying $1 < q < 2$ as well as $k'C(q/(q-1)) < 1$ with k' being a bound < 1 for $|\nu| + |\mu|$ in $U(0)$ and $C(l)$ the norm of the Hilbert transformation in L_l ($C(l) \to 1$ as $l \to 2$); K denotes a constant depending only on k', q and $\sup\{|z|: z \in U(0)\}$.

We want an estimate

(7) $$\max\{|F \circ \chi(z)|: |z| = 2r\} \leqslant \text{const} \cdot (2r)^{-\tau}$$

with τ a real constant. To this end we have to estimate the L_q-norm of $F\circ\chi$ on the right-hand side of (6). By (5) we first have

$$\|F\circ\chi\|_{L_q(G')} \leqslant K_2 \|f_z(z)\|_{L_q(G')}, \tag{8}$$

and by the Hölder inequality

$$\|f_z(z)\|_{L_q(G')} \leqslant \|f_z(z)\|_{L_2(G')} \cdot [\pi(r_0^2 - r^2)]^{(2-q)/q}. \tag{9}$$

$f(z)$ itself, of course, also admits a representation according to the representation theorem, namely

$$f(z) = H\circ\varphi(z)$$

with φ being a quasiconformal mapping of $U(0)$, $\varphi(0) = 0$, and H analytic in $\varphi(U(0)\setminus\{0\})$. Since $z_0 = 0$ is a point of order n of f, $\varphi(0) = 0$ is a point of order n of H. Consequently there exists a special representation

$$f(z) = (\varphi(z))^n \tag{10}$$

where φ is a quasiconformal mapping of some neighbourhood $U'(0)$ of $z_0 = 0$. The r_0 of G' may be chosen in such a way that $U'(0) = \{|z| < r_0\}$. Then

$$\begin{aligned}\iint_{G'} |f_z(z)|^2 \, d\sigma_z &= \iint_{G'} n^2 |\varphi(z)|^{2n-2} |\varphi_z(z)|^2 \, d\sigma_z \\ &\leqslant \frac{n^2}{1-k'^2} \iint_{\varphi(G')} |\varphi|^{2n-2} \, d\sigma_\varphi.\end{aligned} \tag{11}$$

Because φ and its inverse, being quasiconformal mappings, are in particular Hölder continuous at $z = 0$, there exist positive constants D_1, D_2, γ, δ, such that

$$D_1 |z|^\gamma < |\varphi(z)| < D_2 |z|^\delta \quad \text{when } z \in U'(0).$$

Thus

$$\{D_1 r^\gamma < |\varphi| < D_2 r_0^\delta\} \supseteq \varphi(G'). \tag{12}$$

The estimate (7) is now a consequence of (5), (6), (8), (9), (11) and (12). Furthermore, since also $\chi(z)$ is Hölder continuous at $z = 0$, (7) implies that $\chi(0)$ cannot be an essential singularity of F, and this fact, along with (4), means that $f_z(z)$ in any case has an expansion

$$f_z(z) = Az^j + O(|z|^{j+\alpha'}) \quad \text{in } U(0)\setminus\{0\}, \tag{13}$$

with $A \neq 0$, $\alpha' > 0$, j an integer. It still remains to prove that $A = nc$ and $j = n-1$. For this we put $m+1 = \min\{j, n-1\}$. Then the function $g(z) = f(z)z^{-(m+1)} = cz^{n-(m+1)} + O(|z|^{n-(m+1)+\alpha})$ is continuous in $U(0)$, $g(0) = 0$, and differentiable in $U(0)\setminus\{0\}$ with

$$\begin{aligned} g_z(z) &= -(m+1)f(z)z^{-m-2} + f_z(z)z^{-m-1}, \\ g_{\bar z}(z) &= z^{-m-1}(\nu f_z + \mu \overline{f_z}),\end{aligned}$$

and g_z, $g_{\bar z}$ can be extended continuously to the origin, that is, to all of $U(0)$. Thus,

$$f(z)z^{-m-1} = \int_0^z g_z dz + g_{\bar z} d\bar z, \qquad \forall z \in \{|z| < r_0\},$$

where the straight line joining 0 and z can be chosen as the path of integration. This gives

$$\begin{aligned} f(z)z^{-m-1} &= \int_0^z [-(m+1)ct^{n-m-2} + At^{j-m-1} + O(|t|^{\alpha''})]\,dt \\ &\qquad + [\nu A t^{j-m-1} + \mu \bar A t^{-m-1}\bar t^{j} + O(|t|^{1+\alpha''})]\,d\bar t \\ &= -(m+1)c\frac{z^{n-m-1}}{n-m-1} + A\frac{z^{j-m}}{j-m} + O(|z|^{1+\alpha''}) + O(|z|^2) \end{aligned}$$

(note that $\nu(z) = O(|z|)$, $\mu(z) = O(|z|)$, after the affine transformations mentioned at the beginning of the proof). This is $cz^{n-m-1} + O(|z|^{n-m-1+\alpha})$, which is possible only if $j-m = n-m-1$ and $A = nc$.

3. Further preparations. Next we prove a preliminary step towards Theorem 4.

LEMMA 1. *Let $f(z)$ be a (ν, μ)-solution in $\{0 < |z| < R_0\}$ and let ν, μ have continuous second order partial derivatives in $\{|z| < R_0\}$. Suppose $f(z)$ has a point of order n at $z = 0$,*

$$f(z) = a(z + \mathfrak{b}\bar z)^n - b\bar a(\bar z + \bar{\mathfrak{b}} z)^n + O(|z|^{n+\alpha}),$$

and $[cz^{-n}]_*$ is the generalized power of degree $-n$ with respect to a (ν^*, μ^*)-system that coincides with the adjoint $(\nu, \bar\mu)$-system in $\{|z| < R_0\}$, ν^* and μ^* satisfying condition (1). Then with any $r \in (0, R_0)$ we have

(i) $$\operatorname{Re}\frac{1}{2\pi i}\oint_{|z|=r} f(z)\,d[cz^{-n}]_* = (1-|b|^2)(-n)\operatorname{Re}(ac)$$

(b, $\mathfrak{b}$ as in Theorem 1 with $z_0 = 0$).

Proof. Let us denote the left-hand term in (i) by I_n. By the generalized Cauchy integral theorem (cf. e.g. [5], § II.6), we obtain, after putting $\zeta = z + \mathfrak{b}\bar z$,

(14) $$I_n = \lim_{r\to 0}\operatorname{Re}\frac{1}{2\pi i}\oint_{|\zeta|=r}\{a\zeta^n - b\bar a\bar\zeta^n + O(|\zeta|^{n+\alpha})\}\,dh(\zeta)$$

where $h(\zeta) = [cz(\zeta)^{-n}]_*$ is a $(\nu_1, \bar\mu_1)$-solution with some $\nu_1(\zeta)$, $\mu_1(\zeta)$, $\nu_1(0) = 0$ and $\overline{\mu_1(0)} = -\bar b$ (cf. [5], § II.3). Since $[cz^{-n}]_*$ has the expansion

$$c(z + \mathfrak{b}\bar z)^{-n} - \bar b\bar c(\bar z + \bar{\mathfrak{b}} z)^{-n} + O(|z|^{-n+\alpha})$$

at $z = 0$, $h(\zeta)$ admits the asymptotic expansion $c\zeta^{-n} - \bar b\bar c\bar\zeta^{-n} + O(|\zeta|^{-n+\alpha})$ at $\zeta = 0$; also note that b (and $\bar b$) is invariant under the substitution $\zeta = z + \mathfrak{b}\bar z$.

Thus, when applying Theorem 5 and neglecting all the terms on the right-hand side of (14) whose limits are at once recognized to be zero, we obtain

$$I_n = \lim_{r\to 0} \operatorname{Re} \frac{-n}{2\pi i} \oint_{|\zeta|=r} \{a\zeta^n - b\bar{a}\bar{\zeta}^n\}\{c\zeta^{-n-1}\,d\zeta + \overline{\mu_1(0)}\bar{c}\overline{\zeta^{-n-1}\,d\zeta}\}$$
$$= -n\cdot \operatorname{Re}\big(ac + \bar{a}\bar{c}b\overline{\mu_1(0)}\big) = (1-|b|^2)(-n)\operatorname{Re}(ac),$$

which we had to prove.

Let $m_\varrho(z)$ be mollifiers as in, e.g., [5], §0.3,

(15) $$m_\varrho(z) = m_1(z/\varrho)\varrho^{-2}, \quad \varrho \text{ any positive number.}$$

Any integrable function, in particular every ν, $\mu \in HL_p(D)$, will be carried into a C^∞-function by

(16) $$\nu_\varrho(z) = \iint_{|t|<\varrho} \nu(t+z)m_\varrho(t)\,d\sigma_t, \qquad \mu_\varrho(z) = \iint_{|t|<\varrho} \mu(t+z)m_\varrho(t)\,d\sigma_t.$$

We want to know how the HL_p-norm changes under this smoothing procedure.

Lemma 2. *Let $\nu \in HL_p(\{|z| < R\})$, $p > 2$, $R > 0$. Then*

$$\|\nu_\varrho\|_{HL_p(\{|z|<R-\varrho\})} \leqslant \|\nu\|_{HL_p(\{|z|<R\})}$$

for every positive $\varrho < R$, and $\nu_\varrho(z) \to \nu(z)$ (even) locally uniformly in $\{|z| < R\}$ as $\varrho \to 0$.

Proof. By the definition of ν_ϱ and the Hölder inequality, for any positive T we have, with $q = p/(p-1)$,

$$\iint_{|z-z_0|<T} \left|\frac{\nu_\varrho(z)-\nu_\varrho(z_0)}{z-z_0}\right|^p d\sigma_z$$
$$= \iint_{|z-z_0|<T} \frac{1}{|z-z_0|^p}\Big|\iint_{|t|<\varrho} \big(\nu(t+z)-\nu(t+z_0)\big)m_\varrho(t)\,d\sigma_t\Big|^p d\sigma_z$$
$$\leqslant \iint_{|z-z_0|<T} \frac{1}{|z-z_0|^p}\Big(\iint_{|t|<\varrho} |\nu(t+z)-\nu(t+z_0)|\,m_\varrho(t)^{1/p}m_\varrho(t)^{1/q}\,d\sigma_t\Big)^p d\sigma_z$$
$$\leqslant \iint_{|z-z_0|<T} \frac{1}{|z-z_0|^p}\Big\{\iint_{|t|<\varrho} |\nu(t+z)-\nu(t+z_0)|^p m_\varrho(t)\,d\sigma_t\Big(\iint_{|t|<\varrho} m_\varrho(t)\,d\sigma_t\Big)^{p/q}\Big\} d\sigma_z$$
$$= \iint_{|z-z_0|<T} \frac{1}{|z-z_p|^p}\Big(\iint_{|t|<\varrho} |\nu(t+z)-\nu(t+z_0)|^p m_\varrho(t)\,d\sigma_t\Big) d\sigma_z,$$

since $\iint_{|t|<\varrho} m_\varrho(t)\,d\sigma_t = 1$. Furthermore,

$$(17)\quad \iint\limits_{|t|<\varrho}\Bigg(\iint\limits_{|z+t-(z_0+t)|<T}\left|\frac{v(t+z)-v(t+z_0)}{t+z-(t+z_0)}\right|^p d\sigma_z\Bigg)m_\varrho(t)\,d\sigma_t$$
$$\leqslant \sup\Bigg\{\iint\limits_{|h-(z_0+t)|<T}\left|\frac{v(h)-v(t+z_0)}{h-(t+z_0)}\right|^p d\sigma_h\colon\ |t|<\varrho\Bigg\}\cdot 1,$$

which is finite for every T, as long as $|z_0+t|<R$. Since the integrand on the left-hand side in (17) is positive and measurable, the order of integration can be changed, which, after letting T tend to infinity, finally gives

$$(18)\quad \sup\Bigg\{\iint\limits_{\mathbf{C}}\left|\frac{v_\varrho(z)-v_\varrho(z_0)}{z-z_0}\right|^p d\sigma_z\colon\ |z_0|<R-\varrho\Bigg\}$$
$$\leqslant \sup\Bigg\{\iint\limits_{\mathbf{C}}\left|\frac{v(z)-v(z_0)}{z-z_0}\right|^p d\sigma_z\colon\ |z_0|<R\Bigg\}.$$

The L_∞-part of the asserted norm estimate is obvious. Concerning the last statement note that the condition $v\in HL_p(\{|z|<R\})$, $p>2$, implies the continuity of v in $\{|z|<R\}$, by [5], Theorem II.5.49. The lemma is proved.

A corresponding statement holds with $v\in HL_p(\{\infty\})$.

LEMMA 3. *Let $v\in HL_p(\{\infty\})$, $p>2$. Then $v_\varrho\in HL_p(\{\infty\})$ with $v_\varrho(\infty)=v(\infty)$ for every positive ϱ and*

$$\|v_\varrho\|_{HL_p(\{\infty\})}\leqslant K_1\|v\|_{HL_p(\{\infty\})}$$

for every positive $\varrho\leqslant 1$, where the constant K_1 depends only on p and $\|v\|_{L_\infty}$ (i.e. K_1 depends only on p for all v satisfying $|v(z)|<1$ $\forall z\in\mathbf{C}$).

Proof. By hypothesis $v(1/z)-A\in HL_p(\{0\})$, $A=v(\infty)$. By a reasoning similar to the one in the proof of Lemma 2 we obtain

$$(19)\quad \iint\limits_{\mathbf{C}}\left|\frac{v_\varrho(1/z)-A}{z}\right|^p d\sigma_z\leqslant \iint\limits_{|t|<\varrho} m_\varrho(t)\iint\limits_{\mathbf{C}}\left|\frac{v(t+1/z)-A}{z}\right|^p d\sigma_z\,d\sigma_t$$
$$\leqslant \sup\Bigg\{\iint\limits_{\mathbf{C}}\left|\frac{v(t+1/z)-A}{z}\right|^p d\sigma_z\colon\ |t|<\varrho\Bigg\},$$

from which our assertions follow.

LEMMA 4. *Let v_m, μ_m be sequences of functions satisfying*

$$|v_m(z)|+|\mu_m(z)|\leqslant k=\text{const}<1,\qquad \forall z\in\mathbf{C},$$
$$\|v_m\|_{HL_p(\{|z|<R\})}+\|\mu_m\|_{HL_p(\{|z|<R\})}\leqslant K=\text{const},$$

with any fixed $R>0$, $p>2$, v_m, $\mu_m\in HL_p(\{\infty\})$ for every $m=1,2,$

Moreover, v_m, μ_m are supposed to converge to v, $\mu\in HL_p(\{\infty\})$, respectively, pointwise almost everywhere in $\mathbf{C}$ as $m\to\infty$. Further, let a be any fixed constant

$\neq 0$ and l be any fixed integer $\neq 0$. Then $[az^l]_{(\nu_m,\mu_m)}$ tends to $[az^l]_{(\nu,\mu)}$ pointwise in $\mathbf{C}\backslash\{0\}$, and the derivatives of the $[az^l]_{(\nu_m,\mu_m)}$ are uniformly bounded in each domain $\{d < |z| < D\}$ with any d, D satisfying $0 < d < D < R$.

Proof. Note first that by means of affine transformations not affecting the assertions of the lemma we can additionally obtain $\nu_m(0) = \mu_m(0) = 0$ for every m. Now put $H_m(z) = [az^l]_{(\nu_m,\mu_m)}$, $h_m(z) = H_m(z)/|H_m(1)|$. By the representation theorem every h_m admits a representation $h_m(z) = (\varphi_m(z))^l$ (cf. [6], Corollary 2.3) where $\varphi_m(z)$ is a quasiconformal mapping of the extended plane with $\varphi_m(0) = 0$, $\varphi_m(\infty) = \infty$, $|\varphi_m(1)| = 1$. By well-known compactness criteria (cf. [3], p. 73) the sequence φ_m is relatively compact in the set of $(1+k)/(1-k)$-quasiconformal mappings of $\mathbf{C}$ onto itself. Thus, a subsequence of φ_m tends to a quasiconformal mapping φ of $\mathbf{C}$ onto itself, $\varphi(0) = 0$, $\varphi(\infty) = \infty$, $|\varphi(1)| = 1$, $h(z) = (\varphi(z))^l$ is the limit function of the corresponding subsequence of h_m, and h is a (ν, μ)-solution in $\mathbf{C}\backslash\{0\}$, cf. e.g. [5], Theorem II.4.1(V). Every φ_m admits an asymptotic expansion

$$\varphi_m(z) = \frac{a^{1/l}}{|H_m(1)|^{1/l}} \cdot z + O(|z|^{1+\gamma})$$

with a positive γ, the 0-term, of course, depending on m. Since the φ_m are locally uniformly bounded in $\mathbf{C}$, the derivatives

$$\varphi_{mz}(0) = \frac{a^{1/l}}{|H_m(1)|^{1/l}}$$

are bounded away from 0 and ∞, cf. [5], Theorem II.5.2(II) and Theorem II.5.47. Therefore the $H_m(1)$, too, are bounded away from 0 and ∞, and thus the $H_m(z)$ themselves are locally uniformly bounded in $\mathbf{C}\backslash\{0\}$. Every $H_m(z)$ admits a representation

$$H_m(z) = (\Phi_m(z))^l$$

where the quasiconformal mappings Φ_m of $\mathbf{C}$ onto itself admit an asymptotic expansion

$$\Phi_m(z) = a^{1/l} z + O_m(|z|^{1+\gamma})$$

each, with one and the same $\gamma > 0$, and there is an O-term $O^*(|z|^{1+\gamma})$ such that for every m

$$|O_m(|z|^{1+\gamma})| \leqslant O^*(|z|^{1+\gamma}), \qquad \text{say, for } |z| \leqslant 1,$$

cf. [5], Theorem II.5.2(I). Thus, every limit function Φ of Φ_m, again being a quasiconformal mapping of $\mathbf{C}$ onto $\mathbf{C}$ with $\Phi(0) = 0$ ($\Phi(\infty) = \infty$), has an asymptotic expansion

$$\Phi(z) = a^{1/l} z + O(|z|^{1+\gamma}).$$

Hence, every corresponding limit function

(20) $$H(z) = (\Phi(z))^l$$

admits an expansion

$$H(z) = az^l + O(|z|^{l+\gamma}) \quad \text{at } z = 0.$$

H is a (ν, μ)-solution in $\mathbf{C}\backslash\{0\}$ and has a point of order $-l$ at ∞, by (20). Since the limits ν and μ of ν_m and μ_m, respectively, do also satisfy the conditions posed on ν_m, μ_m in Lemma 4, by the uniqueness statement of Theorem 2

$$H(z) = [az^l]_{(\nu,\mu)}.$$

Moreover, since every convergent subsequence of H_m has the same limit function, the original sequence must be convergent.

The boundedness of the derivatives of H_m, locally in $\mathbf{C}\backslash\{0\}$, is again a consequence of Theorem II.5.2(II) from [5]. The lemma is proved.

4. Proof of Theorems 3 and 4. We start with proving Theorem 4. Without loss of generality we may assume $z_0 = 0$. Let $h(z) = [az^n]_{(\nu,\mu)}$, $g(z) = [cz^l]_{(\nu,\bar{\mu})}$, $h^{(\varrho)}(z) = [az^n]_{(\nu_\varrho,\mu_\varrho)}$, $g^{(\varrho)}(z) = [cz^l]_{(\nu_\varrho,\bar{\mu}_\varrho)}$, where ν_ϱ, μ_ϱ correspond to ν, μ, respectively, according to (16). For a suitable null sequence $\varrho_m \to 0$, on putting $h^{(\varrho_m)} = h_m$, $g^{(\varrho_m)} = g_m$, we have

(21) $$\lim_{m\to\infty} \oint_{|z|=r} h_m dg_m = \oint_{|z|=r} h dg,$$

r any positive number such that $\{|z| \leqslant r\} \subset G$, by Lemmas 4 and 2. Thus, Theorem 4 holds if it holds under the additional condition $\nu, \mu \in C^\infty$ which we shall now suppose.

Furthermore, let λ_m be a function with continuous second order partial derivatives, satisfying $\lambda_m(z) = 1$ for $|z| \leqslant m$, $|\lambda_m(z)| \leqslant 1$ for $m \leqslant |z| \leqslant m+1$, $\lambda_m(z) = 0$ for $|z| > m+1$, $m = 1, 2, \ldots$ We now put

$$\nu_m = \lambda_m \cdot \nu, \quad \mu_m = \lambda_m \cdot \mu, \quad h_m(z) = [az^n]_{(\nu_m,\mu_m)}, \quad g_m(z) = [cz^l]_{(\nu_m,\bar{\mu}_m)}.$$

The sequences ν_m, μ_m again satisfy the assumptions of Lemma 4. Thus we again have (21), at present even with $\mathbf{C}$ in place of G, and, consequently, Theorem 4 holds if it holds with every $\nu, \mu \in C_0^2(\mathbf{C})$. In order to prove Theorem 4 for such ν, μ we distinguish three cases: $n+l = 0$, $n+l \leqslant -1$ and $n+l \geqslant 1$. The first case is settled by Lemma 1. For the second case fix an $R > 0$ such that $\nu(z) = \mu(z) = 0$ when $|z| > R$. Since h and g are analytic there, they admit Laurent expansions

(22) $$h(z) = A_n z^n + A_{n-1} z^{n-1} + \ldots \qquad g(z) = B_l z^l + B_{l-1} z^{l-1} + \ldots$$

for $|z| > R$. By the generalized Cauchy integral theorem,

$$\mathrm{Re}\frac{1}{2\pi i}\oint_C h\,dg = \mathrm{Re}\frac{1}{2\pi i}\oint_{|z|=R+1} h\,dg,$$

which is at once recognized to be zero in the case $n+l \leqslant -1$, because of (22). Finally, in the case $n+l \geqslant 1$, we have

$$\operatorname{Re}\frac{1}{2\pi i}\int_C h\,dg = \lim_{r\to 0}\operatorname{Re}\frac{1}{2\pi i}\oint_{|z|=r} h\,dg = \lim_{r\to 0}\operatorname{Re}\frac{1}{2\pi i}\oint_{|z|=r} O(z^n)O(z^{l-1})\,|dz| = 0.$$

Theorem 4 is proved.

Proof of Theorem 3. $f(z)$ has an expansion

(23) $f(z) = f(z_0) + a\big(z - z_0 + \mathfrak{b}\overline{(z-z_0)}\big) - b\bar{a}\big(\overline{z-z_0} + \bar{\mathfrak{b}}(z-z_0)\big) + O(|z-z_0|^{1+\alpha})$

at z_0 with some constant a (not necessarily $\neq 0$), thus

(24) $\quad f_z(z_0) = a - \bar{a}b\bar{\mathfrak{b}} \quad$ and $\quad a = \big(f_z(z_0) + \bar{\mathfrak{b}}b\overline{f_z(z_0)}\big)/(1-|b\mathfrak{b}|^2).$

Let us first consider the case $a = 0$. This means $f_z(z_0) = 0$. Then

$$\int_C f(z)\,d[c(z-z_0)^{-1}]_{(\nu,\bar{\mu})} = \lim_{r\to 0}\Big\{-\oint_{|z-z_0|=r}[c(z-z_0)^{-1}]_{(\nu,\bar{\mu})}\,df(z)\Big\} = 0,$$

because

$$\Big|\oint_{|z-z_0|=r}[c(z-z_0)^{-1}]_{(\nu,\bar{\mu})}\,df(z)\Big|$$

$$\leqslant \oint_{|z-z_0|=r}\frac{\text{const}}{|z-z_0|}|dz|(1+k)\max\{|f_z(z)|:\ |z-z_0| = r\}$$

and

$$\lim_{z\to z_0} f_z(z) = 0 = f_z(z_0),$$

by the continuity of $f_z(z)$ in G, cf. [5], Theorem II.5.2. Thus the asserted formula holds for $f_z(z_0) = 0$.

Now let $f_z(z_0) \neq 0$. This means that we also have $a \neq 0$ in (23), (24). The function

$$f^*(z) = f(z) - [a(z-z_0)]_{(\nu,\mu)}$$

is a (ν, μ)-solution in G, and $f^*_z(z_0) = 0$. According to the first part of our proof

$$\operatorname{Re}\frac{1}{2\pi i}\int_C f^*(z)\,d[c(z-z_0)^{-1}]_{(\nu,\bar{\mu})} = 0,$$

and along with Theorem 4 this finishes the proof of Theorem 3.

References

[1] L. Bers, *Partial differential equations and generalized analytic functions*, Proc. Nat. Acad. Sci. U.S.A. 36 (1950), 130–136; Second note, ibid. 37 (1951), 42–47.

[2] B. Bojarski, *Generalized solutions of a system of differential equations of first order and elliptic type with discontinuous coefficients*, Mat. Sb. (N.S.) 43 (85) (1957), 451 503 (in Russian).

[3] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, 2nd ed., Springer, Berlin 1973.

[4] G. N. Polozhii, *A generalization of Cauchy's integral formula*, Mat. Sb. (N.S.) 24 (66) (1949), 375–384 (in Russian).

[5] H. Renelt, *Elliptic Systems and Quasiconformal Mappings*, Wiley, 1988.

[6] —, *Generalized powers in the theory of* (ν, μ)*-solutions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 40 (1986), 217–235.

[7] B. V. Shabat, *Cauchy's theorem and formula for quasiconformal mappings of linear classes*, Dokl. Akad. Nauk SSSR (N.S.) 69 (1949), 305–308 (in Russian).

MARTIN-LUTHER-UNIVERSITÄT HALLE-WITTENBERG, SEKTION MATHEMATIK
O-4010 Halle/S., Germany

Reçu par la Rédaction le 04.12.1989
Révise le 15.03.1990