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Existence-uniqueness and iterative methods for right focal point boundary value problems for differential equations with deviating arguments

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Abstract. For nth order delay differential equations with right focal point boundary conditions we provide necessary and sufficient conditions for the existence and uniqueness of solutions. Next, we give a priori conditions so that the Picard iterative method converges to the unique solution of the problem. Necessary and sufficient conditions for the convergence of an approximate Picard method are also obtained. Finally, we supply sufficient conditions so that the Picard method converges monotonically to a solution of the problem.

1. Introduction. We shall consider the nth order ordinary differential equation with deviating arguments

$$x^{(n)}(t) = f(t, x\circ w(t)), \quad a \leqslant t \leqslant b, \tag{1.1}$$

where $x\circ w(t)$ stands for $\left(x(w_{0,1}(t)), \ldots, x(w_{0,p(0)}(t)), \ldots, x^{(q)}(w_{q,p(q)}(t))\right)$, $0 \leqslant q \leqslant n-1$ (but fixed), and $p(i)$, $0 \leqslant i \leqslant q$, are positive integers. The function $f(t, \langle x\rangle)$ is assumed to be continuous on $[a, b]\times R^N$, where $\langle x\rangle$ represents $(x_{0,1}, \ldots, x_{0,p(0)}, \ldots, x_{q,p(q)})$, and $N = \sum_{i=0}^{q} p(i)$. The functions $w_{i,j}$, $1 \leqslant j \leqslant p(i)$, $0 \leqslant i \leqslant q$, are continuous on $[a, b]$, and $w_{i,j}(t) \leqslant b$ for all $t\in[a, b]$; also, they assume the value a at most a finite number of times as t ranges over $[a, b]$. Let

$$\alpha = \min\{a, \inf_{a\leqslant t\leqslant b} w_{i,j}(t),\ 1 \leqslant j \leqslant p(i),\ 0 \leqslant i \leqslant q\}.$$

If $\alpha < a$, we assume that a function $\varphi\in C^{(q)}[\alpha, a]$ is given. Let k be a fixed integer such that $1 \leqslant k \leqslant n-1$, and let $r = \min\{q, k-1\}$. We seek a function $x\in\mathscr{B} = C^{(r)}[\alpha, b]\cap C^{(q)}[\alpha, a]\cap C^{(q)}[a, b]$ having at least a piecewise continuous nth derivative on $[a, b]$, and such that:

(1.2) if $\alpha < a$ and $q \geqslant k-1$, then

$$x^{(i)}(t) = \varphi^{(i)}(t), \quad 0 \leqslant i \leqslant q,\ t\in[\alpha, a];$$

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if $\alpha < a$ and $q < k-1$, then

$$x^{(i)}(t) = \varphi^{(i)}(t), \quad 0 \leqslant i \leqslant q, \; t \in [\alpha, a]; \quad x^{(i)}(a) = A_i, \quad q+1 \leqslant i \leqslant k-1;$$

if $\alpha = a$, then

$$x^{(i)}(a) = A_i, \quad 0 \leqslant i \leqslant k-1;$$

and

$$x^{(i)}(b) = B_i, \quad k \leqslant i \leqslant n-1; \tag{1.3}$$

also, x is a solution of (1.1) on $[a, b]$.

Second order boundary value problems with deviating arguments arise naturally in the study of variational problems in control theory. Sufficient conditions for the existence and uniqueness of a solution and several constructive methods for these boundary value problems have been given in [5], [7]–[17]. While for higher order differential equations with deviating arguments initial value problems have been studied extensively, not much seems to be known for boundary value problems except some existence and uniqueness results are obtained in [1]. Due to their importance in several engineering applications, right focal point boundary value problems for ordinary differential equations have recently been studied in [2]–[4], and references therein; the motivation of the present paper is in line with this work.

2. Some basic lemmas.

LEMMA 2.1 [2]. *The Green's function $g(t, s)$ of the boundary value problem*

$$x^{(n)} = 0, \quad x^{(i)}(a) = 0, \; 0 \leqslant i \leqslant k-1, \quad x^{(i)}(b) = 0, \; k \leqslant i \leqslant n-1, \tag{2.1}$$

can be written as

$$g(t, s) = \begin{cases} \dfrac{1}{(n-1)!}\displaystyle\sum_{i=0}^{k-1}\binom{n-1}{i}(t-a)^i(a-s)^{n-i-1}, & s \leqslant t, \\ -\dfrac{1}{(n-1)!}\displaystyle\sum_{i=k}^{n-1}\binom{n-1}{i}(t-a)^i(a-s)^{n-i-1}, & s \geqslant t, \end{cases}$$

and

$$(-1)^{n-k} g^{(i)}(t, s) \geqslant 0, \quad 0 \leqslant i \leqslant k, \; (t, s) \in [a, b] \times [a, b];$$

$$(-1)^{n-i} g^{(i)}(t, s) \geqslant 0, \quad k+1 \leqslant i \leqslant n-1, \; (t, s) \in [a, b] \times [a, b];$$

$$\sup_{a \leqslant t \leqslant b} \int_a^b |g^{(i)}(t, s)|\, ds \leqslant C_{n,i}(b-a)^{n-i}, \quad 0 \leqslant i \leqslant n-1,$$

where $g^{(i)}(t, s) = \partial^i g(t, s)/\partial t^i$ and

$$C_{n,i} = \begin{cases} \dfrac{1}{(n-i)!}\left|\displaystyle\sum_{j=0}^{k-i-1}\binom{n-1}{j}(-1)^{n-1-j}\right|, & 0 \leqslant i \leqslant k-1, \\ \dfrac{1}{(n-i)!}, & k \leqslant i \leqslant n-1. \end{cases}$$

LEMMA 2.2 [2]. *The unique polynomial $P_{n-1}(t)$ of degree $n-1$ satisfying $P_{n-1}^{(i)}(a) = \alpha_i$, $0 \leqslant i \leqslant k-1$, and $P_{n-1}^{(i)}(b) = \beta_i$, $k \leqslant i \leqslant n-1$, can be written as*

$$(2.2) \qquad P_{n-1}(t) = \sum_{i=0}^{k-1}\frac{(t-a)^i}{(i)!}\alpha_i + \sum_{j=0}^{n-k-1}\left(\sum_{i=0}^{j}\frac{(t-a)^{k+i}(a-b)^{j-i}}{(k+i)!(j-i)!}\right)\beta_{k+j}.$$

LEMMA 2.3 [1]. *Let B be a Banach space and let $\mu > 0$, $\mu \in \mathbf{R}$, $\bar{S}(u_0, \mu) = \{u \in B: \|u-u_0\| \leqslant \mu\}$. Further, let T be an operator which maps $\bar{S}(u_0, \mu)$ into B, and*

(i) *for all $u, v \in \bar{S}(u_0, \mu)$, $\|Tu-Tv\| \leqslant \lambda\|u-v\|$, where $0 \leqslant \lambda < 1$,*

(ii) $\mu_0 = (1-\lambda)^{-1}\|Tu_0-u_0\| \leqslant \mu$.

Then

(1) *T has a fixed point u^* in $\bar{S}(u_0, \mu_0)$,*

(2) *u^* is the unique fixed point of T in $\bar{S}(u_0, \mu)$,*

(3) *the sequence $\{u_m\}$, where $u_{m+1} = Tu_m$, $m = 0, 1, \ldots$, converges to u^* with $\|u^*-u_m\| \leqslant \lambda^m\mu_0$, and $\|u^*-u_m\| \leqslant \lambda(1-\lambda)^{-1}\|u_m-u_{m-1}\|$,*

(4) *for any $u \in \bar{S}(u_0, \mu_0)$, $u^* = \lim_{m\to\infty} T^m u$.*

LEMMA 2.4. *Let $(E, \leqslant)$ be a partially ordered space, let $x_0 \leqslant y_0$ be two elements of E, and let $[x_0, y_0]$ denote the interval $\{x \in E: x_0 \leqslant x \leqslant y_0\}$. Let $T: [x_0, y_0] \to E$ be an isotone operator ($T(x) \leqslant T(y)$ whenever $x \leqslant y$) and let it have properties* (i)–(iv) *or* (i)′–(iv)′ *below*:

(i) $x_0 \leqslant T(x_0)$;

(ii) *the (nondecreasing) sequence $\{T^m(x_0)\}$ where $T^0(x_0) = x_0$, $T^{m+1}(x_0) = T[T^m(x_0)]$ for each $m = 0, 1, \ldots$ is well defined, i.e., $T^m(x_0) \leqslant y_0$ for each natural m;*

(iii) *the sequence $\{T^m(x_0)\}$ has* $\sup x \in E$, *i.e., $T^m(x_0)\uparrow x$;*

(iv) $T^{m+1}(x_0)\uparrow T(x)$;

(i)′ $T(y_0) \leqslant y_0$;

(ii)′ *the (nonincreasing) sequence $\{T^m(y_0)\}$ is well defined, i.e., $T^m(y_0) \geqslant x_0$ for each natural m;*

(iii)′ *the sequence $\{T^m(y_0)\}$ has* $\inf y \in E$, *i.e., $T^m(y_0)\downarrow y$;*

(iv)′ $T^{m+1}(y_0)\downarrow T(y)$.

Then $x = T(x)$ and $x \leqslant z$ for any other fixed point $z \in [x_0, y_0]$ of T, or, respectively, $y = T(y)$ and $z \leqslant y$ for any other fixed point $z \in [x_0, y_0]$ of T

Moreover, if T has both properties (i) *and* (i)′, *then the sequences $\{T^m(x_0)\}$,*

$\{T^m(y_0)\}$ *are well defined and if, further,* T *has the properties* (iii), (iii)$'$ *and* (iv), (iv)$'$ *then*

$$x_0 \leqslant T(x_0) \leqslant \ldots \leqslant T^m(x_0) \leqslant \ldots \leqslant x \leqslant y \leqslant \ldots \leqslant T^m(y_0) \leqslant \ldots \leqslant T(y_0) \leqslant y_0,$$

and $x = T(x)$, $y = T(y)$; *also any other fixed point* $z \in [x_0, y_0]$ *satisfies* $x \leqslant z \leqslant y$.

Proof. This is stated in [18] and its proof is based on Viswanatham's lemma [20] and an existence theorem due to Bange [6].

LEMMA 2.5 [19]. *Let* $M > 0$ *and let* $\{x_m(t)\}$ *be a sequence of functions in* $C^{(n)}[a, b]$ *such that* $|x_m(t)| \leqslant M$ *and* $|x_m^{(n)}(t)| \leqslant M$ *for all m. Then there exists a subsequence* $\{x_{m(j)}(t)\}$ *such that* $\{x_{m(j)}^{(i)}(t)\}$ *converges uniformly on* $[a, b]$ *for each* i, $0 \leqslant i \leqslant n-1$.

3. Existence and uniqueness. To prove the existence and uniqueness of solutions of the boundary value problem (1.1)–(1.3) we shall convert it to its equivalent integral equation representation. For this, we define functions θ and ψ as follows:

$$\theta(t) = \begin{cases} 0, & t \in [\alpha, a], \\ 1, & \text{otherwise}. \end{cases}$$

If $\alpha < a$ and $q \geqslant k-1$, then

$$\psi(t) = \begin{cases} \varphi(t), & t \in [\alpha, a], \\ P_{n-1}(t), & t \in [a, b], \end{cases}$$

where $\alpha_i = \varphi^{(i)}(a)$, $0 \leqslant i \leqslant k-1$, and $\beta_i = B_i$, $k \leqslant i \leqslant n-1$.

If $\alpha < a$ and $q < k-1$, then

$$\psi(t) = \begin{cases} \varphi(t), & t \in [\alpha, a], \\ P_{n-1}(t), & t \in [a, b], \end{cases}$$

where $\alpha_i = \varphi^{(i)}(a)$, $0 \leqslant i \leqslant q$; $\alpha_i = A_i$, $q+1 \leqslant i \leqslant k-1$, and $\beta_i = B_i$, $k \leqslant i \leqslant n-1$.

If $\alpha = a$, then

$$\psi(t) = P_{n-1}(t), \quad t \in [\alpha, a],$$

where $\alpha_i = A_i$, $0 \leqslant i \leqslant k-1$, and $\beta_i = B_i$, $k \leqslant i \leqslant n-1$.

It is clear that $\psi \in \mathscr{B}$, and for all $t \in [a, b]$ with $w_{i,j}(t) = a$, $\psi^{(i)}(w_{i,j}(t)) = P_{n-1}^{(i)}(a+0)$. Further, the boundary value problem (1.1)–(1.3) is equivalent to the integral equation

$$x(t) = \psi(t) + \theta(t) \int_a^b g(t, s) f(s, x \circ w(s)) ds. \tag{3.1}$$

THEOREM 3.1. *Suppose that*

(i) $K_i > 0$, $0 \leqslant i \leqslant q$, *are given real numbers and* Q *is the maximum of* $|f(t, \langle x\rangle)|$ *on the compact set* $[a, b]\times D_0$, *where*

$$D_0 = \{\langle x\rangle\colon |x_{i,1}|, \ldots, |x_{i,p(i)}| \leqslant 2K_i,\ 0 \leqslant i \leqslant q\},$$

(ii) $\sup_{a\leqslant t\leqslant b}|\psi^{(i)}(t)| \leqslant K_i$, $0 \leqslant i \leqslant q$, *wherever* $\psi^{(i)}(t)$ *exists,*

(iii) $$(b-a) \leqslant \left(\frac{K_i}{QC_{n,i}}\right)^{1/(n-i)} \qquad 0 \leqslant i \leqslant q.$$

Then the boundary value problem (1.1)–(1.3), *or equivalently* (3.1), *has a solution in* $\mathscr{B}$.

Proof. We define an operator T from the Banach space $(\mathscr{B}, \|\cdot\|)$, where

$$\|x\| = \max_{0\leqslant i\leqslant q}\{\sup_{a\leqslant t\leqslant b}|x^{(i)}(t)|,\ \text{wherever } x^{(i)}(t) \text{ exists}\},$$

into $\mathscr{B}$ by

$$(Tx)(t) = \psi(t)+\theta(t)\int_a^b g(t, s)f(s, x\circ w(s))ds. \tag{3.2}$$

The following properties of T may easily be established:

(1) $(Tx)(t)$ satisfies the boundary conditions (1.2), (1.3),

(2) $(Tx)^{(n)}(t) = f(t, x\circ w(t))$ at all points of continuity of $f(t, x\circ w(t))$ with $t\in[a, b]$,

(3) T is a completely continuous operator,

(4) fixed points of T are solutions of the boundary value problem (1.1)–(1.3).

Consider the closed convex subset $\mathscr{B}_1$ of $(\mathscr{B}, \|\cdot\|)$ defined by $\mathscr{B}_1 = \{x\in\mathscr{B}\colon |x^{(i)}(t)| \leqslant 2K_i,\ 0 \leqslant i \leqslant q,$ wherever $x^{(i)}(t)$ exists$\}$. We shall show that T maps $\mathscr{B}_1$ into itself. For this, let $x\in\mathscr{B}_1$. If $t\in[\alpha, a]$, then from (1) and (ii) it is obvious that $|(Tx)^{(i)}(t)| \leqslant K_i$, $0 \leqslant i \leqslant q$. If $t\in[a, b]$, then since $w_{i,j}(t) \in[\alpha, b]$, $1 \leqslant j \leqslant p(i)$, $0 \leqslant i \leqslant q$, it is clear that $x\circ w(t)\in D_0$, and hence from Lemma 2.1 and (ii) and (iii), we find

$$\begin{aligned}|(Tx)^{(i)}(t)| &\leqslant \sup_{a\leqslant t\leqslant b}|\psi^{(i)}(t)|+\sup_{a\leqslant t\leqslant b}\int_a^b|g^{(i)}(t, s)||f(s, x\circ w(s))|ds\\ &\leqslant K_i+QC_{n,i}(b-a)^{n-i} \leqslant 2K_i, \qquad 0 \leqslant i \leqslant q.\end{aligned}$$

Thus, if $t\in[\alpha, b]$, we have $|(Tx)^{(i)}(t)| \leqslant 2K_i$, $0 \leqslant i \leqslant q$, wherever $(Tx)^{(i)}(t)$ exists, and hence $T\mathscr{B}_1 \subseteq \mathscr{B}_1$. Using the Ascoli–Arzelà theorem one may easily prove that $T\mathscr{B}_1$ is sequentially compact. Hence by the Schauder–Tikhonov fixed point theorem T has a fixed point in $\mathscr{B}_1$. From (4) this fixed point is a solution of (1.1)–(1.3).

COROLLARY 3.2. *Suppose that the hypotheses* (i), (iii) *of Theorem* 3.1 *are satisfied and let* $l \in C^{(q)}[\alpha, a] \cap C^{(n-1)}[a, b]$ *be given. Then the differential equation* (1.1) *together with the conditions*:

(3.3) *if* $\alpha < a$ *and* $q \geqslant k-1$, *then*

$$x^{(i)}(t) = l^{(i)}(t), \quad 0 \leqslant i \leqslant q, \ t \in [\alpha, a];$$

if $\alpha < a$ *and* $q < k-1$, *then*

$$x^{(i)}(t) = l^{(i)}(t), \quad 0 \leqslant i \leqslant q, \ t \in [\alpha, a]; \quad x^{(i)}(a) = l^{(i)}(a), \quad q+1 \leqslant i \leqslant k-1;$$

if $\alpha = a$, *then*

$$x^{(i)}(a) = l^{(i)}(a), \quad 0 \leqslant i \leqslant k-1;$$

and

$$x^{(i)}(b) = l^{(i)}(b), \quad k \leqslant i \leqslant n-1, \tag{3.4}$$

has a solution in $\mathscr{B}$ *if*

$$\sup_{\alpha \leqslant t \leqslant a} |l^{(i)}(t)| \leqslant K_i, \quad 0 \leqslant i \leqslant q, \tag{3.5}$$

$$\sum_{p=j}^{k-2} \frac{(b-a)^p}{(p)!} M_p + \frac{(b-a)^{k-1-j}}{(k-1-j)!} \left(\sum_{p=k-1}^{n-1} \frac{(b-a)^{p-k+1}}{(p-k+1)!} M_p \right) \leqslant K_j, \quad 0 \leqslant j \leqslant k-2, \tag{3.6}$$

$$\sum_{p=j}^{n-1} \frac{(b-a)^{p-j}}{(p-j)!} M_p \leqslant K_j, \quad k-1 \leqslant j \leqslant q, \tag{3.7}$$

where $\sup_{a \leqslant t \leqslant b} |l^{(j)}(t)| \leqslant M_j$, $0 \leqslant j \leqslant n-1$.

Proof. We need to verify the hypothesis (ii) of Theorem 3.1 for the function ψ_1 defined by

$$\psi_1(t) = \begin{cases} l(t), & t \in [\alpha, a], \\ P_{n-1}(t), & t \in [a, b], \end{cases}$$

where $\alpha_i = l^{(i)}(a)$, $0 \leqslant i \leqslant k-1$, and $\beta_i = l^{(i)}(b)$, $k \leqslant i \leqslant n-1$.

If $t \in [\alpha, a]$, then (3.5) is the hypothesis (ii) of Theorem 3.1. If $t \in [a, b]$, then since $P_{n-1}(t)$ is a polynomial of degree $n-1$, we find

$$|P_{n-1}^{(n-1)}(t)| = |P_{n-1}^{(n-1)}(b)| = |g^{(n-1)}(b)| \leqslant \sup_{a \leqslant t \leqslant b} |g^{(n-1)}(t)| \leqslant M_{n-1}.$$

Next, from

$$P_{n-1}^{(n-2)}(t) = P_{n-1}^{(n-2)}(b) - \int_t^b P_{n-1}^{(n-1)}(s)ds = g^{(n-2)}(b) - \int_t^b P_{n-1}^{(n-1)}(s)ds$$

we get

$$|P_{n-1}^{(n-2)}(t)| \leqslant M_{n-2} + (b-t)M_{n-1}.$$

Using the same arguments repeatedly, we obtain

$$(3.8) \qquad |P_{n-1}^{(j)}(t)| \leqslant \sum_{p=j}^{n-1} \frac{(b-t)^{p-j}}{(p-j)!} M_p, \quad k \leqslant j \leqslant n-1.$$

Further, we have

$$P_{n-1}^{(k-1)}(t) = P_{n-1}^{(k-1)}(a) + \int_a^t P_{n-1}^{(k)}(s)ds = g^{(k-1)}(a) + \int_a^t P_{n-1}^{(k)}(s)ds$$

and hence from (3.8) it follows that

$$(3.9) \quad |P_{n-1}^{(k-1)}(t)| \leqslant M_{k-1} + \sum_{p=k}^{n-1} \frac{(b-a)^{p-k+1}}{(p-k+1)!} M_p = \sum_{p=k-1}^{n-1} \frac{(b-a)^{p-k+1}}{(p-k+1)!} M_p.$$

Finally, since

$$P_{n-1}^{(j)}(t) = \sum_{p=j}^{k-2} \frac{(t-a)^p}{(p)!} P_{n-1}^{(p)}(a) + \int_a^t \frac{(t-s)^{k-2-j}}{(k-2-j)!} P_{n-1}^{(k-1)}(s)ds, \quad 0 \leqslant j \leqslant k-2,$$

from (3.9) we find

$$(3.10) \quad |P_{n-1}^{(j)}(t)| = \sum_{p=j}^{k-2} \frac{(t-a)^p}{(p)!} M_p + \frac{(t-a)^{k-1-j}}{(k-1-j)!} \left(\sum_{p=k-1}^{n-1} \frac{(b-a)^{p-k+1}}{(p-k+1)!} M_p \right).$$

Inequalities (3.8)–(3.10) together with (3.6), (3.7) imply that $|P_{n-1}^{(j)}(t)| \leqslant K_j$, $0 \leqslant j \leqslant q$. Thus, $|\psi_1^{(j)}(t)| \leqslant K_j$, $0 \leqslant j \leqslant q$, for all $t \in [\alpha, b]$, wherever $\psi_1^{(j)}(t)$ exists.

COROLLARY 3.3. *Suppose that*

$$(3.11) \qquad |f(t, \langle x \rangle)| \leqslant L + \sum_{i=0}^{q} \sum_{j=1}^{p(i)} L_{i,j} |x_{i,j}|^{\alpha(i,j)}$$

for all $(t, \langle x \rangle) \in [a, b] \times \mathbf{R}^N$, where $0 \leqslant \alpha(i, j) < 1$ and L, $L_{i,j}$ are nonnegative constants. Then the boundary value problem (1.1)–(1.3) *has a solution in $\mathscr{B}$.*

Proof. (3.11) implies that on $[a, b] \times D_0$

$$|f(t, \langle x \rangle)| \leqslant L + \sum_{i=0}^{q} \sum_{j=1}^{p(i)} L_{i,j} (2K_i)^{\alpha(i,j)} = Q_1, \quad \text{say}.$$

Next it is easy to verify that $K_i/Q_1 \to \infty$, $0 \leqslant i \leqslant q$, as $K_i \to \infty$. Thus, Theorem 3.1 is applicable by choosing K_i, $0 \leqslant i \leqslant q$, so large that the hypotheses (ii) and (iii) are satisfied.

THEOREM 3.4. *Suppose that*

$$(3.12) \qquad |f(t, \langle x \rangle)| \leqslant L + \sum_{i=0}^{q} \sum_{j=1}^{p(i)} L_{i,j} |x_{i,j}|$$

for all $(t, \langle x\rangle)\in[a, b]\times D_1$, *where*

$$D_1 = \{\langle x\rangle: |x_{i,1}|, \ldots, |x_{i,p(i)}| \leqslant \sup_{a\leqslant t\leqslant b} |\psi^{(i)}(t)| + (L+c)(1-\lambda)^{-1} C_{n,i}(b-a)^{n-i},$$

$$0 \leqslant i \leqslant q, \text{ wherever } \psi^{(i)}(t) \text{ exists}\},$$

(3.13)
$$\lambda = \sum_{i=0}^{q} \sum_{j=1}^{p(i)} L_{i,j} C_{n,i}(b-a)^{n-i} < 1,$$

(3.14)
$$c = \sum_{i=0}^{q} \sum_{j=1}^{p(i)} L_{i,j} \sup_{a\leqslant t\leqslant b} |\psi^{(i)}(t)|, \quad \text{wherever } \psi^{(i)}(t) \text{ exists.}$$

Then the boundary value problem (1.1)–(1.3) *has a solution in* $\mathscr{B}$.

Proof. Consider the closed convex subset $\mathscr{B}_2$ of $(\mathscr{B}, \|\cdot\|)$ defined by

$$\mathscr{B}_2 = \{x\in\mathscr{B}: |x^{(i)}(t)| \leqslant \sup_{a\leqslant t\leqslant b} |\psi^{(i)}(t)| + (L+c)(1-\lambda)^{-1} C_{n,i}(b-a)^{n-i},$$

$$0 \leqslant i \leqslant q, \text{ wherever } \psi^{(i)}(t) \text{ exists}\}.$$

As in Theorem 3.1 it suffices to show that the operator T defined in (3.2) maps $\mathscr{B}_2$ into itself. For this, let $x\in\mathscr{B}_2$. If $t\in[\alpha, a]$, then $|(Tx)^{(i)}(t)| = |\psi^{(i)}(t)|$, $0\leqslant i\leqslant q$, and hence $T\mathscr{B}_2 \subseteq \mathscr{B}_2$ is obvious. If $t\in[a, b]$, then since $w_{i,j}(t)\in[\alpha, b]$, $1\leqslant j\leqslant p(i)$, $0\leqslant i\leqslant q$, it is obvious that $x\circ w(t)\in D_1$, and hence from Lemma 2.1 it follows that

$$|(Tx)^{(m)}(t)| \leqslant \sup_{a\leqslant t\leqslant b} |\psi^{(m)}(t)| + \sup_{a\leqslant t\leqslant b} \int_a^b |g^{(m)}(t, s)| \Big[L + \sum_{i=0}^{q} \sum_{j=1}^{p(i)} L_{i,j} |x^{(i)}(w_{i,j}(s))|\Big]\, ds$$

$$\leqslant \sup_{a\leqslant t\leqslant b} |\psi^{(m)}(t)| + C_{n,m}(b-a)^{n-m}\Big[L + \sum_{i=0}^{q} \sum_{j=1}^{p(i)} L_{i,j}\Big\{ \sup_{a\leqslant t\leqslant b} |\psi^{(i)}(t)| + (L+c)(1-\lambda)^{-1} C_{n,i}(b-a)^{n-i}\Big\}\Big]$$

$$= \sup_{a\leqslant t\leqslant b} |\psi^{(m)}(t)| + C_{n,m}(b-a)^{n-m}[(L+c) + (L+c)(1-\lambda)^{-1}\lambda]$$

$$= \sup_{a\leqslant t\leqslant b} |\psi^{(m)}(t)| + (L+c)(1-\lambda)^{-1} C_{n,m}(b-a)^{n-m}, \quad 0\leqslant m\leqslant q,$$

$$\text{wherever } \psi^{(m)}(t) \text{ exists.}$$

Thus, if $t\in[a, b]$ then $T\mathscr{B}_2 \subseteq \mathscr{B}_2$. Hence, for all $t\in[\alpha, b]$ we find that $T\mathscr{B}_2 \subseteq \mathscr{B}_2$.

THEOREM 3.5. *Suppose that*

(i) $x_1\in\mathscr{B}$ *is a solution of the boundary value problem* (1.1)–(1.3) *different from* ψ, *so that*

$$\|x_1-\psi\| = \max_{0\leqslant i\leqslant q}\left\{\sup_{a\leqslant t\leqslant b}\frac{(b-a)^i}{C_{n,i}}|x_1^{(i)}(t)-\psi^{(i)}(t)|,\right.$$

$$\left.\text{wherever } x_1^{(i)}(t) \text{ and } \psi^{(i)}(t) \text{ exist}\right\}\neq 0,$$

(ii) *the function f is such that*

$$|f(t,\langle x\rangle)|\leqslant\sum_{i=0}^{q}\sum_{j=1}^{p(i)}L_{i,j}|x_{i,j}-\psi^{(i)}(w_{i,j}(t))| \tag{3.15}$$

for all $(t,\langle x\rangle)\in[a,b]\times D_2$, where

$$D_2=\{\langle x\rangle:\ |x_{i,j}-\psi^{(i)}(w_{i,j}(t))|\leqslant C_{n,i}(b-a)^{-i}\|x_1-\psi\|,$$
$$1\leqslant j\leqslant p(i),\ 0\leqslant i\leqslant q\}.$$

Then it is necessary that $\lambda\geqslant 1$.

Proof. Since $x_1(t)$ is a solution of the boundary value problem (1.1)–(1.3), and $w_{i,j}(t)\in[\alpha,b]$ for all $t\in[a,b]$, it follows that

$$|x_1^{(i)}(w_{i,j}(t))-\psi^{(i)}(w_{i,j}(t))|\leqslant\sup_{a\leqslant t\leqslant b}\frac{(b-a)^i}{C_{n,i}}|x_1^{(i)}(t)-\psi^{(i)}(t)|\,C_{n,i}(b-a)^{-i} \tag{3.16}$$
$$\leqslant\|x_1-\psi\|\,C_{n,i}(b-a)^{-i},$$
$$1\leqslant j\leqslant p(i),\ 0\leqslant i\leqslant q,$$

and hence $(t,x_1\mathrm{o}w(t))\in[a,b]\times D_2$. Thus, on using the hypothesis (ii) in (3.1), we find

$$|x_1^{(m)}(t)-\psi^{(m)}(t)|$$
$$\leqslant\sup_{a\leqslant t\leqslant b}\int_a^b|g^{(m)}(t,s)|\sum_{i=0}^{q}\sum_{j=1}^{p(i)}L_{i,j}|x_1^{(i)}(w_{i,j}(s))-\psi^{(i)}(w_{i,j}(s))|\,ds,\quad 0\leqslant m\leqslant q.$$

Hence, from Lemma 2.1 and (3.16), we get

$$|x_1^{(m)}(t)-\psi^{(m)}(t)|\leqslant C_{n,m}(b-a)^{n-m}\sum_{i=0}^{q}\sum_{j=1}^{p(i)}L_{i,j}C_{n,i}(b-a)^{-i}\|x_1-\psi\|,$$
$$0\leqslant m\leqslant q,$$

which gives

$$\frac{(b-a)^m}{C_{n,m}}|x_1^{(m)}(t)-\psi^{(m)}(t)|\leqslant\lambda\|x_1-\psi\|,\quad 0\leqslant m\leqslant q,$$

and finally $\|x_1-\psi\|\leqslant\lambda\|x_1-\psi\|$, which implies that $\lambda\geqslant 1$.

Remark 3.1. If (3.15) is satisfied, then obviously $\psi \in \mathscr{B}$ is a solution of the boundary value problem (1.1)–(1.3). Hence, if $D_2 = \boldsymbol{R}^N$ and $\lambda < 1$, then $\psi(t)$ is the unique solution of the boundary value problem (1.1)–(1.3) in $\mathscr{B}$.

4. Picard method. Let $\bar{A}_i$, $0 \leqslant i \leqslant k-1$, and $\bar{B}_i$, $k \leqslant i \leqslant n-1$, be given fixed numbers. For a given $\bar{x} \in \mathscr{B}$, we define a new function $\psi_2 \in \mathscr{B}$ as follows:

if $\alpha < a$ and $q \geqslant k-1$, then

$$\psi_2(t) = \begin{cases} \bar{x}(t), & t \in [\alpha, a], \\ P_{n-1}(t), & t \in [a, b], \text{ where } \alpha_i = \bar{x}^{(i)}(a),\ 0 \leqslant i \leqslant k-1, \text{ and} \end{cases}$$
$$\beta_i = \bar{B}_i,\ k \leqslant i \leqslant n-1;$$

if $\alpha < a$ and $q < k-1$, then

$$\psi_2(t) = \begin{cases} \bar{x}(t), & t \in [\alpha, a], \\ P_{n-1}(t), & t \in [a, b], \text{ where } \alpha_i = \bar{x}^{(i)}(a),\ 0 \leqslant i \leqslant q, \end{cases}$$
$$\alpha_i = \bar{A}_i,\ q+1 \leqslant i \leqslant k-1, \text{ and}$$
$$\beta_i = \bar{B}_i,\ k \leqslant i \leqslant n-1;$$

if $\alpha = a$, then

$$\psi_2(t) = P_{n-1}(t), \quad t \in [a, b], \text{ where } \alpha_i = \bar{A}_i,\ 0 \leqslant i \leqslant k-1, \text{ and}$$
$$\beta_i = \bar{B}_i,\ k \leqslant i \leqslant n-1.$$

A function $\bar{x} \in \mathscr{B}$ is called an *approximate solution* of (3.1) if there exist nonnegative constants ε and δ such that wherever $\psi^{(i)}(t)$, $\psi_2^{(i)}(t)$ and $\bar{x}^{(i)}(t)$ are defined,

$$\sup_{a \leqslant t \leqslant b} |\psi_2^{(i)}(t) - \psi^{(i)}(t)| \leqslant \varepsilon C_{n,i}(b-a)^{n-i}, \quad 0 \leqslant i \leqslant q, \tag{4.1}$$

$$\sup_{a \leqslant t \leqslant b} \left|\bar{x}^{(i)}(t) - \psi_2^{(i)}(t) - \theta(t)\int_a^b g^{(i)}(t, s) f(s, \bar{x} \circ w(s))ds\right| \leqslant \delta C_{n,i}(b-a)^{n-i}, \quad 0 \leqslant i \leqslant q. \tag{4.2}$$

Inequalities (4.2) imply that there exist functions $\eta_i(t)$, $0 \leqslant i \leqslant q$, defined on $[\alpha, b]$ such that

$$\bar{x}^{(i)}(t) = \psi_2^{(i)}(t) + \theta(t)\int_a^b g^{(i)}(t, s) f(s, \bar{x} \circ w(s))ds + \eta_i(t), \quad 0 \leqslant i \leqslant q, \tag{4.3}$$

and

$$\sup_{a \leqslant t \leqslant b} |\eta_i(t)| \leqslant \delta C_{n,i}(b-a)^{n-i}.$$

The function f is said to be of *Lipschitz class* if for all $(t, \langle x \rangle)$, $(t, \langle y \rangle) \in [a, b] \times D_3$, $D_3 \subseteq \boldsymbol{R}^N$, (4.4) $$|f(t, \langle x\rangle)-f(t, \langle y\rangle)| \leqslant \sum_{i=0}^{q} \sum_{j=1}^{p(i)} L_{i,j}|x_{i,j}-y_{i,j}|.$$

In what follows we shall consider the following norm in the space $\mathscr{B}$:

$$\|x\| = \max_{0 \leqslant i \leqslant q} \left\{\frac{C_{n,0}(b-a)^i}{C_{n,i}} \sup_{a \leqslant t \leqslant b} |x^{(i)}(t)| \text{ wherever } x^{(i)}(t) \text{ exists}\right\}.$$

THEOREM 4.1. *Suppose that* (3.1) *has an approximate solution* $\bar{x} \in \mathscr{B}$, *and*

(i) f *is of Lipschitz class on* $[a, b] \times D_4$, *where*

$$D_4 = \left\{\langle x\rangle: |x_{i,j}-\bar{x}^{(i)}(w_{i,j}(t))| \leqslant \mu \frac{C_{n,i}}{C_{n,0}(b-a)^i}, \ 1 \leqslant j \leqslant p(i), \ 0 \leqslant i \leqslant q\right\},$$

(ii) $\lambda < 1$ *and*

(4.5) $$\mu_0 = (1-\lambda)^{-1}(\varepsilon+\delta)C_{n,0}(b-a)^n \leqslant \mu.$$

Then

(1) *there exists a solution* $x^*(t)$ *of* (1.1)–(1.3) *in* $\bar{S}(\bar{x}, \mu_0)$,

(2) $x^*(t)$ *is the unique solution of* (1.1)–(1.3) *in* $\bar{S}(\bar{x}, \mu)$,

(3) *the Picard sequence* $\{x_m(t)\}$ *defined by*

(4.6) $$x_{m+1}(t) = \psi(t)+\theta(t)\int_a^b g(t, s)f(s, x_m \circ w(s))ds, \quad m = 0, 1, \ldots,$$
$$x_0(t) = \bar{x}(t),$$

converges to $x^*(t)$ *with*

$$\|x^*-x_m\| \leqslant \lambda^m \mu_0, \quad \|x^*-x_m\| \leqslant \lambda(1-\lambda)^{-1}\|x_m-x_{m-1}\|,$$

(4) *for any* $x_0(t) = x(t)$, *where* $x \in \bar{S}(\bar{x}, \mu_0)$, *the iterative process converges to* $x^*(t)$.

Proof. Define an operator T: $\bar{S}(\bar{x}, \mu) \to \mathscr{B}$ as in (3.2). If $x \in \bar{S}(\bar{x}, \mu)$, i.e., $\|x-\bar{x}\| \leqslant \mu$, then whenever $x^{(i)}(t)$ and $\bar{x}^{(i)}(t)$ exist,

$$\sup_{a \leqslant t \leqslant b} |x^{(i)}(t)-\bar{x}^{(i)}(t)| \leqslant \mu \frac{C_{n,i}}{C_{n,0}(b-a)^i}, \quad 0 \leqslant i \leqslant q,$$

and hence

$$\sup_{a \leqslant t \leqslant b} |x^{(i)}(w_{i,j}(t))-\bar{x}^{(i)}(w_{i,j}(t))| \leqslant \mu \frac{C_{n,i}}{C_{n,0}(b-a)^i}, \quad 0 \leqslant i \leqslant q.$$

Thus, if $t \in [a, b]$, then $x \circ w(t) \in D_4$. Now let $x, y \in \bar{S}(\bar{x}, \mu)$. If $t \in [\alpha, a]$, then from (3.2) we have $(Tx)^{(i)}(t)-(Ty)^{(i)}(t) = 0$, $0 \leqslant i \leqslant q$, which implies that

(4.7) $$\sup_{a \leqslant t \leqslant b} \frac{C_{n,0}(b-a)^i}{C_{n,i}}|(Tx)^{(i)}(t)-(Ty)^{(i)}(t)| \leqslant \lambda\|x-y\|.$$

If $t\in[a, b]$, then from (3.2) and the Lipschitz condition (4.4) we find

$$\begin{aligned}|(Tx)^{(i)}(t)-(Ty)^{(i)}(t)| &\leqslant \int_a^b |g^{(i)}(t, s)| \sum_{i=0}^{q}\sum_{j=1}^{p(i)} L_{i,j}\left|x^{(i)}(w_{i,j}(s))-y^{(i)}(w_{i,j}(s))\right| ds \\ &\leqslant C_{n,i}(b-a)^{n-i}\sum_{i=0}^{q}\sum_{j=1}^{p(i)} L_{i,j}\frac{C_{n,i}}{C_{n,0}(b-a)^i}\|x-y\| \\ &= \frac{C_{n,i}}{C_{n,0}(b-a)^i}\lambda\|x-y\|\end{aligned}$$

and hence

$$\sup_{a\leqslant t\leqslant b}\frac{C_{n,0}(b-a)^i}{C_{n,i}}|(Tx)^{(i)}(t)-(Ty)^{(i)}(t)| \leqslant \lambda\|x-y\|. \tag{4.8}$$

Combining (4.7) and (4.8), we get $\|Tx-Ty\|\leqslant\lambda\|x-y\|$.

Next, for all $t\in[\alpha, b]$ equations (3.2) and (4.3) give

$$(T\bar{x})^{(i)}(t)-\bar{x}^{(i)}(t) = \psi^{(i)}(t)-\psi_2^{(i)}(t)-\eta_i(t), \quad 0\leqslant i\leqslant q, \tag{4.9}$$

wherever the derivatives exist.

Using (4.1) and (4.2) in (4.9), we obtain

$$\sup_{a\leqslant t\leqslant b}|(T\bar{x})^{(i)}(t)-\bar{x}^{(i)}(t)| \leqslant (\varepsilon+\delta)C_{n,i}(b-a)^{n-i},$$

which gives

$$\sup_{a\leqslant t\leqslant b}\frac{C_{n,0}(b-a)^i}{C_{n,i}}|(T\bar{x})^{(i)}(t)-\bar{x}^{(i)}(t)| \leqslant (\varepsilon+\delta)C_{n,0}(b-a)^n$$

and hence from (4.5) it follows that

$$(1-\lambda)^{-1}\|T\bar{x}-\bar{x}\| \leqslant (1-\lambda)^{-1}(\varepsilon+\delta)C_{n,0}(b-a)^n = \mu_0 \leqslant \mu.$$

Thus the assumptions of Lemma 2.3 are satisfied and the conclusions (1)–(4) follow.

5. Approximate Picard method. Theorem 4.1 has an important feature of being constructive: moreover, a priori as well as a posteriori bounds on the difference between the iterates and the solution are available. However, in practical evaluation of the sequence $\{x_m(t)\}$ generated by (4.6) only an approximate sequence, say $\{y_m(t)\}$, is computed. To find $y_{m+1}(t)$, the function f is approximated by f_m. Therefore, the computed sequence $\{y_m(t)\}$ satisfies the recurrence relation

$$\begin{aligned}y_{m+1}(t) &= \psi(t)+\theta(t)\int_a^b g(t, s)f_m(s, y_m\mathrm{o}w(s))ds, \quad m = 0, 1, \ldots, \\ y_0(t) &= x_0(t) = \bar{x}(t).\end{aligned} \tag{5.1}$$

For $y_m^{(i)}(t)$, $\alpha \leqslant t \leqslant b$, $0 \leqslant i \leqslant q$ (wherever exist), obtained from (5.1) we shall assume that f_m satisfies

$$\sup_{\alpha \leqslant t \leqslant b} \left| f_m(t, y_m \circ w(t)) - f(t, y_m \circ w(t)) \right| \leqslant \Delta \sup_{\alpha \leqslant t \leqslant b} \left| f(t, y_m \circ w(t)) \right|, \quad m = 0, 1, \ldots, \tag{5.2}$$

where Δ is a nonnegative constant.

(5.2) corresponds to the relative error in approximating the function f by f_m for the $(m+1)$th iteration.

THEOREM 5.1. *Suppose that* (3.1) *has an approximate solution* $\bar{x} \in \mathscr{B}$, *and*

(i) *inequality* (5.2) *is satisfied,*

(ii) *hypothesis* (i) *of Theorem* 4.1 *holds,*

(iii) $\lambda_1 = (1+\Delta)\lambda < 1$,

(iv) $\mu_1 = (1-\lambda_1)^{-1}(\varepsilon + \delta + \Delta F) C_{n,0}(b-a)^n \leqslant \mu$,

where $F = \sup_{\alpha \leqslant t \leqslant b} \left| f(t, \bar{x} \circ w(t)) \right|$.

Then

(1) *all the conclusions* (1)–(4) *of Theorem* 4.1 *hold,*

(2) *the sequence* $\{y_m(t)\}$ *constructed from* (5.1) *remains in* $\bar{S}(\bar{x}, \mu_1)$,

(3) *the sequence* $\{y_m(t)\}$ *converges to the solution* $x^*(t)$ *of* (1.1)–(1.3) *if and only if* $\lim_{m\to\infty} a_m = 0$, *where*

$$a_m = \left\| y_{m+1}(t) - \psi(t) - \theta(t) \int_a^b g(t, s) f(s, y_m \circ w(s)) ds \right\|,$$

(4) *the following error estimate holds:*

$$\|x^* - y_{m+1}\| \leqslant (1-\lambda)^{-1} \Big[\lambda \|y_{m+1} - y_m\| + \Delta C_{n,0}(b-a)^n \sup_{\alpha \leqslant t \leqslant b} \left| f(t, y_m \circ w(t)) \right| \Big]. \tag{5.3}$$

Proof. Since $\lambda_1 < 1$ implies that $\lambda < 1$ and obviously $\mu_0 \leqslant \mu_1$, the hypotheses of Theorem 4.1 are satisfied and the conclusion (1) follows.

To prove (2) obviously $\bar{x} = y_0 \in \bar{S}(\bar{x}, \mu_1)$, and for $\alpha \leqslant t \leqslant a$ equations (5.1) and (4.3) give

$$y_{m+1}^{(i)}(t) - \bar{x}^{(i)}(t) = \psi^{(i)}(t) - \psi_2^{(i)}(t) - \eta_i(t), \quad 0 \leqslant i \leqslant q, \ m = 0, 1,$$

and hence from (4.1) and (4.2) we get

$$|y_{m+1}^{(i)}(t) - \bar{x}^{(i)}(t)| \leqslant (\varepsilon + \delta) C_{n,i}(b-a)^{n-i},$$

which implies that

$$\sup_{\alpha \leqslant t \leqslant a} \frac{C_{n,0}(b-a)^i}{C_{n,i}} |y_{m+1}^{(i)}(t) - \bar{x}^{(i)}(t)| \leqslant (\varepsilon + \delta) C_{n,0}(b-a)^n. \tag{5.4}$$

Now assume that $a \leqslant t \leqslant b$; then from (5.1) and (4.3) we have

$$\begin{aligned}
&|y_1^{(i)}(t)-\bar{x}^{(i)}(t)| \\
&\quad \leqslant |\psi^{(i)}(t)-\psi_2^{(i)}(t)|+\int_a^b |g^{(i)}(t,s)|\,|f_0(s, y_0 \circ w(s))-f(s, \bar{x}\circ w(s))|\,ds+|\eta_i(t)| \\
&\quad \leqslant (\varepsilon+\delta)C_{n,i}(b-a)^{n-i}+C_{n,i}(b-a)^{n-i}\Delta F \\
&\quad = (\varepsilon+\delta+\Delta F)C_{n,i}(b-a)^{n-i},
\end{aligned}$$

which implies that

$$\sup_{a\leqslant t\leqslant b}\frac{C_{n,0}(b-a)^i}{C_{n,i}}|y_1^{(i)}(t)-\bar{x}^{(i)}(t)| \leqslant (\varepsilon+\delta+\Delta F)C_{n,0}(b-a)^n. \tag{5.5}$$

Combining (5.4) and (5.5), we obtain $\|y_1-\bar{x}\| \leqslant \mu_1$ and hence $y_1 \in \bar{S}(\bar{x}, \mu_1)$.
Next we assume that $y_m \in \bar{S}(\bar{x}, \mu_1)$ and show that $y_{m+1} \in \bar{S}(\bar{x}, \mu_1)$. For this, from (5.1) and (4.3) we have

$$\begin{aligned}
&|y_{m+1}^{(i)}(t)-\bar{x}^{(i)}(t)| \\
&\quad \leqslant |\psi^{(i)}(t)-\psi_2^{(i)}(t)|+\int_a^b |g^{(i)}(t,s)|\,|f_m(s, y_m \circ w(s))-f(s, \bar{x}\circ w(s))|\,ds+|\eta_i(t)| \\
&\quad \leqslant (\varepsilon+\delta)C_{n,i}(b-a)^{n-i}+C_{n,i}(b-a)^{n-i}\sup_{a\leqslant t\leqslant b}\big[|f_m(t, y_m\circ w(t))-f(t, y_m\circ w(t))| \\
&\qquad\qquad +|f(t, y_m\circ w(t))-f(t, \bar{x}\circ w(t))|\big] \\
&\quad \leqslant (\varepsilon+\delta)C_{n,i}(b-a)^{n-i}+C_{n,i}(b-a)^{n-i} \\
&\qquad \times \sup_{a\leqslant t\leqslant b}\big[(1+\Delta)|f(t, y_m\circ w(t))-f(t, \bar{x}\circ w(t))|+\Delta|f(t, \bar{x}\circ w(t))|\big] \\
&\quad \leqslant (\varepsilon+\delta+\Delta F)C_{n,i}(b-a)^{n-i}+C_{n,i}(b-a)^{n-i}(1+\Delta) \\
&\qquad \times \sup_{a\leqslant t\leqslant b}\sum_{i=0}^{q}\sum_{j=1}^{p(i)} L_{i,j}\frac{C_{n,i}}{C_{n,0}(b-a)^i}\|y_m-\bar{x}\|
\end{aligned}$$

and hence

$$\begin{aligned}
\sup_{a\leqslant t\leqslant b}\frac{C_{n,0}(b-a)^i}{C_{n,i}}|y_{m+1}^{(i)}(t)-\bar{x}^{(i)}(t)| &\leqslant (\varepsilon+\delta+\Delta F)C_{n,0}(b-a)^n+(1+\Delta)\lambda\|y_m-\bar{x}\| \\
&\leqslant (1-\lambda_1)\mu_1+\lambda_1\mu_1 = \mu_1.
\end{aligned} \tag{5.6}$$

Combining (5.4) and (5.6), we find $\|y_{m+1}-\bar{x}\| \leqslant \mu_1$ and hence $y_{m+1} \in \bar{S}(\bar{x}, \mu_1)$. This completes the proof of (2).
Next, from the definition of $x_{m+1}(t)$ and $y_{m+1}(t)$ for $\alpha \leqslant t \leqslant a$, we have

$$x_{m+1}^{(i)}(t)-y_{m+1}^{(i)}(t) = 0, \quad 0 \leqslant i \leqslant q,\ m = 0, 1,$$

and hence

$$\sup_{\alpha \leqslant t \leqslant a} \frac{C_{n,0}(b-a)^i}{C_{n,i}} |x_{m+1}^{(i)}(t) - y_{m+1}^{(i)}(t)| = 0. \tag{5.7}$$

Further, if $a \leqslant t \leqslant b$, then

$$x_{m+1}(t) - y_{m+1}(t) = \psi(t) + \int_a^b g(t, s) f(s, y_m \circ w(s)) ds - y_{m+1}(t)$$
$$+ \int_a^b g(t, s)[f(s, x_m \circ w(s)) - f(s, y_m \circ w(s))] ds$$

and as earlier, we find

$$\sup_{a \leqslant t \leqslant b} \frac{C_{n,0}(b-a)^i}{C_{n,i}} |x_{m+1}^{(i)}(t) - y_{m+1}^{(i)}(t)| \leqslant a_m + \lambda \|x_m - y_m\|. \tag{5.8}$$

Combining (5.7) and (5.8), we obtain

$$\|x_{m+1} - y_{m+1}\| \leqslant a_m + \lambda \|x_m - y_m\|. \tag{5.9}$$

Since $x_0(t) = y_0(t)$, $\alpha \leqslant t \leqslant b$, inequality (5.9) gives

$$\|x_{m+1} - y_{m+1}\| \leqslant \sum_{i=0}^{m} \lambda^{m-i} a_i. \tag{5.10}$$

Using (5.10) in the triangle inequality, we get

$$\|x^* - y_{m+1}\| \leqslant \sum_{i=0}^{m} \lambda^{m-i} a_i + \|x_{m+1} - x^*\|. \tag{5.11}$$

Theorem 4.1 ensures that $\lim_{m\to\infty} \|x_{m+1} - x^*\| = 0$. That the condition $\lim_{m\to\infty} a_m = 0$ is necessary and sufficient for the convergence of the sequence $\{y_m(t)\}$ now follows from Toeplitz' lemma: *For any* $0 \leqslant \alpha < 1$, *let* $s_m = \sum_{i=0}^{m} \alpha^{m-i} d_i$, $m = 0, 1, \ldots$ *Then* $\lim_{m\to\infty} s_m = 0$ *if and only if* $\lim_{m\to\infty} d_m = 0$".

Finally, we shall prove (5.3). For this, if $\alpha \leqslant t \leqslant a$, then obviously $x^{*(i)}(t) - y_{m+1}^{(i)}(t) = 0$ and hence

$$\sup_{\alpha \leqslant t \leqslant a} \frac{C_{n,0}(b-a)^i}{C_{n,i}} |x^{*(i)}(t) - y_{m+1}^{(i)}(t)| = 0. \tag{5.12}$$

Further, if $a \leqslant t \leqslant b$, then we have

$$x^*(t) - y_{m+1}(t)$$
$$= \int_a^b g(t, s)[f(s, x^* \circ w(s)) - f(s, y_m \circ w(s)) + f(s, y_m \circ w(s)) - f_m(s, y_m \circ w(s))] ds$$

and as earlier, we find

$$(5.13)\qquad \sup_{a\leqslant t\leqslant b} \frac{C_{n,0}(b-a)^i}{C_{n,i}} |x^{*(i)}(t)-y^{(i)}_{m+1}(t)| \leqslant \lambda\|x^*-y_m\|+\Delta C_{n,0}(b-a)^n \sup_{a\leqslant t\leqslant b} |f(t,\, y_m\circ w(t))|.$$

Combining (5.12) and (5.13), we get

$$(5.14)\qquad \|x^*-y_{m+1}\| \leqslant \lambda\|x^*-y_m\|+\Delta C_{n,0}(b-a)^n \sup_{a\leqslant t\leqslant b} |f(t,\, y_m\circ w(t))|.$$

From (5.14) inequality (5.3) follows immediately.

Remark 5.1. If $\Delta < 1$, then from (5.2) it is easy to obtain

$$\sup_{a\leqslant t\leqslant b} |f(t,\, y_m\circ w(t))| \leqslant (1-\lambda)^{-1} \sup_{a\leqslant t\leqslant b} |f(t,\, y_m\circ w(t))|.$$

Thus, in this case (5.3) can be replaced by a more practical error estimate

$$\|x^*-y_m\| \leqslant (1-\lambda)^{-1}[\lambda\|y_{m+1}-y_m\|+\Delta(1-\Delta)^{-1}C_{n,0}(b-a)^n \times \sup_{a\leqslant t\leqslant b} |f_m(t,\, y_m\circ w(t))|].$$

In our next result we shall assume that for $y^{(i)}_m(t)$, $\alpha\leqslant t\leqslant b$, $0\leqslant i\leqslant q$ (wherever exist), obtained from (5.1) the function f_m satisfies

$$(5.15)\qquad \sup_{a\leqslant t\leqslant b} |f_m(t,\, y_m\circ w(t))-f(t,\, y_m\circ w(t))| \leqslant V, \qquad m=0,\,1,$$

where V is a nonnegative constant.

(5.15) corresponds to an absolute error in approximating the function f by f_m for the $(m+1)$th iteration.

THEOREM 5.2. *Suppose that* (3.1) *has an approximate solution* $\bar{x}\in\mathscr{B}$, *and*

(i) *inequality* (5.15) *is satisfied,*

(ii) *hypothesis* (i) *of Theorem* 4.1 *holds,*

(iii) $\lambda < 1$ *and* $\mu_2 = (1-\lambda)^{-1}(\varepsilon+\delta+V)C_{n,0}(b-a)^n \leqslant \mu$.

Then

(1) *all the conclusions* (1)–(4) *of Theorem* 4.1 *hold,*

(2) *the sequence* $\{y_m(t)\}$ *constructed from* (5.1) *remains in* $\bar{S}(\bar{x},\,\mu_2)$,

(3) *conclusion* (3) *of Theorem* 5.1 *holds,*

(4) *the following error estimate holds:*

$$\|x^*-y_{m+1}\| \leqslant (1-\lambda)^{-1}[\lambda\|y_{m+1}-y_m\|+\Delta_1 C_{n,0}(b-a)^n].$$

Proof. The proof is similar to that of Theorem 5.1.

6. Monotone convergence. In this section we shall assume that $r=q$ or $\alpha=a$, so that the space $\mathscr{B}$ is $C^{(q)}[\alpha,\,b]$. In $C^{(q)}[\alpha,\,b]$ we shall introduce a partial ordering. For this, we need to consider the following four cases:

(1) n is even, k is odd,
(2) n is even, k is even,
(3) n is odd, k is odd,
(4) n is odd, k is even.

We shall consider only case (1); the other three cases are analogous. For $x, y \in C^{(q)}[\alpha, b]$ we say that $x \leqslant y$ if and only if $x^{(i)}(t) \leqslant y^{(i)}(t)$ for $0 \leqslant i \leqslant k$ and for $k < i$ (odd) $\leqslant q$, and $y^{(i)}(t) \leqslant x^{(i)}(t)$ for $k < i$ (even) $\leqslant q$ for all $t \in [\alpha, b]$. Since n is even and k is odd, Lemma 2.1 implies that $g^{(i)}(t, s) \leqslant 0$ if $0 \leqslant i \leqslant k$ or $k < i$ (odd) $\leqslant q$, and $g^{(i)}(t, s) \geqslant 0$ if $k < i$ (even) $\leqslant q$ for all $(t, s) \in [a, b] \times [a, b]$.

A function $x_0 \in C^{(q)}[\alpha, b] \cap C^{(n)}[a, b]$ is called a *lower solution* of (1.1) provided

$$x_0^{(n)}(t) \geqslant f(t, x_0 \circ w(t)), \quad t \in [a, b]. \tag{6.1}$$

Similarly, a function $y_0 \in C^{(q)}[\alpha, b] \cap C^{(n)}[a, b]$ is called an *upper solution* of (1.1) if

$$y_0^{(n)}(t) \leqslant f(t, y_0 \circ w(t)), \quad t \in [a, b]. \tag{6.2}$$

THEOREM 6.1. *For the boundary value problem* (1.1)–(1.3), *suppose that n is even, k is odd and the function $f(t, \langle x \rangle)$ is nonincreasing in $x_{i,j}$ for all $1 \leqslant j \leqslant p(i)$, $0 \leqslant i \leqslant k$, $k < i$ (odd) $\leqslant q$ and nondecreasing in $x_{i,j}$ for all $1 \leqslant j \leqslant p(i)$, $k < i$ (even) $\leqslant q$. Further, assume that there exist lower and upper solutions x_0, y_0 of* (1.1) *such that*

$$x_0 \leqslant y_0, \tag{6.3}$$

$$\psi_{x_0} \leqslant \psi \leqslant \psi_{y_0}, \tag{6.4}$$

where ψ_{x_0} is defined as ψ with φ replaced by x_0, $\alpha_i = x_0^{(i)}(a)$, $0 \leqslant i \leqslant k-1$, and $\beta_i = x_0^{(i)}(b)$, $k \leqslant i \leqslant n-1$; and similarly for ψ_{y_0}. Then the sequences $\{x_m\}$, $\{y_m\}$ are well defined by the iterative schemes

$$x_{m+1}(t) = \psi(t) + \theta(t) \int_a^b g(t, s) f(s, x_m \circ w(s)) ds, \tag{6.5}$$

$$y_{m+1}(t) = \psi(t) + \theta(t) \int_a^b g(t, s) f(s, y_m \circ w(s)) ds, \quad m = 0, 1, \tag{6.6}$$

and $\{x_m\}$ converges to an element $x \in C^{(q)}[\alpha, b]$, $\{y_m\}$ converges to an element $y \in C^{(q)}[\alpha, b]$ (the convergence being in the norm of $C^{(q)}[\alpha, b]$, which is defined as $\|x\| = \max_{0 \leqslant i \leqslant q} \{\sup_{\alpha \leqslant t \leqslant b} |x^{(i)}(t)|\}$). Further,

$$x_0 \leqslant x_1 \leqslant \ldots \leqslant x_m \leqslant \ldots \leqslant x \leqslant y \leqslant \quad \leqslant y_m \leqslant \quad \leqslant y_1 \leqslant y_0,$$

x and y are solutions of (1.1)–(1.3) *and each solution z of this problem such that $z \in [x_0, y_0]$ satisfies $x \leqslant z \leqslant y$.*

Proof. First, we shall show that the operator T defined in (3.2) is isotone. Let $x, y \in C^{(q)}[\alpha, b]$ and $x \leqslant y$. Then from the partial ordering it follows that $x^{(i)}(t) \leqslant y^{(i)}(t)$, $0 \leqslant i \leqslant k$, $k < i$ (odd) $\leqslant q$, $t \in [\alpha, b]$, and $y^{(i)}(t) \leqslant x^{(i)}(t)$, $k < i$ (even) $\leqslant q$, $t \in [\alpha, b]$. If $t \in [\alpha, a]$, then from (3.2) we have $(Tx)(t) = (Ty)(t) = \psi(t)$, and hence $T(x) \leqslant T(y)$ is obvious. If $t \in [a, b]$, then the monotonicity property of f implies that $f(t, y \circ w(t)) \leqslant f(t, x \circ w(t))$. Thus, from the sign property of the Green's function $g(t, s)$, we find

$$g^{(i)}(t, s) f(s, x \circ w(s)) \leqslant g^{(i)}(t, s) f(s, y \circ w(s))$$

$$\text{for all } 0 \leqslant i \leqslant k,\ k < i \text{ (odd)} \leqslant q,\ (t, s) \in [a, b] \times [a, b],$$

$$g^{(i)}(t, s) f(s, x \circ w(s)) \geqslant g^{(i)}(t, s) f(s, y \circ w(s))$$

$$\text{for all } k < i \text{ (even)} \leqslant q,\ (t, s) \in [a, b] \times [a, b].$$

From these inequalities it follows that $T(x) \leqslant T(y)$, and this completes the proof of T being isotone.

Next, since x_0 is a lower solution, for $0 \leqslant i \leqslant k$, $k < i$ (odd) $\leqslant q$ we have

$$x_0^{(i)}(t) = \psi_{x_0}^{(i)}(t) + \theta(t) \int_a^b g^{(i)}(t, s) x_0^{(n)}(s) ds$$

$$\leqslant \psi^{(i)}(t) + \theta(t) \int_a^b g^{(i)}(t, s) f(s, x_0 \circ w(s)) ds = T^{(i)}(x_0)(t).$$

This together with the inverse inequality for $k < i$ (even) $\leqslant q$ implies that $x_0 \leqslant T(x_0)$ in $C^{(q)}[\alpha, b]$. The inequality $T(y_0) \leqslant y_0$ can be proved analogously. Thus, the conditions (i) and (i)′ of Lemma 2.4 hold, and in conclusion the sequences $\{T^m(x_0)\}$, $\{T^m(y_0)\}$ are well defined.

Since $T^m(x_0) = T[T^{m-1}(x_0)]$, we have $T^m(x_0) = x_m$ and $T^m(y_0) = y_m$. For each i, $0 \leqslant i \leqslant k$, $k < i$ (odd) $\leqslant q$, the sequence $\{x_m^{(i)}(t)\}$ is nondecreasing and bounded from above by $y_0^{(i)}(t)$, $t \in [\alpha, b]$. Also, for each i, $k < i$ (even) $\leqslant q$, $\{x_m^{(i)}(t)\}$ is increasing and bounded from below by $y_0^{(i)}(t)$, $t \in [\alpha, b]$. A similar argument holds for $\{y_m^{(i)}(t)\}$. Hence, the sequences $\{x_m^{(i)}(t)\}$, $\{y_m^{(i)}(t)\}$, $0 \leqslant i \leqslant q$, are uniformly bounded on $[\alpha, b]$.

Now on using the above monotonicity properties, it is easy to verify that

$$y_0^{(n)}(t) \leqslant f(t, y_0 \circ w(t)) \leqslant x_{m+1}^{(n)}(t) \leqslant f(t, x_0 \circ w(t)) \leqslant x_0^{(n)}(t),$$

$t \in [a, b]$, for all m. A similar argument holds for $\{y_m^{(n)}(t)\}$. Hence, $\{x_m^{(n)}(t)\}$, $\{y_m^{(n)}(t)\}$ are also uniformly bounded on $[a, b]$. Thus, from Lemma 2.5 there exist subsequences $\{x_{m(j)}^{(i)}(t)\}$, $\{y_{m(j)}^{(i)}(t)\}$, $0 \leqslant i \leqslant q$, which converge uniformly on $[a, b]$. However, since x_m and y_m are in $C^{(q)}[\alpha, b]$ and $x_m(t) = y_m(t) = \varphi(t)$, $t \in [\alpha, b]$, this uniform convergence is in fact on $[\alpha, b]$. Further, since for each i the sequences $\{x_m^{(i)}(t)\}$, $\{y_m^{(i)}(t)\}$ are monotonic, we conclude that the whole sequences $\{x_m(t)\}$, $\{y_m(t)\}$ converge uniformly to some $x(t)$, $y(t)$ such that $x, y \in C^{(q)}[\alpha, b]$, i.e., $T^m(x_0) \uparrow x$ and $T^m(y_0) \downarrow y$.

Finally, from the continuity of T it is obvious that $T^{m+1}(x_0) = T[T^m(x_0)] \uparrow T(x)$ and $T^{m+1}(y_0) \downarrow T(y)$.

Hence, the assumptions of Lemma 2.4 are satisfied and the conclusions of Theorem 6.1 follow.

We conclude this paper with the remark that from the computational point of view monotone convergence has advantage over ordinary convergence proved in Section 4.

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