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On some problems in the theory of eigenvalues and eigenfunctions associated with linear elliptic partial differential equations of the second order

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Introduction. Although in mathematical literature the problem of eigenvalues and eigenfunctions is mostly considered for boundary conditions of the Dirichlet type, in applications also other boundary condition often have to be dealt with.

The purpose of the present paper is:

1° To determine boundary conditions in such a way that the existence of eigenvalues and eigenfunctions is guaranteed for a possibly large class of domains;

2° To transfer some theorems on eigenvalues and eigenfunctions known for the Dirichlet boundary condition to some other homogeneous boundary conditions.

Our theorems will be formulated and their proofs will be carried out in such a way that they will involve as a special case boundary conditions the most often used in applications, including those of the Dirichlet type.

In this paper we mainly deal with the number of nodal domains of the nth eigenfunction (Theorem 1), with some properties of the first eigenfunction and eigenvalue (Corollaries 1 and 2, Theorem 3), with multiplicity of zeros of the nth eigenfunction and the number of singular points of nodal lines of this function (Theorems 4 and 5, Corollary 3) and lastly with the alternation of zeros of eigenfunctions (Theorems 6 and 7).

§ 1. Introductory concepts.

DEFINITION I. The curve C: $x_1 = x_1(t)$, $x_2 = x_2(t)$, $a \leqslant t \leqslant b$ is said to be *regular* (or *smooth*) if there exist continuous derivatives $x_1'(t)$ and $x_2'(t)$ and, moreover, $[x_1'(t)]^2 + [x_2'(t)]^2 > 0$ for $a \leqslant t \leqslant b$.

DEFINITION II. The curve C is said to be *piecewise regular* if it can be represented as a sum of the finite number of closed regular arcs.

DEFINITION III. We shall say that a curve C, contained in a domain G, *divides* G if $G\setminus C$ is a union of disjoined subdomains such that in an arbitrary neighbourhood of any point of C there are points belonging to two (at least) different subdomains.

DEFINITION IV. We shall say that a *function* $f(x_1, x_2)$ *is of class* C_σ^n $(n \geqslant 1)$ *in the closure of* G, if:

1° $f(x_1, x_2)$ is of the class C^{n-1} in $\bar{G}$,

2° $\bar{G}$ may be subdivided into a finite number of closed subdomains in each of which $f(x_1, x_2)$ is of class C^n.

We shall say that $f(x_1, x_2)$ *is of class* C_σ^0 if $\bar{G}$ may be so subdivided into a finite number of closed subdomains that $f(x_1, x_2)$ is continuous in each of these subdomains.

DEFINITION V. We shall say that $f(x_1, x_2)$ *is of class* C_σ^n $(n \geqslant 0)$ *in a domain* G (open connected set) if it is of class C_σ^n in every closed domain contained in G.

Let G be a bounded Jordan measurable domain (simply or multiply connected) in the plane (x_1, x_2). G may be approximated by an increasing sequence of domains G_n with regular boundaries (i.e. the boundary $F(G_n)$ of G_n is a piecewise regular curve). We do not require any regularity properties from the boundary of G.

In the sequel we shall consider

1° a selfadjoined elliptic partial differential equation of the second order, i.e.

(1) $$L(u) + \mu\varrho u = 0,$$

where

$$L(u) = \sum_{i,k=1}^{2} \frac{\partial}{\partial x_i}\left[a_{ik}(x_1, x_2)\frac{\partial u}{\partial x_k}\right] - q(x_1, x_2)u$$

is a selfadjoined differential operator and μ is a real parameter. Concerning the coefficients we assume that: $q(x_1, x_2) \geqslant 0$, $\varrho(x_1, x_2) > 0$ are defined and continuous in $\bar{G}$, $a_{ik}(x_1, x_2) = a_{ki}(x_1, x_2)$ $(i, k = 1, 2)$ are of class C^1 in $\bar{G}$, and the quadratic form $\sum_{i,k=1}^{2} a_{ik}(x_1, x_2)\xi_i\xi_k$ in positive definite in $\bar{G}$.

2° a generalized boundary condition corresponding to the boundary condition of the form

(1a) $$\frac{du}{d\nu} - h(x_1, x_2)u = 0 \text{ on } F(G) - \Gamma, \qquad u = 0 \text{ on } \Gamma,$$

where Γ denotes a part of $F(G)$ (Γ being connected or not); in extreme cases Γ may be the whole boundary of G, or the empty set. Here $h(x_1, x_2)$

is a non-negative continuous function defined in $\bar{G}$ and $du/d\nu$ is a transversal derivative of u with respect to equation (1), i.e.

$$\frac{du}{d\nu} = \sum_{i,k=1}^{2} a_{ik}(x_1, x_2)\frac{\partial u}{\partial x_k}\cos(n, x_i), \qquad (x_1, x_2) \in F(G), \tag{2}$$

n being the interior normal to $F(G)$.

The object of the following considerations are eigenvalues and eigenfunctions corresponding to equation (1) and condition (1a). The boundary condition (1a) comprises as special cases all the boundary conditions useful in applications.

However, condition (1a) may have no meaning without further explanations. Namely

1° we have not assumed the existence of the tangent to $F(G)$,

2° we have no information about the behaviour on $F(G)$ of the solutions of (1), or about their derivatives.

It might be thought the natural to assume that the boundary of G is a piecewise regular curve and the solutions of (1) are of class C^1 in $\bar{G}$. However, as A. Pliś has shown, there are examples of regular domains G (e.g., concave polygons) such that the solutions of (1) with the condition $u = 0$ on $F(G)$ need not be of class C^1 in $\bar{G}$. In the case of boundary condition (1a), the solutions of (1) need not be of class C^1 even for a circle.

Therefore, we shall now give the meaning of condition (1a). Accordingly, following Courant-Hilbert ([5], vol. II, Chapter VII), we introduce some bilinear functionals and some linear function spaces, in which these functionals will be defined. Namely, we put

$$D(\varphi, \psi) = \iint_G \Big[\sum_{i,k=1}^{2} a_{ik}(x_1, x_2)\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\psi}{\partial x_k} + q(x_1, x_2)\varphi\psi\Big]dx_1 dx_2 + \int_{F(G)-\Gamma} h(x_1, x_2)\varphi\psi ds, \tag{3}$$

$$H(\varphi, \psi) = \iint_G \varrho(x_1, x_2)\varphi\psi dx_1 dx_2. \tag{4}$$

It immediately follows from (3) and (4) that the functionals D and H are symmetric,

$$D(\varphi, \psi) = D(\psi, \varphi) \quad \text{and} \quad H(\varphi, \psi) = H(\psi, \varphi).$$

Let us put

$$D(\varphi) = D(\varphi, \varphi) \quad \text{and} \quad H(\varphi) = H(\varphi, \varphi). \tag{5}$$

Observe that

$$D(\varphi) \geqslant 0, \quad H(\varphi) \geqslant 0 \tag{6}$$

and

$$\begin{aligned} D(a\varphi+\beta\psi) &= a^2 D(\varphi)+2a\beta D(\varphi,\psi)+\beta^2 D(\psi), \\ H(a\varphi+\beta\psi) &= a^2 H(\varphi)+2a\beta H(\varphi,\psi)+\beta^2 H(\psi). \end{aligned} \tag{7}$$

The equality $H(\varphi) = 0$ may occur only if $\varphi \equiv 0$. (7) is valid for arbitrary real numbers a and β and for all functions φ, ψ for which D and H have meaning.

The functionals (3) and (4) are defined as follows. Given φ and ψ, we take an increasing sequence of domains G_n contained with its regular boundary in G and converging to G. Suppose that the expressions (3) and (4) are defined for φ and ψ and every G_n, $n = 1, 2, \ldots$ If these expressions have finite limits for every sequence $\{G_n\}$, these limits are taken as the values of D and H, respectively.

Definition VI. We denote by $\mathfrak{H}$ the space of all functions φ of class C_σ^0 in G such that $H(\varphi) < \infty$.

Definition VII. Denote by $\mathfrak{D}$ the space of all functions φ of class C_σ^1 in G such that $H(\varphi) < \infty$ and $D(\varphi) < \infty$.

Definition VIII. Denote by $\dot{\mathfrak{D}}$ the subspace of $\mathfrak{D}$ of functions which vanish at all points of G whose distance from Γ is less than or equal to ε (ε being a fixed positive number).

Definition IX. Denote by $\mathring{\mathfrak{D}}$ the subspace of $\mathfrak{D}$ of functions φ for which there exists a sequence $\varphi_\nu \in \dot{\mathfrak{D}}$ such that $H(\varphi_\nu - \varphi) \to 0$ and $D(\varphi_\nu - \varphi) \to 0$ for $\nu \to \infty$.

In the sequel by the boundary condition $u = 0$ on Γ we shall mean that $u \in \mathring{\mathfrak{D}}$.

Lemma A. *If a function $\varphi(x_1, x_2)$ 1° belongs to $\mathfrak{D}$, 2° is continuous on $G+\Gamma$, 3° $\varphi = 0$ on Γ (in the ordinary sense), then φ belongs to $\mathring{\mathfrak{D}}$.*

Proof. Denote by $G_{\delta,\Gamma}$ the set of all points P of G such that the distance P from Γ is greater than δ. Following [8], p. 102, one may construct by the integral mean procedure the function χ_δ with the following properties: 1° $\chi_\delta \equiv 1$ in $G_{3\delta,\Gamma}$, 2° $\chi_\delta \equiv 0$ in $G - G_{\delta,\Gamma}$, 3° χ_δ is of class C^n in G, n being arbitrarily large. It is obvious that $\varphi_\delta = \chi_\delta \varphi$ belongs to $\dot{\mathfrak{D}}$. Further, one easily checks that $D(\varphi_\delta - \varphi) \to 0$ and $H(\varphi_\delta - \varphi) \to 0$, when $\delta \to 0$, i.e. $\varphi \in \mathring{\mathfrak{D}}$. Q.E.D.

Definition X. Let $\mathcal{F}$ denote the subspace of $\mathfrak{D}$ of all functions φ of class C^2 in G such that $L(\varphi) \in \mathfrak{H}$.

We want to give meaning to the boundary condition $du/d\nu - hu = 0$ on $F(G) - \Gamma$. Accordingly consider domain G_ε contained with its regular boundary $F(G_\varepsilon)$ in G, such that the distance of $F(G_\varepsilon)$ from $F(G)$ is less than ε. Let $\varphi \in \mathfrak{F}$ and $\psi \in \dot{\mathfrak{D}}$. We shall prove that

$$(8) \qquad D_\varepsilon(\varphi, \psi) + H_\varepsilon\left(\frac{1}{\varrho}, L(\varphi), \psi\right) + \int\limits_{F(G_\varepsilon)-\Gamma_\varepsilon} \psi\left(\frac{d\varphi}{d\nu} - h\varphi\right) ds = 0\,,$$

Γ_ε denoting the set of points of $F(G_\varepsilon)$ whose distance from Γ is less than or equal to ε.

Indeed, integrating the identity

$$(9) \qquad \sum_{i,k=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial\varphi}{\partial x_k}\psi\right) = \sum_{i,k=1}^{2} a_{ik}\frac{\partial\varphi}{\partial x_k}\cdot\frac{\partial\psi}{\partial x_i} + \psi\sum_{i,k=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial\varphi}{\partial x_k}\right)$$

over the domain G_ε, we obtain

$$(10) \qquad \iint\limits_{G_\varepsilon} \sum_{i,k=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial\varphi}{\partial x_k}\psi\right) dx_1\,dx_2$$
$$= \iint\limits_{G_\varepsilon} \psi \sum_{i,k=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial\varphi}{\partial x_k}\right) dx_1\,dx_2 + \iint\limits_{G_\varepsilon}\sum_{i,k=1}^{2} a_{ik}\frac{\partial\varphi}{\partial x_k}\cdot\frac{\partial\psi}{\partial x_i}\,dx_1\,dx_2\,.$$

By (2) (definition of transversal derivative) and since $\psi \in \dot{\mathfrak{D}}$ and $\psi = 0$ on Γ_ε we get

$$(11) \qquad \iint\limits_{G_\varepsilon}\sum_{i,k=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial\varphi}{\partial x_k}\psi\right) dx_1\,dx_2 = -\int\limits_{F(G_\varepsilon)-\Gamma_\varepsilon} \psi\frac{d\varphi}{d\nu}\,ds\,.$$

Combining this with (10), we obtain

$$(12) \qquad \iint\limits_{G_\varepsilon}\psi\sum_{i,k=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial\varphi}{\partial x_k}\right) dx_1\,dx_2 +$$
$$+\iint\limits_{G_\varepsilon}\sum_{i,k=1}^{2} a_{ik}\frac{\partial\varphi}{\partial x_k}\cdot\frac{\partial\psi}{\partial x_i}\,dx_1\,dx_2 + \int\limits_{F(G_\varepsilon)-\Gamma_\varepsilon}\psi\frac{d\varphi}{d\nu}\,ds = 0\,.$$

Adding now equality (12) to the equality

$$\iint\limits_{G_\varepsilon} q(\varphi\psi - \varphi\psi)\,dx_1 dx_2 + \int\limits_{F(G_\varepsilon)-\Gamma_\varepsilon} h(\varphi\psi - \varphi\psi)\,ds = 0\,,$$

we get (8). Let $\varepsilon \to 0$ in (8). Since there exists limits $D(\varphi, \psi) = \lim\limits_{\varepsilon\to 0} D_\varepsilon(\varphi, \psi)$ and

$$H\left(\frac{1}{\varrho}L(\varphi), \psi\right) = \lim_{\varepsilon\to 0} H_\varepsilon\left(\frac{1}{\varrho}L(\varphi), \psi\right),$$

then there exists also a limit

$$\int_{F(G)-\Gamma} \left(\frac{d\varphi}{d\nu} - h\varphi\right) \psi ds = \lim_{\varepsilon\to 0} \int_{F(G_\varepsilon)-\Gamma_\varepsilon} \left(\frac{d\varphi}{d\nu} - h\varphi\right) \psi ds .$$

All the limits involved are connected by the formula

$$D(\varphi, \psi) + H\left(\frac{1}{\varrho} L(\varphi), \psi\right) + \int_{F(G)-\Gamma} \left(\frac{d\varphi}{d\nu} - h\varphi\right) \psi ds = 0 . \tag{13}$$

One may prove (compare an analogous theorem in [5]) that if (13) is valid for every $\varphi \in \mathfrak{F}$ and $\psi \in \mathring{\mathfrak{D}}$, then it is also valid for $\varphi \in \mathfrak{F}$ and $\psi \in \mathring{\mathfrak{D}}$.

The boundary condition $du/d\nu - hu = 0$ on $F(G) - \Gamma$ for $u \in \mathfrak{F}$ is now defined by the requirement that the equality

$$\int_{F(G)-\Gamma} \left(\frac{du}{d\nu} - hu\right) \psi ds = 0 \tag{14}$$

be valid for all $\psi \in \mathring{\mathfrak{D}}$. The boundary condition (1a) is defined by the requirements that $u \in \mathfrak{F} \cap \mathring{\mathfrak{D}}$ and (14) be valid for all $\psi \in \mathring{\mathfrak{D}}$.

Definition XI. $\mathfrak{F}_{h,\Gamma}(G)$ will denote the space of all functions $\varphi \in \mathfrak{F}$ satisfying (1a) in the above sense.

We define eigenvalues and eigenfunctions of problem (1) (1a) in the following way (variationally):

The first eigenvalue λ_1 of problem (1) (1a) is defined by

$$\lambda_1 = \min_{\varphi \in \mathring{\mathfrak{D}}} \frac{D(\varphi)}{H(\varphi)}, \tag{15}$$

and the first eigenfunction u_1 is that φ which realizes minimum (15).

Having defined eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the corresponding eigenfunctions $u_1, u_2, \ldots, u_n$, we put

$$\lambda_{n+1} = \min_{\varphi \in K_n} \frac{D(\varphi)}{H(\varphi)}, \tag{16}$$

where K_n is a subclass of $\mathring{\mathfrak{D}}$ of the functions φ that satisfy the orthogonality conditions

$$H(\varphi, u_i) = 0, \quad i = 1, 2, \ldots, n, \tag{17}$$

and the u_{n+1} is that $\varphi \in K_n$ which gives minimum (16).

Hypothesis Z_k. *Given* (1) *and* (1a) *there exist a sequence of* k $(1 \leqslant k \leqslant \infty)$ *eigenvalues*

$$0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_k , \tag{18}$$

and a corresponding sequence of eigenfunctions

$$u_1(x_1, x_2), \ldots, u_k(x_1, x_2) \tag{19}$$

which belong to $\mathfrak{F}$.

The problem whether the hypothesis Z_k is satisfied under the assumptions which we have made concerning the coefficients of equation (1) will not be dealt with in this paper.

The existence of eigenvalues and eigenfunctions defined as above is proved in [5], Ch. VII, for a less general equation than (1) and under some assumptions on the coefficients of the equation. There is also proved in [5] the following theorem:

"*Eigenfunctions* (19) *satisfy equation* (1) *and boundary condition* (1a) *for the corresponding eigenvalues*".

We shall now prove the following

THEOREM A. *If hypothesis* Z_k *is satisfied, then each function* u_n *of sequence* (19) *satisfies equation* (1) *for* $\mu = \lambda_n$, $n = 1, \ldots, k$, *and* $u_n \in \mathcal{F}_{h,\Gamma}(G)$.

Proof. To begin with, observe that by (13) if $\varphi \in \mathcal{F}_{h,\Gamma}(G)$ satisfies (1) for $\mu = t$, then

$$D(\varphi, \psi) - tH(\varphi, \psi) = 0 , \tag{20}$$

for every $\psi \in \mathring{\mathfrak{D}}$. We shall now show that if (20) holds for a fixed $\varphi \in \mathcal{F}_{h,\Gamma}(G)$ and for every $\psi \in \mathring{\mathfrak{D}}$, then φ satisfies (1) for $\mu = t$. Indeed, by (13) and (20) we have

$$H\left(t\varphi + \frac{1}{\varrho} L(\varphi), \psi\right) + \int\limits_{F(G)-\Gamma} \left(\frac{d\varphi}{d\nu} - h\varphi\right) \psi ds = 0 . \tag{21}$$

By the arbitrariness of $\psi \in \mathring{\mathfrak{D}}$ and by Lagrange's Lemma ([7], p. 275) we have

$$L(\varphi) + t\varrho\varphi = 0, \qquad \int\limits_{F(G)-\Gamma} \left(\frac{d\varphi}{d\nu} - h\varphi\right) \psi ds = 0 .$$

Therefore the only thing left to be proved is the equality

$$D(u_n, \psi) - \lambda_n H(u_n, \psi) = 0, \qquad n = 1, \ldots, k , \tag{22}$$

for any $\psi \in \mathring{\mathfrak{D}}$. This will be done by induction with respect to n.

1° $n = 1$. Let ψ be an arbitrary function of $\mathring{\mathfrak{D}}$ and let τ be an arbitrary real number. Put $\varphi = u_1 + \tau\psi$. Then by (15)

$$D(\varphi) \geqslant \lambda_1 H(\varphi) , \tag{23}$$

whence by (7) and by the definition of u_1 we get

$$2\tau[D(u_1, \psi) - \lambda_1 H(u_1, \psi)] + \tau^2[D(\psi) - \lambda_1 H(\psi)] \geqslant 0 . \tag{24}$$

The τ being arbitrary, this is possible only if

$$D(u_1, \psi) - \lambda_1 H(u_1, \psi) = 0 , \tag{25}$$

which is equivalent to (22) for $n = 1$.

2° Assume that (22) holds for $n = 1, \ldots, s < k$ and for any $\psi \in \mathring{\mathfrak{D}}$. Because of (16) we have

$$D(\varphi) \geqslant \lambda_{s+1} H(\varphi), \qquad \varphi = u_{s+1} + \tau\zeta, \tag{26}$$

τ being an arbitrary real number and ζ an arbitrary function belonging to K_s. Hence, as in 1°,

$$D(u_{s+1}, \zeta) - \lambda_{s+1} H(u_{s+1}, \zeta) = 0 . \tag{27}$$

We have to show that this equality holds also for every $\psi \in \mathring{\mathfrak{D}}$. Indeed, given $\psi \in \mathring{\mathfrak{D}}$, take s real numbers $a_1, \ldots, a_s$ such that $\zeta = \psi + a_1 u_1 + \ldots + a_s u_s$ belongs to K_s. This is always possible by putting

$$a_l = -\frac{H(\psi, u_l)}{H(u_l)}, \qquad l = 1, \ldots, s .$$

For such a ζ, by the induction assumption and because of the symmetry property of D and H, we get

$$D(u_{s+1}, \psi) - \lambda_{s+1} H(u_{s+1}, \psi) = 0, \qquad \text{Q.E.D.} \tag{28}$$

§ 2. It follows from the definition of u_n, $n = 1, \ldots, k$, that

$$H(u_i, u_j) \begin{cases} = 0, & \text{if} \quad i \neq j, \\ \neq 0, & \text{if} \quad i = j, \end{cases} \qquad i, j = 1, \ldots, k . \tag{29}$$

This implies that all functions (19), except at most one, change their sign in G. And since the functions are continuous, there exists a zero set in G for these functions.

It was proved in [4] that any function u satisfying (1) and changing its sign in G vanishes along lines which are called nodal lines. The nodal lines divide G into so called nodal domains. The nodal lines are oscillation lines, i.e. the function u changes its sign in every neighbourhood of every point of its nodal line.

LEMMA 1. *No eigenfunction of* (1) (1a) *can vanish identically in any subdomain of domain G.*

This is a direct consequence of the general theorem of [1], corresponding to which no solution of an elliptic equation can have zero of infinite order in G without vanishing identically in G.

DEFINITION XII. Every subdomain $G_1 \subset G$ bounded by nodal line of an eigenfunction $u(x_1, x_2)$ and by a part of $F(G)$ such that $u(x_1, x_2) \neq 0$ in G_1 will be called the *nodal domain of u*, associated with (1) (1a) if there are no points of $F(G)$ on the boundary of G_1, G_1 will be called the *interior nodal domain.*

LEMMA 2. *Under the assumption Z_k the nodal lines of any function $u \in \mathcal{F}_{h,\Gamma}(G)$ satisfying* (1) *with $\mu < \lambda_n$ $(n \leqslant k)$ divide G into less than n nodal domains.*

Proof. Let $u \in \mathcal{F}_{h,\Gamma}(G)$ and let

$$L(u) + \mu\varrho u = 0 . \tag{30}$$

Suppose the nodal lines divide G into subdomains $G_1, G_2, \ldots, G_n, \ldots$ Put

$$U_i = \begin{cases} u(x_1, x_2) & \text{in} \quad \bar{G}_i , \\ 0 & \text{in} \quad \bar{G} - \bar{G}_i , \end{cases} \qquad i = 1, \ldots, n . \tag{31}$$

It is obvious that $U_1, \ldots, U_n$ are linearly independent in G. Put

$$F(x_1, x_2) = a_1 U_1 + \ldots + a_n U_n , \tag{32}$$

where $a_1, \ldots, a_n$ are real numbers such that $a_1^2 + \ldots + a_n^2 > 0$, and the function F is orthogonal to $u_1, u_2, \ldots, u_{n-1}$ with respect to the functional H. The choice of a_k, $k = 1, \ldots, n$, is always possible; they are a solution of the system of $n-1$ linear equations. The function F belongs to K_n, thus by (16)

$$D(F) \geqslant \lambda_n H(F) . \tag{33}$$

Since $u \in \mathcal{F}_{h,\Gamma}(G)$ and u satisfies (1), then by (20)

$$D(u, \psi) = \mu H(u, \psi) \tag{34}$$

for every $\psi \in \mathring{\mathfrak{D}}$. Let $\psi = a_1^2 U_1 + \ldots + a_n^2 U_n$. Then $\psi \in \mathring{\mathfrak{D}}$. The definition of $U_1, \ldots, U_n$ implies that they are orthogonal with respect to $D(\varphi)$ and $H(\varphi)$. Hence by (7)

$$D(F) = \sum_{i=1}^{n} a_i^2 D(U_i), \qquad H(F) = \sum_{i=1}^{n} a_i^2 H(U_i) . \tag{35}$$

Further, $D(\varphi, \psi)$ and $H(\varphi, \psi)$ being bilinear, we have

$$D(u, \psi) = \sum_{i=1}^{n} a_i^2 D(U_i), \qquad H(u, \psi) = \sum_{i=1}^{n} a_i^2 H(U_i) . \tag{36}$$

In virtue of (34), (35) and (36) we have

$$D(F) = \mu H(F) , \tag{37}$$

whence by (33)

$$\lambda_n H(F) \leqslant D(F) = \mu H(F) .$$

Thus $\lambda_n < \mu$, which contradicts our assumption.

LEMMA 3. *Under the assumption Z_k, if $\lambda_n = \lambda_{n+1} = \ldots = \lambda_{n+s-1} < \lambda_{n+s}$ $(n+s \leqslant k)$ (i.e. λ_n is an s-fold eigenvalue of* (1) (1a)), *then the nodal lines of each function $u \in \mathcal{F}_{h,\Gamma}(G)$ satisfying* (1) *with $\mu = \lambda_n$ divide G into at most n domains.*

Proof. Let $G_1, G_2, \ldots, G_n, G_{n+1}, \ldots$ be nodal domains of u. As in the proof of Lemma 2, we put

$$(38)\qquad U_i = \begin{cases} u & \text{in} \quad \bar{G}_i, \\ 0 & \text{in} \quad \bar{G}-\bar{G}_i, \end{cases} \qquad i = 1, \ldots, n,$$

and $F = a_1 U_1 + \ldots + a_n U_n$, the coefficient $a_1, \ldots, a_n$ being chosen in such a way that $a_1^2 + \ldots + a_n^2 > 0$ and F is orthogonal to the functions $u_1, \ldots, u_{n-1}$ of system (19) with respect to the functional H. Repeating the reasoning used for deriving (37) we conclude that

$$(39)\qquad D(F) = \lambda_n H(F).$$

By the construction of F we have $F \equiv 0$ in $G_{n+1} + \ldots$ The functions $u_n, \ldots, u_{n+s-1}$ are by Lemma 1 linearly independent in each subdomain of G which has been considered. Thus $F, u_n, \ldots, u_{n+s-1}$ are linearly independent in G. Put

$$\Phi(x_1, x_2) = F + c_n u_n + \ldots + c_{n+s-1} u_{n+s-1}.$$

Then $\Phi \not\equiv 0$ in G. Let $c_l = -H(F, u_l)/H(u_l)$, $l = n, \ldots, n+s-1$. Then Φ is orthogonal to $u_n, \ldots, u_{n+s-1}$. The definition of F and the orthogonality of eigenfunctions imply that $\Phi \in K_{n+s-1}$. Hence

$$(40)\qquad D(\Phi) \geqslant \lambda_{n+s} H(\Phi).$$

On the other hand, $\varphi(x_1, x_2) = c_n u_n + \ldots + c_{n+s-1} u_{n+s-1}$ belongs to $\mathcal{F}_{h,r}(G)$ and satisfies (1) with $\mu = \lambda_n$. Therefore by (20) we have

$$(41)\qquad D(\varphi, \psi) = \lambda_n H(\varphi, \psi)$$

for every $\psi \in \mathring{\mathfrak{D}}$. As a special case of (41) we get

$$(42)\qquad D(\varphi, F) = \lambda_n H(\varphi, F),$$

$$(43)\qquad D(\varphi) = \lambda_n H(\varphi).$$

Because of the equality $\Phi = F + \varphi$, (42), (43), (39) and (7) imply

$$(44)\qquad D(\Phi) = \lambda_n H(\Phi),$$

whence by (40) $\lambda_n \geqslant \lambda_{n+s}$, which is a contradiction. Lemmas 2 and 3 imply the following

THEOREM 1. *Under the assumption* Z_k, *if* $N(n)$ *denotes the number of nodal domains of the* n*-th* ($n \leqslant k$) *eigenfunction associated with* (1) (1a), *then for each* n *we have*

$$(45)\qquad N(n) \leqslant n,$$

the equality occurring only when $\lambda_{n-1} < \lambda_n$.

Proof. Inequality (45) follows immediately from Lemmas 2 and 3. Suppose that in (45) the equality takes place and $\lambda_{n-1} = \lambda_n$. Then, by Lemma 3, the number of nodal domains of the nth eigenfunction would be less than or equal to $n-1$, which is a contradiction. Thus $\lambda_{n-1} < \lambda_n$.

§ 3. If $n = 1$, Theorem 1 implies the following

COROLLARY 1. *The first eigenfunction of* (1) (1a) *does not vanish at any point of* G.

LEMMA 4. *Under the assumption* Z_k, *if* μ_1 *is a real number such that there exists a function* $u(x_1, x_2) \in \mathcal{F}_{h,\Gamma}(G)$ *which is different from zero in the whole* G *and which satisfies* (1) *with* $\mu = \mu_1$, *then* μ_1 *is the first eigenvalue* λ_1 *of* (1) (1a).

Proof. We shall use formula (13) first for the pair u, u_1 and then for the pair u_1, u. Since both u and u_1 belong to $\mathcal{F}_{h,\Gamma}(G)$ and satisfy (1) with $\mu = \mu_1$ and $\mu = \lambda_1$, respectively, we get by (13)

$$D(u, u_1) = \mu_1 H(u, u_1) \quad \text{and} \quad D(u_1, u) = \lambda_1 H(u_1, u),$$

whence by the symmetry property of D and H

$$(\mu_1 - \lambda_1) H(u, u_1) = 0.$$

Since u and u_1 do not change their sign in G, then $H(u, u_1) \neq 0$, and thus $\mu_1 = \lambda_1$.

THEOREM 2. *Under the assumption* Z_k *the* n*-th eigenvalue* λ_n $(n \leqslant k)$ *is the first eigenvalue for each nodal domain of* $u_n(x_1, x_2)$.

Proof. This follows from Lemma 4 by observing that in every nodal domain G_n the equation

$$L(u_n) + \lambda_n \varrho u_n = 0$$

is satisfied and that $u_n \in \mathcal{F}_{h,\Gamma_n}(G)$, where $\Gamma_n = F(G_n) - F(G_n) \cap [F(G) - \Gamma]$.

THEOREM 3. *Each function* $\varphi(x_1, x_2) \in \mathcal{F}_{h,\Gamma}(G)$, *not vanishing identically in* G *and satisfying* (1) *with* $\mu = \lambda_1$, *is equal to the first eigenfunction of* (1) (1a) *multiplied by a constant* $c \neq 0$.

Proof. Suppose φ and u_1 are linearly independent. According to E. Schmidt there are constants c_1 and c_2, $c_1^2 + c_2^2 > 0$ such that

$$\Phi(x_1, x_2) = c_1 u_1 + c_2 \varphi$$

is orthogonal to u_1. Since $u_1 \neq 0$ in G, Φ, being orthogonal to u_1, changes its sign in G. Since $\Phi \not\equiv 0$ and Φ satisfies (1), then its nodal lines divide G. On the other hand, Φ satisfying (1) with $\mu = \lambda_1$ and $\Phi \in \mathcal{F}_{h,\Gamma}(G)$, by Lemma 3 we have $\Phi \neq 0$ in G or $\Phi \equiv 0$. We get a contradiction. So $\varphi = cu_1$, $c \neq 0$.

COROLLARY 2. *The first eigenvalue of* (1) (1a) *is a single eigenvalue, i.e.* $\lambda_1 < \lambda_2$.

LEMMA 5. *Under the assumption* Z_k *the* n*-th eigenfunction of* (1) (1a) ($n \leqslant k$) *depends in each of its nodal domains* G_j *linearly on the first eigenfunction of* G_j *belonging to* $\mathcal{F}_{h,\Gamma_j}(G_j)$, *where* $\Gamma_j = F(G_j) - F(G_j) \cap [F(G) - \Gamma]$.

The proof follows from Theorems 2 and 3.

LEMMA 6. *Under the assumption* Z_k, *if two eigenfunctions* u *and* v *corresponding to the same eigenvalue* λ_n ($n \leqslant k$) *of* (1) (1a) *are linearly dependent in a subdomain* G^* *of* G, *they are also linearly dependent in the whole* G.

Proof. Let $u = cv$ in G^*, $c \neq 0$ being constant. Put $\Phi = u - cv$. Since the equation and the boundary condition are linear, the function Φ satisfies (1) with $\mu = \lambda_n$ and $\Phi \in \mathcal{F}_{h,\Gamma}(G)$. But $\Phi \equiv 0$ in G^*, and so, by Lemma 1, Φ vanishes in the whole G. Thus $u = cv$ in G.

§ 4. In this section we shall state some results concerning the multiplicity of the nth eigenfunction of (1) (1a) and some results concerning the number of singular points of nodal lines of this function. I am indebted to A. Pliś for drawing my attention to the possibility of such results.

DEFINITION XIII. We shall say that the *function* $u(x_1, x_2)$ *has zero of order* N *at the point* $P_0(\mathring{x}_1, \mathring{x}_2)$ if u satisfies the following conditions

$$\frac{u(x_1, x_2)}{r^i} \to 0, \quad i = 0, \ldots, N-1, \quad \frac{u(x_1, x_2)}{r^N} \not\to 0, \tag{46}$$

when $r = \sqrt{(x_1 - \mathring{x}_1)^2 + (x_2 - \mathring{x}_2)^2} \to 0$.

DEFINITION XIV. We shall say that the *point* P_0 *is a singular point of a nodal line of* u if u has zero of order $N \geqslant 2$ at P_0.

In the sequel we consider a family R of curves dividing domain G in the sense of Definition III. Besides, we assume that every point of an arbitrary curve of R is an origin of an even number of smooth arcs.

DEFINITION XV. We shall say that P *is a point of intersection of* k *locally different curves of* R if there exists a neighbourhood $O(P)$ of P such that P is an origin of $2k$ smooth arcs of R (restricted to $O(P)$), no two of which have common points in $O(P)$ except P.

DEFINITION XVI. We shall say that the *point* P *is a singular point of multiplicity* p *of the family* R if P is an intersection point of $p+1$ locally different curves of R.

We shall prove the following

LEMMA 7. *If* R *is not empty and the total amount of multiplicities of singular points of all curves of* R (*belonging to the interior of* G) *is equal to* s, *then the curves of* R *divide domain* G *into at most* $s+2$ *subdomains.*

Proof. We shall use induction with respect to s. Lemma 7 being true for $s = 0$, suppose it is true for $s = k$ and let $s = k+1$. Let P be a fixed singular point of a curve of R. By Definitions XV and XVI there is a neighbourhood $O(P)$ of P contained in G such that the curves of R restricted to $O(P)$ have no common points except P. Let us take a circle K_r with centre P and radius r contained together with its interior in $O(P)$. Take two points A and B of K_r belonging to two different subdomains in such a way that one of the subarcs of K_r (obtained by taking A and B) meets exactly two arcs, l_1 and l_2, of R starting from point P. Such points certainly exist.

Let A' and B' be the points of K_r common with l_1 and l_2, respectively. Now delete the parts of l_1 and l_2 contained between the points $A'P$ and $B'P$, respectively, and add the arc $A'B'$ of K_r belonging to the arc AB as a curve of the family R.

After this modification of R both the total number of multiplicities of singular points of R and the number of subdomains diminish by one. By the induction assumption there are at least $k+2$ modified subdomains; thus if $s = k+1$ the curves of R divide G into at least $k+3$ subdomains. The proof is completed.

THEOREM 4. *Under the assumption* Z_k *the sum of the orders of zeros of the* n*-th eigenfunction* ($n \leqslant k$) *associated with* (1) (1a) *at the singular points of its nodal lines does not exceed* $n+r-2$, *where* r *denotes the number of the singular points of the function.*

Proof. According to Barański's results ([3]), if an eigenfunction of (1) (1a) has zero of order N at a point P, then there are exactly N nodal lines of the eigenfunction passing through the point P, so that P is a singular point of multiplicity $N-1$ (cf. Definition XVI) of the nodal lines. Hence, because of Theorem 1, Lemma 7 and the definition and properties of nodal lines, we have obtained the required proof.

THEOREM 5. *The* n*-th eigenfunction of* (1) (1a) *cannot assume zeros of order greater than* $n-1$ *in the interior of* G.

Proof. Let P_0 be a point of the nodal lines of $u_n(x_1, x_2)$ with the greatest multiplicity. Denote by N the order of zero of $u_n(x_1, x_2)$ at P_0. Let r be the number of the singular points of the nodal lines of $u_n(x_1, x_2)$. Since at any other singular point the function $u_n(x_1, x_2)$ has zero of order at least 2, then by Theorem 4 and by the results of [3] we have $n+r-2 \geqslant N+2(r-1)$, whence $N \leqslant n-r \leqslant n-1$.

COROLLARY 3. *The family of nodal lines of the* n*-th eigenfunction of* (1) (1a) *cannot have more than* $n-2$ *singular points in the interior of* G.

Indeed, since at each singular point the eigenfunction has zero of order at least 2, the sum of the orders at r singular points is at least $2r$. Thus by Theorem 4 we have $r \leqslant n-2$.

§ 5. In the following two theorems we shall suppose that the assumption Z_k, $k > 1$, is satisfied.

THEOREM 6. *If $\lambda_i < \lambda_j$, then in each nodal subdomain of $u_i(x_1, x_2)$ there are nodes of the eigenfunction $u_j(x_1, x_2)$, where $i, j \leqslant k$.*

Proof. Let G_i be an arbitrary fixed nodal subdomain of u_i. Denote by Γ_0 that part of the boundary of G_i which is composed of nodal lines of u_i. By Corollary 3 Γ_0 is a regular curve. Put $\Gamma_i = F(G_i) \cap \Gamma$. Since $u_i, u_j \in \mathcal{F}_{h,\Gamma}(G)$ and $u_i \in \mathcal{F}_{h,\Gamma_i+\Gamma_0}(G_i)$, by applying (13) first for u_i, u_j and then for u_j, u_i we get

$$D(u_i, u_j) = \lambda_i H(u_i, u_j) - \int\limits_{\Gamma_0} \left(\frac{du_i}{d\nu} - hu_i\right) u_j ds ,$$

(47)

$$D(u_j, u_i) = \lambda_j H(u_j, u_i) .$$

But u_i, u_j are of class C^1 on Γ_0 and $u_i = 0$ on Γ_0. Therefore

$$D(u_i, u_j) = \lambda_i H(u_i, u_j) - \int\limits_{\Gamma_0} u_j \frac{du_i}{d\nu} ds ,$$

(48)

$$D(u_j, u_i) = \lambda_j H(u_j, u_i) ,$$

whence by the symmetry property of D and H we have

$$(\lambda_j - \lambda_i) H(u_i, u_j) = - \int\limits_{\Gamma_0} u_j \frac{du_i}{d\nu} ds . \tag{49}$$

Suppose $u_i > 0$ and $u_j > 0$ in G_i. Then the left-hand side of (49) is positive. On the other hand, the transversal goes into the interior of G_i ([6], p. 147, problem 14), and so $du_i/d\nu \geqslant 0$ on Γ_0. Thus the right-hand side of (49) is non-positive—a contradiction. Analogously, we arrive at a contradiction when assumming any other possible combination of signs u_i and u_j. Therefore u_j has to change its sign in G_i.

THEOREM 7. *If u and v are two eigenfunctions linearly independent and corresponding to the same multiple eigenvalue λ_s ($s \leqslant k$), then in each nodal domain of u there are nodes of v, and, inversely, in each nodal domain of v there exist nodes of u.*

Proof. Let G_i be an arbitrary nodal domain of u. Let Γ_0 and Γ_i have the same meaning as in the proof of Theorem 6. Then

$$D(u, v) = \lambda_s H(u, v) - \int\limits_{\Gamma_0} v \frac{du}{d\nu} ds ,$$

(50)

$$D(v, u) = \lambda H(v, u) .$$

Hence by the symmetry property of D and H we have

$$\int\limits_{\Gamma_0} v \frac{du}{d\nu} ds = 0 . \tag{51}$$

Suppose $u > 0$ in G_t and $v \neq 0$ in G_t, for instance $v > 0$ in G_t. The function v has to be positive on a non-void subarc of Γ_0. Otherwise by Lemmas 4, 5, 6 and Theorem 2 it would be linearly dependent on u, contrary to the assumption. In virtue of Corollary 3, $\text{grad}^2 u$ does not vanish on any subarc of Γ_0, whence by [6], p. 44, $du/d\nu \neq 0$ on Γ_0. Finally, v and $du/d\nu$ are both non-negative on Γ_0. This, however, is contradictory to (51). Similarly, we would get a contradiction, if we assumed any other combination of the constant signs of u and v in G_t.

Remark 1. Theorems 6 and 7 have been formulated and proved by Barański in [2] for the boundary condition $u = 0$ on $F(G)$ and under more restrictive assumptions on the coefficients of (1); namely he has assumed that the coefficients of (1) are analytic and that the solutions considered are of class C^1 in $\bar{G}$.

Remark 2. All the lemmas and theorems of our paper (except those of § 4 and § 5) together with their proofs may be generalized without essential changes to the case of more variables.

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