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Plane wave decompositions of monogenic functions

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Abstract. In this paper, we investigate the decomposition of monogenic functions into plane wave type monogenic functions. Special decompositions are obtained for the monogenic extensions of Riesz potentials in terms of so-called plane wave generalized powers. When taking boundary values, we obtain plane wave decompositions for generalized Hilbert–Riesz kernels and for special classes of homogeneous differential operators.

Introduction. Let C_m be the complex Clifford algebra constructed over C^m. Then in [1], [7] C_m-valued functions f in open subsets Ω of R^{m+1} were investigated, which satisfy the generalized Cauchy–Riemann system $(\partial/\partial x_0 + D)f = 0$, D being a Dirac type operator in R^m. These functions, which are called *left monogenic* in Ω, satisfy properties similar to the holomorphic functions in the plane, where the variable z is being replaced by a "parametrized hypercomplex variable" $\langle \vec{x}, \vec{t}\rangle - x_0\vec{t}$, $\vec{x}, \vec{t} \in R^m$, $x_0 \in R$ (see [5], [9]). In particular, if g is a holomorphic function, then $g(\langle \vec{x}, \vec{t}\rangle - x_0\vec{t})$ is a left monogenic function of plane wave type. Of course not every monogenic function is of this form, but it can be shown (see [9]) that left monogenic functions admit decompositions into plane wave type monogenic functions. In this paper we study these decompositions explicitly for the so-called *axial monogenic functions*, introduced in [10] and [7].

In the first section we give a general formula for plane wave decompositions, based on the Funk–Hecke theorem (see [4]). Furthermore we study the plane wave generalized powers, which may be regarded as functions of the form $(\langle \vec{x}, \vec{t}\rangle - x_0\vec{t})^{\alpha}$, $\alpha \in C$.

In the second section we study in detail the decomposition into "plane wave generalized powers" of the "axial monogenic generalized powers", introduced in [13]. These axial monogenic powers involve Cauchy-type integrals of the form

$$A_{\eta,k}^{\pm}(x) = \frac{1}{\omega_{m+1}} \int_0^{+\infty} \frac{\bar{x} \pm t}{|\bar{x} \pm t|^{m+1+2k}} t^{\eta}\, dt, \qquad \eta \in C,$$

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the boundary values of which correspond to Riesz potentials. Furthermore the residues of $\Lambda_{\eta,k}^{\pm}$ with respect to $\eta\in C$, lead to generalized Hilbert Riesz kernels and were studied in detail in [12], [13] (see also [14]).

In the final section we take the boundary values of the plane wave decompositions obtained before. This leads to the plane wave decompositions of the generalized Hilbert–Riesz kernels and of a special class of homogeneous differential operators. These results may be interpreted in terms of the inverse Radon transforms for special types of spherical harmonics (see [2], [3], [6]).

Preliminaries. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $\boldsymbol{R}^m$. Then $\boldsymbol{C}_m$ denotes the complex Clifford algebra constructed by means of this basis. The product in $\boldsymbol{C}_m$ is determined by the relations $e_i e_j + e_j e_i = -2\delta_{ij}$ and a general element of $\boldsymbol{C}_m$ is of the form $a = \sum_{A\subseteq N} a_A e_A$, $a_A\in \boldsymbol{C}$, where $N = \{1, \ldots, m\}$ and where for $A = \{\alpha_1, \ldots, \alpha_k\}$, $\alpha_1 < \ldots < \alpha_k$, $e_A = e_{\alpha_1}\ldots e_{\alpha_k}$.

An involution in $\boldsymbol{C}_m$ is given by $\bar{a} = \sum_{A\subseteq N} \bar{a}_A \bar{e}_A$, where $\bar{a}_A$ is complex conjugation and $\bar{e}_A = \bar{e}_{\alpha_k}\ldots\bar{e}_{\alpha_1}$, $\bar{e}_j = -e_j$, $j = 1, \ldots, m$.

$\boldsymbol{R}^{m+1}$ is naturally inbedded in $\boldsymbol{C}_m$ as follows: $x = (x_0, \ldots, x_m)\in\boldsymbol{R}^{m+1}$ is identified with $x = x_0+\vec{x} = x_0 e_0 + \sum_{j=1}^{m} x_j e_j$, where $e_0 = e_\emptyset = 1$. Hence $\boldsymbol{R}^m$ is the subspace of vectors $\vec{x}$ in $\boldsymbol{R}^{m+1}$ and is identified with the hyperspace $\{x\colon x_0 = 0\}$. We also use the notations $\boldsymbol{R}^{m+1}_{\pm} = \{x\in\boldsymbol{R}^{m+1}\colon x_0 \gtrless 0\}$. Notice that $\bar{x} = x_0 - \vec{x}$. The norm of $x\in\boldsymbol{R}^{m+1}$ is denoted by $|x| = |x_0+\vec{x}|$. The inner product between vectors $\vec{t}, \vec{x}\in\boldsymbol{R}^m$ is given by

$$\langle \vec{t}, \vec{x}\rangle = -\tfrac{1}{2}(\vec{t}\vec{x}+\vec{x}\vec{t}) = \sum_{j=1}^{m} t_j x_j.$$

By Δ we denote the m-dimensional Laplacian, whereas the gradient is as usual denoted by ∇.

ω_m denotes the surface area of the unit sphere S^{m-1} in $\boldsymbol{R}^m$.

Let $\Omega\subseteq\boldsymbol{R}^{m+1}$ be open; then $f\in C_1(\Omega; \boldsymbol{C}_m)$ is called *left* (*right*) *monogenic* in Ω if $(\partial/\partial x_0 + D)f = 0$ ($f(\partial/\partial x_0+D) = 0$), where $D = \sum_{j=1}^{m} e_j\,\partial/\partial x_j$ is a Dirac type operator.

A function $P_k(\vec{\xi})$, $\vec{\xi}\in S^{m-1}$, $\vec{x} = r\vec{\xi}$, is called *inner spherical monogenic of degree k* if $P_k(\vec{x}) = r^k P_k(\vec{\xi})$, is left monogenic in $\boldsymbol{R}^m$. A function $Q_k(\vec{\xi})$ is called *outer spherical monogenic of degree k* if $r^{-(k+m-1)}Q_k(\vec{\xi})$ is left monogenic in $\boldsymbol{R}^m\setminus\{0\}$. Let $\mathbb{P}_k$, $\mathbb{Q}_k$, $\mathbb{S}_k$ be the projection operators onto the spaces of inner and outer spherical monogenics and spherical harmonics respectively, then we have that for every $f\in L_2(S^{m-1})$ (see [8]),

$$\mathbb{S}_k(f) = \mathbb{P}_k(f)+\mathbb{Q}_{k-1}(f), \quad \mathbb{Q}_k(f)(\vec{t}) = -\vec{t}\,\mathbb{P}_k(\vec{t}f)(\vec{t}).$$

1. Basic plane wave decompositions. Let g be a $\boldsymbol{C}$-valued function on $\boldsymbol{R}$ and let h be a $\boldsymbol{C}_m$-valued function of S^{m-1}. Then we consider integrals of the form

$$\int_{S^{m-1}} g(\langle \vec{x}, \vec{u}\rangle) h(\vec{u}) dS_u.$$

To evaluate these integrals, we shall use the expansion of h into spherical monogenics (see [1], [8])

$$h(\vec{u}) = \sum_{k=0}^{\infty} \big(P_k(h)(\vec{u}) - \vec{u}P_k(\vec{u}h)(\vec{u})\big).$$

As $P_k(h)(\vec{u})$ and $Q_{k-1}(h)(\vec{u}) = -\vec{u}P_{k-1}(\vec{u}h)(\vec{u})$ are spherical harmonics of degree k, we obtain by using the Funk–Hecke theorem (see [4]) the following

LEMMA 1. *Let $P_k(\vec{u})$ be inner spherical monogenic of degree k. Then we have the identities*

(i) $$\int_{S^{m-1}} g(\langle \vec{x}, \vec{u}\rangle) P_k(\vec{u}) dS_u = \omega_{m-1}\Big(\int_{-1}^{1} g(\varrho t) P_{k,m}(t)(1-t^2)^{(m-3)/2} dt\Big) P_k(\vec{\xi}),$$

(ii) $$\int_{S^{m-1}} g(\langle \vec{x}, \vec{u}\rangle) \vec{u}P_k(\vec{u}) dS_u = \omega_{m-1}\Big(\int_{-1}^{1} g(\varrho t) P_{k+1,m}(t)(1-t^2)^{(m-3)/2} dt\Big) \vec{\xi}P_k(\vec{\xi}),$$

where $\vec{x} = \varrho\vec{\xi}$, $\varrho = |\vec{x}|$ and $P_{k,m}$ is the k-th Legendre polynomial in m dimensions.

Using the notation

$$I_k(g)(\varrho) = \int_{-1}^{1} g(\varrho t) P_{k,m}(t)(1-t^2)^{(m-3)/2} dt,$$

we hence obtain that

$$\int_{S^{m-1}} g(\langle \vec{x}, \vec{u}\rangle) P_k(\vec{u}) dS_u = \omega_{m-1} I_k(g)(\varrho) P_k(\vec{\xi}),$$

$$\int_{S^{m-1}} g(\langle \vec{x}, \vec{u}\rangle) \vec{u}P_k(\vec{u}) dS_u = \omega_{m-1} I_{k+1}(g)(\varrho) \vec{\xi}P_k(\vec{\xi}).$$

Next, let g_1, g_2 be $\boldsymbol{C}$-valued functions in $\Omega \subseteq \boldsymbol{C}$ satisfying the Cauchy–Riemann system

$$\frac{\partial}{\partial x} g_1 - \frac{\partial}{\partial y} g_2 = \frac{\partial}{\partial x} g_2 + \frac{\partial}{\partial y} g_1 = 0.$$

Then for every $\vec{t} \in \boldsymbol{R}^m$, the function

$$g_1(\langle \vec{x}, \vec{t}\rangle, x_0|\vec{t}|) - \frac{\vec{t}}{|\vec{t}|} - g_2(\langle \vec{x}, \vec{t}\rangle, x_0|\vec{t}|)$$

is left monogenic in $\Omega_t = \{(x_0, \vec{x}) : (\langle \vec{x}_0, \vec{t}\rangle, x_0|\vec{t}|) \in \tilde{\Omega}\}$. Hence, for suitable domains $\tilde{\Omega}$, we consider

$$\int_{S^{m-1}} (g_1(\langle \vec{x}, \vec{t}\rangle, x_0) - \vec{t} g_2(\langle \vec{x}, \vec{t}\rangle, x_0)) P_k(\vec{t})\, dS_t,$$

which is left monogenic in $\Omega = \bigcap_{\vec{t} \in S^{m-1}} \Omega_{\vec{t}}$ and which, by Lemma 1, is the monogenic extension of

$$\omega_{m-1}\big(I_k(g_1| \; \boldsymbol{R})(\varrho) - \vec{\xi} I_{k+1}(g_2| \; \boldsymbol{R})(\varrho)\big) P_k(\vec{\xi}).$$

Hence, this integral is left monogenic and of the form $\big(A(x_0, \varrho) + \vec{\xi} B(x_0, \varrho)\big) P_k(\vec{\xi})$. Functions of this type are called *axial monogenics of degree k* and satisfy the plane elliptic system (see [10])

$$\frac{\partial}{\partial x_0} A - \frac{\partial}{\partial \varrho} B = \frac{k+m-1}{\varrho} B, \qquad \frac{\partial}{\partial x_0} B + \frac{\partial}{\partial \varrho} A = \frac{k}{\varrho} A.$$

Hence we obtain plane wave decompositions of axial monogenic functions. These decompositions may also be applied to boundary values of monogenic functions (see [1], [11], [12]). Let f_1, f_2 be $\boldsymbol{C}$-valued functions in $\boldsymbol{C} \setminus \boldsymbol{R}$, satisfying the Cauchy–Riemann equations, and put

$$g_j(s) = f_j(s+i0) - f_j(s-i0), \qquad j = 1, 2.$$

Then g_1, g_2 are hyperfunctions on $\boldsymbol{R}$, so that

$$g_1(\langle \vec{x}, \vec{t}\rangle) - \frac{\vec{t}}{|\vec{t}|} g_2(\langle \vec{x}, \vec{t}\rangle)$$

is a plane wave hyperfunction in $\boldsymbol{R}^m$, which is the boundary value of the left and right monogenic function

$$f_1(\langle \vec{x}, \vec{t}\rangle, x_0|\vec{t}|) - \frac{\vec{t}}{|\vec{t}|} f_2(\langle \vec{x}, \vec{t}\rangle, x_0|\vec{t}|).$$

From Lemma 1 it hence follows that

$$\int_{S^{m-1}} (f_1(\langle \vec{x}, \vec{t}\rangle, x_0) - \vec{t} f_2(\langle \vec{x}, \vec{t}\rangle, x_0)) P_k(\vec{t})\, dS_t$$

is a left monogenic representation of the hyperfunction

$$\int_{S^{m-1}} (g_1(\langle \vec{x}, \vec{t}\rangle) - \vec{t} g_2(\langle \vec{x}, \vec{t}\rangle)) P_k(\vec{t})\, dS_t = \omega_{m-1}\big(I_k(g_1)(\varrho) - \vec{\xi} I_{k+1}(g_2)(\varrho)\big) P_k(\vec{\xi}).$$

In view of [11], this hyperfunction is well defined.

Example. Let us start from the generalized power functions

$$p^\alpha(z) = |z|^\alpha (\cos(\alpha \arg z) + i \sin(\alpha \arg z)), \qquad \alpha \in \boldsymbol{C}.$$

Then, using the above procedure, we may construct the so-called plane wave generalized powers by

$$p^\alpha(\vec{t}, x) = (\langle \vec{x}, \vec{t}\rangle^2 + x_0^2|\vec{t}|^2)^{\alpha/2}\left(\cos(\alpha \arg(\vec{t}, \vec{x})) - \frac{\vec{t}}{|\vec{t}|}\sin(\alpha \arg(\vec{t}, x))\right).$$

where

$$\arg(\vec{t}, x) = \begin{cases} \arctan\dfrac{x_0|\vec{t}|}{\langle \vec{x}, \vec{t}\rangle} + 2k\pi & \text{if } \langle \vec{x}, \vec{t}\rangle > 0, \\ \arctan\dfrac{x_0|\vec{t}|}{\langle \vec{x}, \vec{t}\rangle} + (2k+1)\pi & \text{if } \langle \vec{x}, \vec{t}\rangle < 0. \end{cases}$$

Notice that $\arg(\vec{t}, x)$, and hence $p^\alpha(\vec{t}, x)$, is extendable to a multivalued function in $\mathbf{R}^{m+1}\setminus\{x\colon x_0 = \langle \vec{x}, \vec{t}\rangle = 0\}$.

Let $\theta \in \mathbf{R}$. Then by $\arg_\theta(\vec{t}, x)$ we denote the branch of $\arg(\vec{t}, x)$, determined by the condition $\arg_\theta(\vec{t}, x) \in [\theta - \pi, \theta + \pi[$. Furthermore by $p_\theta^\alpha(\vec{t}, x)$ we denote the branch of $p^\alpha(\vec{t}, x)$, given by

$$p_\theta^\alpha(\vec{t}, x) = (\langle \vec{x}, \vec{t}\rangle^2 + x_0^2|\vec{t}|^2)^{\alpha/2}\left(\cos(\alpha \arg_\theta(\vec{t}, x)) - \frac{\vec{t}}{|\vec{t}|}\sin(\alpha \arg_\theta(\vec{t}, x))\right),$$

which for $\alpha \notin \mathbf{Z}$ is left and right monogenic in $\mathbf{R}^{m+1}\setminus\{x\colon \arg_\theta(\vec{t}, x) = \theta - \pi\}$, while for $k \in \mathbf{Z}$,

$$p^k(\vec{t}, x) = p_\theta^k(\vec{t}, x) = (\langle \vec{x}, \vec{t}\rangle - x_0\vec{t})^k.$$

Notice that, in general

$$p^\alpha(\vec{t}, \vec{x}) = \begin{cases} |\langle \vec{x}, \vec{t}\rangle|^\alpha\left(\cos 2k\pi\alpha - \dfrac{\vec{t}}{|\vec{t}|}\sin 2k\pi\alpha\right), & \langle \vec{x}, \vec{t}\rangle > 0, \\ |\langle \vec{x}, \vec{t}\rangle|^\alpha\left(\cos(2k+1)\pi\alpha - \dfrac{\vec{t}}{|\vec{t}|}\sin(2k+1)\pi\alpha\right), & \langle \vec{x}, \vec{t}\rangle < 0. \end{cases}$$

Furthermore for $\operatorname{Re}\alpha > -1$, we for instance have that

$$p_\pi^\alpha(\vec{t}, \vec{x}+0) - p_\pi^\alpha(\vec{t}, \vec{x}-0) = \chi(\langle \vec{x}, \vec{t}\rangle > 0)|\langle \vec{x}, \vec{t}\rangle|^\alpha\left(1 - \cos 2\pi\alpha + \frac{\vec{t}}{|\vec{t}|}\sin \pi\alpha\right).$$

Finally, notice that, instead of $p^\alpha(\vec{t}, x)$, we may as well write $(\langle \vec{x}, \vec{t}\rangle - \vec{t}x_0)^\alpha$. Indeed, as to the calculations $\langle \vec{x}, \vec{t}\rangle - x_0\vec{t}$ behaves just like the complex variable $z = x + iy$, where x is replaced by $\langle \vec{x}, \vec{t}\rangle$, y by $x_0|\vec{t}|$ and i by $-\vec{t}/|\vec{t}|$.

2. Plane wave decompositions of generalized powers. First we recall the definitions and main properties of the generalized powers $p_{\alpha,k,m}$ and $q_{\alpha,k,m}$ and of the Cauchy type integrals $\Lambda_{\eta,k}^{\pm}(x)$ introduced in [13].

Definition 1. Let $P_k(\vec{\xi})$ be inner spherical monogenic of degree k. Then we have that

(i) $p_{\alpha,k,m}(x_0, \varrho, \vec{\xi})\, P_k(\vec{\xi})$ is the axial monogenic of degree k determined by the condition $p_{\alpha,k,m}(0, \varrho, \vec{\xi}) = \varrho^{\alpha}$,

(ii) $q_{\alpha,k,m}(x_0, \varrho, \vec{\xi})\, P_k(\vec{\xi})$ is the axial monogenic of degree k determined by the condition $q_{\alpha,k,m}(0, \varrho, \vec{\xi}) = \varrho^{\alpha}\, \vec{\xi}$,

(iii) the Cauchy integrals $\Lambda^{\pm}_{\eta,k}(x)$ are given by

$$\Lambda^{\pm}_{\eta,k}(x) = \frac{1}{\omega_{m+1}} \int_0^{+\infty} \frac{\bar{x} \pm t}{|x \pm t|^{m+1+2k}}\, t^{\eta}\, dt.$$

Proposition 1. *For* $-1 < \operatorname{Re} \eta < m + 2k - 1$, *we have that*

$$\varrho^k \Lambda^{\pm}_{\eta,k}(x) = \sigma_{m,k}(\eta+1)\, p_{\eta-m+1-k,k,m} \mp \sigma_{m,k}(\eta)\, q_{\eta-m+1-k,k,m},$$

where

$$\sigma_{m,k}(\eta) = \frac{1}{2\omega_{m+1}} B\left(\frac{2k+m-\eta}{2}, \frac{\eta+1}{2}\right).$$

In this section we shall express these power functions as plane wave integrals of the form

$$\int_{S^{m-1}} p^{\alpha}_{\theta}(\vec{t}, x)\, P_k(\vec{t})\, dS_t \qquad \text{and} \qquad \int_{S^{m-1}} p^{\alpha}_{\theta}(\vec{t}, x)\, \vec{t} P_k(\vec{t})\, dS_t,$$

where $P_k(\vec{t})$ is inner spherical monogenic of degree k and $\theta = \pm\pi/2$.

Notice that $p^{\alpha}_{\pm\pi/2}(\vec{t}, \vec{x}) = p^{\alpha}_{\pm\pi/2}(\vec{t}, x)|_{x_0=0}$ are given by

$$p^{\alpha}_{\pm\pi/2}(\vec{t}, \vec{x}) = |\langle \vec{x}, \vec{t}\rangle|^{\alpha} \left(\cos \pi\alpha\chi(\langle \vec{x}, \vec{t}\rangle < 0) \mp \frac{\vec{t}}{|\vec{t}|} \sin \pi\alpha\chi(\langle \vec{x}, \vec{t}\rangle < 0)\right)$$

and that $p^{\alpha}_{\pm\pi/2}$ are left monogenic in respectively $\boldsymbol{R}^{m+1}_{+}$ and $\boldsymbol{R}^{m+1}_{-}$. First we introduce the following

Definition 2. Let $\operatorname{Re} \alpha > -1$ and let P_k be inner spherical monogenic of degree k. Then we put

$$\mathscr{C}^{\pm}_{\alpha}(P_k)(\vec{x}) = \int_{S^{m-1}} p^{\alpha}_{\pm\pi/2}(\vec{t}, \vec{x})\, P_k(\vec{t})\, dS_t,$$

$$\mathscr{S}^{\pm}_{\alpha}(P_k)(\vec{x}) = \int_{S^{m-1}} p^{\alpha}_{\pm\pi/2}(\vec{t}, \vec{x})\, \vec{t} P_k(\vec{t})\, dS_t.$$

From Lemma 1, we immediately obtain the following

Proposition 2. *The restrictions of* $\mathscr{C}^{\pm}_{\alpha}(P_k)$, $\mathscr{S}^{\pm}_{\alpha}(P_k)$ *to* $\boldsymbol{R}^m$ *are given by*

$$\mathscr{C}^{\pm}_{\alpha}(P_k)(\vec{x}) = \omega_{m-1}\left(C_k(\alpha)\, \varrho^{\alpha} \mp \vec{\xi} S_k(\alpha)\, \varrho^{\alpha}\right) P_k(\vec{\xi}),$$

$$\mathscr{S}^{\pm}_{\alpha}(P_k)(\vec{x}) = \omega_{m-1}\left(C_{k+1}(\alpha)\, \vec{\xi} \varrho^{\alpha} \pm S_{k-1}(\alpha)\, \varrho^{\alpha}\right) P_k(\vec{\xi}),$$

where

$$C_k(\alpha) = \int_{-1}^{1} |t|^\alpha \cos(\pi\alpha\chi(t<0))\, P_{k,m}(t)(1-t^2)^{(m-3)/2}\, dt,$$

$$S_k(\alpha) = \int_{-1}^{0} |t|^\alpha \sin \pi\alpha P_{k+1,m}(t)(1-t^2)^{(m-3)/2}\, dt.$$

As $\mathscr{C}_\alpha^\pm(P_k)$ and $\mathscr{S}_\alpha^\pm(P_k)$ are defined in $\mathbf{R}^{m+1}_\pm$, they are completely determined by their restrictions to $\mathbf{R}^m$. Hence, using the definition of the generalized powers, Proposition 2 immediately leads to

PROPOSITION 3. *We have that in* $\mathbf{R}^{m+1}_+$

$$\int_{S^{m-1}} p^\alpha_{\pi/2}(\vec{t}, x)\, P_k(\vec{t})\, dS_t = \omega_{m-1}\left(C_k(\alpha)\, p_{\alpha,k,m}(x) - S_k(\alpha)\, q_{\alpha,k,m}(x)\right) P_k(\vec{\xi}),$$

while in $\mathbf{R}^{m+1}_-$

$$\int_{S^{m-1}} p^\alpha_{-\pi/2}(\vec{t}, x)\, P_k(\vec{t})\, dS_t = \omega_{m-1}\left(C_k(\alpha)\, p_{\alpha,k,m}(x) + S_k(\alpha)\, q_{\alpha,k,m}(x)\right) P_k(\vec{\xi}).$$

We now come to the plane wave expansion of the Cauchy integrals $\Lambda^\pm_{\eta,k}(x)$.

THEOREM 1. *Let* $\alpha = \eta - m + 1 - k$. *Then we have that for* $m+k-2 < \operatorname{Re}\eta < m+k-1$

$$\Lambda^\pm_{\eta,k}(x)\, P_k(\vec{x}) = \frac{\sigma_{m,k}(\eta+1)}{\omega_{m-1}\, C_k(\alpha)} \int_{S^{m-1}} p^\alpha_{\pm\pi/2}(\vec{t}, x)\, P_k(\vec{t})\, dS_t.$$

Proof. Let $\alpha = \eta - m + 1 - k$. Then it immediately follows from Proposition 3 that

$$\frac{1}{\omega_{m-1}\, C_k(\alpha)} \int_{S^{m-1}} \left(p^\alpha_{\pi/2}(\vec{t}, \vec{x}+0) + p^\alpha_{-\pi/2}(\vec{t}, \vec{x}-0)\right) P_k(\vec{t})\, dS_t$$

$$= \frac{1}{\sigma_{m,k}(\eta+1)} \left[\Lambda^+_{\eta,k}(\vec{x}+0) + \Lambda^-_{\eta,k}(\vec{x}-0)\right] P_k(\vec{x})$$

and that

$$\frac{1}{\omega_{m-1}\, S_k(\alpha)} \int_{S^{m-1}} \left(p^\alpha_{\pi/2}(\vec{t}, \vec{x}+0) - p^\alpha_{-\pi/2}(\vec{t}, \vec{x}-0)\right) P_k(\vec{t})\, dS_t$$

$$= \frac{1}{\sigma_{m,k}(\eta)} \left[\Lambda^+_{\eta,k}(\vec{x}+0) - \Lambda^-_{\eta,k}(\vec{x}-0)\right] P_k(\vec{x}).$$

Hence, in view of [11], there exist entire monogenic functions f_α and g_α such that in $\mathbf{R}^{m+1}_\pm$

$$\frac{1}{\omega_{m-1}\, C_k(\alpha)} \int_{S^{m-1}} p^\alpha_{\pm\pi/2}(\vec{t}, x)\, P_k(\vec{t})\, dS_t - \frac{1}{\sigma_{m,k}(\eta+1)} \Lambda^\pm_{\alpha,k}(x)\, P_k(\vec{x}) = \pm f_\alpha(x),$$

and

$$\frac{1}{\omega_{m-1}S_k(\alpha)}\int_{S^{m-1}} p^{\alpha}_{\pm\pi/2}(\vec{t}, x)\,P_k(\vec{t})\,dS_t - \frac{1}{\sigma_{m,k}(\eta)}\Lambda^{\pm}_{\eta,k}(x)\,P_k(\vec{x}) = g_\alpha(x).$$

Furthermore, for $-1 < \operatorname{Re}\alpha < 0$ or $m+k-2 < \operatorname{Re}\eta < m+k-1$, we have that

$$\lim_{x_0\to\pm\infty}\int_{S^{m-1}} p^{\alpha}_{\pm\pi/2}(\vec{t}, x)\,P_k(\vec{t})\,dS_t = \lim_{x_0\to\pm\infty}\Lambda^{\pm}_{\eta,k}(x) = 0.$$

Hence, by Liouville's theorem (see [1]), $f_\alpha = g_\alpha = 0$, which proves the theorem. ∎

Notice that, at the same time, we obtained the identity

$$\sigma_{m,k}(\eta+1)\,S_k(\alpha) = \sigma_{m,k}(\eta)\,C_k(\alpha).$$

Of course, by holomorphicity of $\Lambda^{\pm}_{\eta,k}(x)\,P_k(\vec{x})$ for $\eta \in C\backslash(\{-1, -2, \ldots\} \cup \cup \{m+2k-1, m+2k, \ldots\})$ (see [13]), the above formula also extends to all these values of η.

In order to calculate $C_k(\alpha)$ and $S_k(\alpha)$, we make use the following recurrence formulae.

THEOREM 2. *The transforms $\mathscr{C}^{\pm}_{\alpha}(P_k)$, $\mathscr{S}^{\pm}_{\alpha}(P_k)$ satisfy the recurrence identities*

$$\mathscr{S}^{\pm}_{\alpha}(P_k) = \frac{D}{\alpha+1}\mathscr{C}^{\pm}_{\alpha+1}(P_k), \qquad \mathscr{C}^{\pm}_{\alpha}(P_k) = -\frac{D}{\alpha+1}\mathscr{S}^{\pm}_{\alpha+1}(P_k).$$

Proof. This theorem follows immediately from the fact that

$$Dp^{\alpha+1}_{\pm\pi/2}(\vec{t}, x) = -\frac{\partial}{\partial x_0}p^{\alpha+1}_{\pm\pi/2}(\vec{t}, x) = \vec{t}(\alpha+1)\,p^{\alpha}_{\pm\pi/2}(\vec{t}, x) = (\alpha+1)\,p^{\alpha}_{\pm\pi/2}(\vec{t}, x)\,\vec{t}. \;\blacksquare$$

COROLLARY 1. *The functions $C_k(\alpha)$ and $S_k(\alpha)$ satisfy the recurrence formulae*

$$C_{k+1}(\alpha) = \frac{\alpha+1-k}{\alpha+1}C_k(\alpha+1), \qquad S_{k-1}(\alpha) = \frac{\alpha+k+m}{\alpha+1}S_k(\alpha+1).$$

Proof. On the one hand we have that

$$\mathscr{S}^{\pm}_{\alpha}(P_k) = \omega_{m-1}\big(C_{k+1}(\alpha)\,\vec{\xi} \pm S_{k-1}(\alpha)\big)\varrho^{\alpha}P_k(\vec{\xi}),$$

while, in view of Theorem 2,

$$\mathscr{S}^{\pm}_{\alpha}(P_k) = \frac{\omega_{m-1}}{\alpha+1}D\varrho^{\alpha+1-k}\big(C_k(\alpha+1) \mp \vec{\xi}S_k(\alpha+1)\big)P_k(\vec{x})$$

$$= \frac{\omega_{m-1}}{\alpha+1}\big((\alpha+1-k)\,\vec{\xi}C_k(\alpha+1) \pm (\alpha+k+m)\,S_k(\alpha+1)\big)\varrho^{\alpha}P_k(\vec{\xi}).$$

Comparing both expressions leads to the stated recurrence formulae. ■

Hence, in order to compute $C_k(\alpha)$ and $S_k(\alpha)$ we only need to calculate $C_0(\alpha)$ and $S_0(\alpha)$. Direct computation yields

$$S_0(\alpha) = -\int_{-1}^{0} |t|^{\alpha+1} \sin \pi\alpha (1-t^2)^{(m-3)/2}\, dt = -\frac{\sin \pi\alpha}{2} B\left(\frac{\alpha}{2}+1, \frac{m-1}{2}\right)$$

and

$$C_0(\alpha) = (1+\cos \pi\alpha) \int_0^1 t^\alpha (1-t^2)^{(m-3)/2}\, dt = \frac{1+\cos \pi\alpha}{2} B\left(\frac{\alpha+1}{2}, \frac{m-1}{2}\right),$$

which leads to the final formulae

$$C_k(\alpha) = \frac{1+\cos \pi(\alpha-k)}{2^{k+1}} \frac{\Gamma(\alpha+1)\Gamma\left(\frac{\alpha-k+1}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}{\Gamma(\alpha-k+1)\Gamma\left(\frac{\alpha+k+m}{2}\right)},$$

$$S_k(\alpha) = \frac{\sin \pi(k-\alpha)}{2^{k+1}} \frac{\Gamma(\alpha+1)\Gamma\left(\frac{\alpha-k}{2}+1\right)\Gamma\left(\frac{m-1}{2}\right)}{\Gamma(\alpha-k+1)\Gamma\left(\frac{\alpha+k+m+1}{2}\right)}.$$

Notice that, as

$$\frac{1+\cos \pi(\alpha-k)}{\sin \pi(k-\alpha)} = \frac{\Gamma\left(\frac{k-\alpha}{2}\right)\Gamma\left(\frac{2+\alpha-k}{2}\right)}{\Gamma\left(\frac{1+k-\alpha}{2}\right)\Gamma\left(\frac{1-k+\alpha}{2}\right)},$$

we reobtain the identity

$$\frac{C_k(\alpha)}{S_k(\alpha)} = \frac{\Gamma\left(\frac{k-\alpha}{2}\right)\Gamma\left(\frac{\alpha+k+m+1}{2}\right)}{\Gamma\left(\frac{1+k-\alpha}{2}\right)\Gamma\left(\frac{\alpha+k+m}{2}\right)} = \frac{\sigma_{m,k}(\eta+1)}{\sigma_{m,k}(\eta)},$$

where $\eta = \alpha+k+m-1$.

By Theorem 2, it is possible to extend $\mathscr{C}_\alpha^{\pm}(P_k)$ and $\mathscr{S}_\alpha^{\pm}(P_k)$ holomorphically to at least $\boldsymbol{C}\backslash\{-1, -2, \ldots\}$. Of course, from the definition of $\mathscr{C}_\alpha^{\pm}(P_k)$ and $\mathscr{S}_\alpha^{\pm}(P_k)$ it is already clear that, at least for $x \in \boldsymbol{R}_{\pm}^{m+1}$, these functions extend holomorphically to the whole complex plane.

Indeed, for $x_0 \neq 0$, $p^\alpha_{\pm\pi/2}(\vec{t}, x)$ has no singularities for $\vec{t} \in S^{m-1}$.

Hence also $S_k(\alpha)$ and $C_k(\alpha)$ are entire functions.

We now study $\mathscr{C}_\alpha^{\pm}(P_k)$ and $\mathscr{S}_\alpha^{\pm}(P_k)$ for the integer values $\alpha = l \in \mathbf{Z}$. Notice that, immediately from the definition of $S_k(\alpha)$ and $C_k(\alpha)$, it follows that $S_k(l) = 0$ for $l \in \mathbf{N}$, while $C_k(l) = 0$ for $l \in \mathbf{N}$ and $k+l$ odd. Furthermore, for $k+l$ even and $l < k$, it follows from the orthogonality properties of $P_{k,m}(t)$ that $C_k(l) = 0$. Hence we obtain

PROPOSITION 4. *For* $l = k + 2s$,

$$\mathscr{C}_l^{\pm}(P_k) = \int_{S^{m-1}} (\langle \vec{x}, \vec{t}\rangle - x_0 \vec{t})^{k+2s} P_k(\vec{t})\, dS_t$$
$$= \omega_{m-1}\, C_k(k+2s)\, p_{k+2s,k,m}(x)\, P_k(\vec{\xi})$$

and for $l = k + 2s + 1$,

$$\mathscr{S}_l^{\pm}(P_k) = \int_{S^{m-1}} (\langle \vec{x}, \vec{t}\rangle - x_0 \vec{t})^{k+2s+1}\, \vec{t} P_k(\vec{t})\, dS_t$$
$$= \omega_{m-1}\, C_{k+1}(k+2s+1)\, q_{k+2s+1,k,m}(x)\, P_k(\vec{\xi}),$$

whereas $\mathscr{C}_l^{\pm}(P_k)$ *and* $\mathscr{S}_l^{\pm}(P_k)$ *vanish for all other values of* $l \in \mathbf{N}$.

This means that (see [13]) the sets of polynomials $\mathscr{C}_{k+2s}^{\pm}(P_k)$ and $\mathscr{S}_{k+2s+1}^{\pm}(P_k)$ form a basis for the spaces of inner spherical monogenics.

As to the negative integer points, we first prove the following

LEMMA 2. (i) *For* $k+l$ *odd and* $l \in \mathbf{N} \setminus \{0\}$, $S_k(-l) = 0$ *and* $C_k(-l) = 0$.
(ii) *For* m *odd and* $l > m+k$, $S_k(-l) = 0$.
(iii) *For* m *even and* $l \geqslant m+k$, $C_k(-l) = 0$.

Proof. (iii) follows immediately from the relation

$$C_k(-l) = \frac{1+\cos\pi(k+l)}{2^{k+1}} \frac{\Gamma(1-l)}{\Gamma(1-l-k)} \frac{\Gamma\left(\dfrac{1-l-k}{2}\right)}{\Gamma\left(\frac{1}{2}(k+m-l)\right)} \Gamma\left(\frac{m-1}{2}\right).$$

Furthermore for $k+l$ odd, $1+\cos\pi(k+z)$ has a second order zero at $z = k$, whereas the other factor may only have a pole of order at most one.

As to the function $S_k(\alpha)$, we have that

$$S_k(-l) = \pi \frac{\cos\dfrac{\pi}{2}(k+l)}{2^k} \cdot \frac{\Gamma(1-l)\,\Gamma\left(\dfrac{m-1}{2}\right)}{\Gamma(1-l-k)\,\Gamma\left(\dfrac{k+l}{2}\right)\Gamma\left(\dfrac{m+k-l+1}{2}\right)},$$

which clearly vanishes for $k+l$ odd.

For m odd and $l > m+k$, $\Gamma((m+k-l+1)/2)$ will have a first order pole, which leads to (ii). ■

In all other cases, $C_k(-l) \neq 0$ and $S_k(-l) \neq 0$. Hence for $k+l$ odd we have that $\mathscr{C}_l^{\pm}(P_k) = \mathscr{S}_{l-1}^{\pm}(P_k) = 0$. As to the other cases, the previous theory leads to

THEOREM 3. (i) *If m is even and $l = k+m+2s$, then*

$$\mathscr{C}^{\pm}_{-l}(P_k) = \mp\omega_{m-1} S_k(-l) q_{-l,k,m} P_k(\vec{\xi}),$$

$$\mathscr{S}^{\pm}_{-l-1}(P_k) = \omega_{m-1} S_{k-1}(-l-1) p_{-l-1,k,m} P_k(\vec{\xi}).$$

(ii) *If m is odd and $l = k+m+2s$, then*

$$\mathscr{S}^{\pm}_{-l}(P_k) = \omega_{m-1} C_{k+1}(-l) q_{-l,k,m} P_k(\vec{\xi}),$$

$$\mathscr{C}^{\pm}_{-l-1}(P_k) = \omega_{m-1} C_k(-l-1) p_{-l,-1,k,m} P_k(\vec{\xi}).$$

A special case of this is the following

COROLLARY 1. (i) *For m even, we have that in $\mathbf{R}^{m+1}_{\pm}$*

$$\frac{1}{\omega_{m+1}} \frac{x_0-\vec{x}}{|x_0-\vec{x}|^{m+1}} = \pm \frac{(-1)^{m/2}(m-1)!}{2(2\pi)^m} \int_{S^{m-1}} (\langle \vec{x}, \vec{t}\rangle - x_0 \vec{t})^{-m} dS_t.$$

(ii) *For m odd, we have that in $\mathbf{R}^{m+1}$*

$$\frac{1}{\omega_{m+1}} \frac{x_0-\vec{x}}{|x_0-\vec{x}|^{m+1}} = \frac{(-1)^{(m+1)/2}(m-1)!}{2(2\pi)^m} \int_{S^{m-1}} (\langle \vec{x}, \vec{t}\rangle - x_0 \vec{t})^{-m} t\, dS_t.$$

The above expressions are the plane wave decompositions of the fundamental solution of the operator $\partial/\partial x_0 + D$, in even and odd dimensions.

3. Boundary value results. Theorem 3 leads to the calculation of the boundary values of $\mathscr{C}_{-l}(P_k) = \mathscr{C}^{\pm}_{-l}(P_k)$ and $\mathscr{S}_{-l}(P_k) = \mathscr{S}^{\pm}_{-l}(P_k)$, which are defined in $\mathbf{R}^{m+1}\backslash\mathbf{R}^m$. We first treat the even dimensional case.

In this case we have that for $l = k+m+2s$,

$$\begin{aligned}
&\mathscr{C}_{-l}(P_k)(\vec{x}+0) - \mathscr{C}_{-l}(P_k)(\vec{x}-0) \\
&\qquad = -\omega_{m-1} S_k(-l)\big(q_{-l,k,m}(\vec{x}+0) + q_{-l,k,m}(\vec{x}-0)\big) P_k(\vec{\xi}) \\
&\qquad = -2\omega_{m-1} S_k(-l) \frac{\vec{x}}{|\vec{x}|^{l+1}} P_k(\vec{\xi}),
\end{aligned}$$

whereas

$$\begin{aligned}
&\mathscr{S}_{-l-1}(P_k)(\vec{x}+0) - \mathscr{S}_{-l-1}(P_k)(\vec{x}-0) \\
&\qquad = \omega_{m-1} S_{k-1}(-l-1)\big(p_{-l-1,k,m}(\vec{x}+0) + p_{-l-1,k,m}(\vec{x}-0)\big) P_k(\vec{\xi}) \\
&\qquad = 2\omega_{m-1} S_{k-1}(-l-1) \frac{1}{|\vec{x}|^{l+1}} P_k(\vec{\xi}).
\end{aligned}$$

Notice that both boundary values are restrictions to R^m of outer spherical monogenics, which may be regarded as singular integral kernels for generalized Hilbert–Riesz transforms (see [12], [14]).

As on the other hand (see [2]),

$$(\langle\vec{x},\vec{t}\rangle-0\vec{t})^{-l}-(\langle\vec{x},\vec{t}\rangle+0\vec{t})^{-l} = 2\pi\frac{\vec{t}(-1)^{l-1}}{(l-1)!}\delta^{(l-1)}(\langle\vec{x},\vec{t}\rangle),$$

we obtain

Proposition 5. *For $l = k+m+2s$, we have the plane wave decompositions*

$$\int_{S^{m-1}} \delta^{(l-1)}(\langle\vec{x},\vec{t}\rangle)\vec{t}P_k(\vec{t})\,dS_t = \frac{(l-1)!(-1)^l}{\pi}\omega_{m-1}S_k(-l)\frac{\vec{x}}{|\vec{x}|^{l+1}}P_k(\vec{\xi})$$

and

$$\int_{S^{m-1}} \delta^{(l)}(\langle\vec{x},\vec{t}\rangle)P_k(\vec{t})\,dS_t = \frac{l!(-1)^{l-1}}{\pi}\omega_{m-1}S_{k-1}(-l-1)\frac{1}{|\vec{x}|^{l+1}}p_k(\vec{\xi}).$$

Taking the Hilbert–Riesz transform of the expressions in Proposition 5, we obtain that (see [11], [12])

$$\int_{S^{m-1}} \frac{1}{\langle\vec{x},\vec{t}\rangle^l}P_k(\vec{t})\,dS_t = -\tfrac{1}{2}\omega_{m-1}\omega_{m+1}S_k(-l)\delta^{(l)}(P_k)$$

and

$$\int_{S^{m-1}} \frac{\vec{t}}{\langle\vec{x},\vec{t}\rangle^{l+1}}P_k(\vec{t})\,dS_t = \tfrac{1}{2}\omega_{m-1}\omega_{m+1}S_{k-1}(-l-1)\delta^{(l+1)}(P_k),$$

where for $l = k+m+2s$,

$$\delta^{(l)}(P_k) = BV\left(\frac{1}{\omega_{m+1}}q_{-l,k,m}(x)\right)P_k(\vec{\xi}),$$

$$\delta^{(l+1)}(P_k) = BV\left(\frac{1}{\omega_{m+1}}p_{-l-1,k,m}(x)\right)P_k(\vec{\xi}).$$

Notice that $\delta^{(l)}(P_k)$ and $\delta^{(l+1)}(P_k)$ correspond to homogeneous differential operators with constant coefficients.

We have that (see [12]),

$$\delta^{(k+m)}(P_k) = BV\left(-\frac{1}{\omega_{m+1}}\frac{\vec{x}}{|x|^{m+1+2k}}P_k(\vec{x})\right) = -\frac{\Gamma\left(\frac{m+1}{2}\right)}{2^k\,\Gamma\left(k+\frac{m+1}{2}\right)}P_k(\vec{\nabla})\delta.$$

By Theorem 2, we have the expressions

$$\delta^{(k+m+2s)}(P_k) = \frac{-\Gamma(k+m)\,\Gamma\left(\frac{m+1}{2}\right)}{2^k\,\Gamma(k+m+2s)\,\Gamma\left(k+\frac{m+1}{2}\right)}\,\Delta^s P_k(\nabla)\,\delta,$$

$$\delta^{(k+m+2s+1)}(P_k) = \frac{\Gamma(k+m)\,\Gamma\left(\frac{m+1}{2}\right)(-1)^s}{2^k\,\Gamma(k+m+2s+1)\,\Gamma\left(k+\frac{m+1}{2}\right)}\,D^{2s+1}\,P_k(\nabla)\,\delta.$$

For $k = 0$, $s = 0$ we reobtain the classical formula (see e.g. [2])

$$\delta = \frac{(-1)^{m/2}(m-1)!}{(2\pi)^m} \int_{S^{m-1}} \langle\vec{x}, \vec{t}\rangle^{-m}\, dS_t.$$

In the odd dimensional case we obtain the following

PROPOSITION 6. *For $l = k+m+2s$ we have that*

$$\int_{S^{m-1}} \delta^{(l-1)}(\langle\vec{x}, \vec{t}\rangle)\, P_k(\vec{t})\, dS_t = \frac{(l-1)!\,(-1)^l}{2\pi}\,\omega_{m-1}\,\omega_{m+1}\,C_{k+1}(-l)\,\delta^{(l)}(P_k)$$

and

$$\int_{S^{m-1}} \delta^{(l)}(\langle\vec{x}, \vec{t}\rangle)\,\vec{t}P_k(\vec{t})\, dS_t = \frac{l!\,(-1)^l}{2\pi}\,\omega_{m-1}\,\omega_{m+1}\,C_k(-l-1)\,\delta^{(l+1)}(P_k).$$

The Hilbert–Riesz transform of this again gives rise to the plane wave decomposition of the generalized Hilbert–Riesz kernels in odd dimensions. The explicit calculations are left to the reader.

Notice that for $k = 0$, $s = 0$, Proposition 6 leads back to the classical decomposition (see [2])

$$\delta = \frac{(-1)^{(m-1)/2}}{2(2\pi)^{m-1}} \int_{S^{m-1}} \delta^{(m-1)}(\langle\vec{x}, \vec{t}\rangle)\, dS_t.$$

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