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Lie derivative of vector fields on a differential space

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Abstract. In 1931 W. Ślebodziński (cf. [4]) introduced the concept of the Lie derivative for smooth tensor fields. This concept plays a great part in differential geometry and global analysis. In 1967 R. Sikorski (cf. [2]) introduced the concept of a differential space as a generalization of a manifold. Independently, S. Mac Lane introduced the same concept in his lectures on foundations of mechanics (cf. [1]). In the present paper the concept of the one-parameter local Lie group for differential spaces is introduced and it is proved that in a differential space the Lie derivative of vector field defined by the Lie brackets is the same as the one defined by means of the well-known limit formula for differentiable manifolds. For some details we refer to book [3] and to papers [5] and [6] as well.

1. Preliminaries. Let (M, C) be a differential space and $p \in M$. The sequence $v_1, v_2, \ldots$ of vectors of the tangent space, $(M, C)_p$, to (M, C) at the point p is said to be *tending to the vector* v of $(M, C)_p$ iff for any function $\alpha \in C$ the sequence $v_1(\alpha), v_2(\alpha), \ldots$ tends to $v(\alpha)$. We recall that by tangent vector to (M, C) at p we mean a linear mapping $v\colon C \to \boldsymbol{R}$ fulfilling the condition: $v(\alpha\beta) = \alpha(p)v(\beta) + \beta(p)v(\alpha)$ for $\alpha, \beta \in C$. Let (N, D) be a differential space and let

$$f\colon (M, C) \to (N, D) \tag{1}$$

be a smooth mapping. If V assigns to every point $p \in M$ the vector $V(p)$ of $(N, D)_{f(p)}$ in such a way that for any $\beta \in D$ the function $\partial_V \beta$, defined by the formula $\partial_V \beta(p) = V(p)(\beta)$ for $p \in M$, belongs to C, then V is said to be a *smooth f-vector field on* (M, C) or, shortly, *f-field on* (M, C). In particular, if the mapping (1) is of the form id: $(A, C_A) \to (M, C)$, where $A \subset M$, then smooth f-field on (M, C) is said to be a *tangent to* (M, C) *vector field on* A.

Denote by E the set of all real functions of class C^∞ on $\boldsymbol{R}$; it is so-called the *natural differential structure* of $\boldsymbol{R}$.

PROPOSITION 1. *If V is a smooth tangent to (M, C) f-field, where $f\colon (I, E_I) \to (M, C)$, $V(t) \xrightarrow[t \to t_0]{} v$, t_0 is any of the open interval I, then for every* *smooth on $(M, C)\times(I, E_I)$ real function α we have*

$$V(t)\left(\frac{\alpha(\cdot, t)-\alpha(\cdot, t_0)}{t-t_0}\right)\xrightarrow[t\to t_0]{} v(\alpha'(\cdot, t_0)),$$

where $\alpha'(q, t_0)$ stands for the derivative of $\alpha(q, \cdot)$ at t_0.

Proof. For such a function α there are the functions $\alpha^1, \ldots, \alpha^m \in C$, a neighbourhood U of the point $p = f(t_0)$ and the open interval J such that $t_0 \in J \subset I$ and

$$\alpha(q, t) = \omega(\alpha^1(q), \ldots, \alpha^m(q), t) \quad \text{for } q\in U,\ t\in J,$$

where ω is a function of class C^∞ on R^{m+1}. Thus, there exist functions ω_i of class C^∞ on R^{2m+2} such that for any reals $t^1, \ldots, t^{m+1}, u^1, \ldots, u^{m+1}$ we have

$$\omega(u^1, \ldots, u^{m+1})-\omega(t^1, \ldots, t^{m+1}) = \omega_i(u^1, \ldots, u^{m+1}, t^1, \ldots, t^{m+1})(u^i-t^i),$$

where

$$(2) \qquad \omega_i(t^1, \ldots, t^{m+1}, t^1, \ldots, t^{m+1}) = \omega_{|i}(t^1, \ldots, t^{m+1}),$$

$\omega_{|i}$ stands for the derivative of ω with respect to ith variable, $i = 1, \ldots, m+1$. Hence

$$\alpha(q, t)-\alpha(q, t_0) = \omega_{m+1}(\alpha^1(q), \ldots, \alpha^m(q), t, \alpha^1(q), \ldots, \alpha^m(q), t_0)(t-t_0)$$

for $t\in J$. Setting $\beta(t) = V(t)\big((\alpha(\cdot, t)-\alpha(\cdot, t_0))/(t-t_0)\big)$ and

$$(3) \qquad \omega_0(u^1, \ldots, u^{m+1}) = \omega_{m+1}(u^1, \ldots, u^m, u^{m+1}, u^1, \ldots, u^m, t_0),$$

we obtain in turn

$$\big(\alpha(q, t)-\alpha(q, t_0)\big)/(t-t_0) = \omega_0(\alpha^1(q), \ldots, \alpha^m(q), t),$$
$$\beta(t) = \omega_{0|j}\big(\alpha^1(f(t)), \ldots, \alpha^m(f(t)), t\big)V(t)(\alpha^j),$$
$$\beta(t)\xrightarrow[t\to t_0]{}\omega_{0|j}(\alpha^1(p), \ldots, \alpha^m(p), t_0)v(\alpha^j) = v\big(\omega_0(\alpha^1(\cdot), \ldots, \alpha^m(\cdot), t_0)\big).$$

From (3) and (2) we have

$$\begin{aligned}\omega_0\big(\alpha^1(q), \ldots, \alpha^m(q), t_0\big) &= \omega_{|m+1}\big(\alpha^1(q), \ldots, \alpha^m(q), t_0\big)\\ &= \big(\omega(\alpha^1(q), \ldots, \alpha^m(q), \cdot)\big)'(t_0) = \big(\alpha(q, \cdot)\big)'(t_0)\\ &= \alpha'(q, t_0).\end{aligned}$$

Thus, $\beta(t)\to v\big(\alpha'(\cdot, t_0)\big)$ when $t\to t_0$. What completes the proof.

For any tangent to (M, C) vector fields X and Y on the set U open in τ_C (see [2]) the vector field $[X, Y]$ defined by the equality

$$(4) \qquad [X, Y](p)(\alpha) = X(p)(\partial_Y\alpha)-Y(p)(\partial_X\alpha) \quad \text{for } \alpha\in C,\ p\in U$$

is called the *Lie product of X and Y.*

Let U be a set open in τ_C. The smooth mapping

$$(5) \qquad \varphi\colon (U, C_U) \times (I, E_I) \to (M, C),$$

where $I = (-\varepsilon; \varepsilon)$ and $\varepsilon > 0$, is said to be a *one-parameter local Lie group* iff the following conditions:

(i) *if* $t, s, t+s \in I$, $p, \varphi(p, t) \in U$, *then* $\varphi(\varphi(p, t), s) = \varphi(p, t+s)$;

(ii) $\varphi(p, 0) = p$ *for* $p \in U$;

are fulfilled.

Setting $U_t = (\varphi(\cdot, t))^{-1}[U]$ and $\varphi_t(p) = \varphi(p, t)$ for $p \in U_t$, we have the diffeomorphism

$$\varphi_t\colon (U_t, C_{U_t}) \to (U_{-t}, C_{U_{-t}})$$

such that the inverse diffeomorphism is of the shape

$$\varphi_{-t}\colon (U_{-t}, C_{U_{-t}}) \to (U_t, C_{U_t}).$$

Moreover, for any $p \in U$ there exists $\delta > 0$ such that $p \in U_t$ for $t \in (-\delta; \delta)$.

The mapping (1) defines the tangent mapping, f_*, which to any vector v tangent to (M, C) assigns the vector $f_* v$ such that $(f_* v)(\beta) = v(\beta \circ f)$ for any $\beta \in D$.

The vector tangent to (M, C) at the point $\varphi(p, t)$ defined by the formula

$$(6) \qquad \dot{\varphi}(p, t)(\alpha) = \big(\alpha \circ \varphi(p, \cdot)\big)'(t) \quad \text{for } \alpha \in C$$

will be denoted by $\dot{\varphi}(p, t)$.

Thus, we have the mapping $\varphi(p, \cdot)\colon (I, E_I) \to (M, C)$ and the smooth $\varphi(p, \cdot)$-vector field for $p \in M$.

PROPOSITION 2. *For every one-parameter local Lie group* (5) *setting*

$$(7) \qquad X(p) = \dot{\varphi}(p, 0) \quad \textit{for } p \in U,$$

we obtain a tangent to (M, C) *vector field on* U *such that*

$$(8) \qquad X\big(\varphi(p, t)\big) = \dot{\varphi}(p, t) \quad \textit{for } p \in U,\ t \in I.$$

Proof. Smoothness of X is obvious. For any $p \in U$, $t \in I$ and $\alpha \in C$ we have

$$\begin{aligned} X\big(\varphi(p, t)\big)(\alpha) &= \dot{\varphi}\big(\varphi(p, t), 0\big)(\alpha) \\ &= \big(\alpha \circ \varphi(\varphi(p, t), \cdot)\big)'(0) = \big(s \mapsto \alpha\big(\varphi(\varphi(p, t), s)\big)\big)'(0) \\ &= \big(s \mapsto \alpha(\varphi(p, t+s))\big)'(0) = \big(s \mapsto \alpha(\varphi(p, s))\big)'(t) \\ &= \dot{\varphi}(p, t)(\alpha). \end{aligned}$$

Hence we obtain (8).

The vector field X defined by (7) is said to be *generated by the one-parameter local Lie group* (5). In the connection with that above it arises the problem of determining the one-parameter local Lie group by the vector field. In particular, it is a question about conditions for a differential space under which for any smooth vector field X and any point p there exists one-parameter local Lie group (5) such that $p \in U$ and $X|U$ is generated by (5).

2. The limit formula. In the section we derive the limit formula for Lie derivative for vector fields. It seems to be interesting that the one-parameter local Lie group which generates the vector field X need not to be uniquely determined by X, but Lie derivative calculated by means of this one-parameter local Lie group is independent of the choice of it.

THEOREM. *If a vector field X is generated by the one-parameter local Lie group* (5) *and Y is any tangent to (M, C) vector field on U, then*

$$\frac{1}{t}\Big(Y(p)-(\varphi(\cdot, t))_*\big(Y(\varphi(p, -t))\big)\Big)\xrightarrow[t\to 0]{}[X, Y](p). \tag{9}$$

Proof. Let us take $\alpha \in C$ and $p \in U$. Then, setting

$$\beta(t) = \frac{1}{t}\Big(Y(p)(\alpha)-(\varphi(\cdot, t))_*\big(Y(\varphi(p, -t))\big)(\alpha)\Big),$$

by (6) and (7), we have

$$\beta(t) = \frac{1}{t}\Big(Y(p)(\alpha)- Y\big(\varphi(p, -t)\big)\big(\alpha \circ \varphi(\cdot, t)\big)\Big)$$

$$= \frac{1}{t}\Big(\partial_Y \alpha(p) - \partial_Y \alpha\big(\varphi(p, -t)\big)\Big)-\frac{1}{t} Y\big(\varphi(p, -t)\big)\big(\alpha \circ \varphi(\cdot, t)-\alpha\big),$$

$$\frac{1}{t}\Big(\partial_Y \alpha(p)- \partial_Y \alpha\big(\varphi(p, -t)\big)\Big)$$

$$= \frac{1}{t}\Big(\partial_Y \alpha\big(\varphi(p, 0)\big)- \partial_Y \alpha\big(\varphi(p, -t)\big)\Big)\xrightarrow[t\to 0]{}\Big(\partial_Y \alpha\big(\varphi(p, \cdot)\big)\Big)'(0)$$

$$= \dot{\varphi}(p, 0)(\partial_Y \alpha) = X(p)(\partial_Y \alpha).$$

By Proposition 1 we get

$$\frac{1}{t} Y\big(\varphi(p, -t)\big)\big(\alpha \circ \varphi(\cdot, t)-\alpha\big) = Y\big(\varphi(p, -t)\big)\frac{\alpha(\varphi(\cdot, t))-\alpha(\varphi(\cdot, 0))}{t}$$

$$\xrightarrow[t\to 0]{} Y(p)\Big(\big(t \mapsto \alpha(\varphi(\cdot, t))\big)'(0)\Big) = Y(p)\big(\dot{\varphi}(\cdot, 0)(\alpha)\big) = Y(p)(\partial_X \alpha).$$

Hence, by (4), it follows that $\beta(t)\xrightarrow[t\to 0]{}[X, Y](p)(\alpha)$. This completes the proof.

It seems to be rather difficult to describe the set of all one-parameter local Lie groups for a given differential space.

References

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