ANNALES
POLONICI MATHEMATICI
XXIX (1974)

Strong maximum principle for non-linear parabolic differential-functional inequalities

by JACEK SZARSKI (Kraków)

Abstract. A diagonal system of second order differential-functional inequalities of the type

(0.1) $$u_t^i(t,x) \leqslant f^i(t,x,u(t,x),u_x^i(t,x),u_{xx}^i(t,x),u(t,\cdot)) \quad (i = 1, \ldots, m)$$

is considered, where $x = (x_1, \ldots, x_n)$, $u = (u^1, \ldots, u^m)$, $u_x^i = (u_{x_1}^i, \ldots, u_{x_n}^i)$ and u_{xx}^i is the matrix of second order derivatives with respect to x. The function $u(t,x)$ is continuous in the closure of an open set $D \subset (t, x_1, \ldots, x_n)$ satisfying adequate assumptions and for fixed $\tilde{t}$ the symbol $u(\tilde{t},\cdot) = (u^1(\tilde{t},\cdot), \ldots, u^m(\tilde{t},\cdot))$ denotes the function $x \to u(\tilde{t},x)$ as an element of the space of continuous functions from $S_{\tilde{t}}$ (the intersection of $\bar{D}$ with the plane $t = \tilde{t}$) to R^m. For parabolic solutions of (0.1) two theorems on differential-functional inequalities as well as the maximum principle and the strong maximum principle are obtained.

The strong maximum principle for solutions of parabolic differential equations was treated by L. Nirenberg [2], W. Walter [5] and P. Besala [1]. The proof of the strong maximum principle is patterned on that given in paper [1] of Besala.

1. Notations and definitions. A set $D \subset (t, x_1, \ldots, x_n)$ will be called a *set of type C* if (see [3]):

a. D is open, contained in the zone $0 < t < T$ ($T \leqslant +\infty$), and the intersection of $\bar{D}$ with any closed zone $0 \leqslant t \leqslant \tilde{t} < T$ is bounded,

b. the projection $S_{\tilde{t}}$ onto the space $(x_1, \ldots, x_n)$ of the intersection of $\bar{D}$ with the plane $t = \tilde{t}$ is, for any $\tilde{t} \in \langle 0, T)$, non-void,

c. the point (t, x) being arbitrary fixed in $\bar{D}$, to every sequence t_ν with $t_\nu \in \langle 0, T)$ and $t_\nu \to t$, there is a sequence x^ν such that $x^\nu \in S_{t_\nu}$ and $x^\nu \to x$.

That part of ∂D which is contained in the open zone $0 < t < T$ is denoted by Σ.

A point $(\tilde{t}, \tilde{x})$ being fixed in D we denote by $S^-(\tilde{t}, \tilde{x})$ the set of points $(t, x) \in D$ that can be joined with $(\tilde{t}, \tilde{x})$ by a polygonal line contained in D along which the t-coordinate is increasing (weakly) from (t, x) to $(\tilde{t}, \tilde{x})$.

Let $C_m(S_t)$ stand for the space of continuous functions $z(x) = (z^1(x), \ldots, z^m(x))$ from S_t to R^m with the norm

$$\|z\| = \max_i \max\{|z^i(x)|: x \in S_t\}.$$

In the space $C_m(S_t)$ the following partial order is introduced: for $z = (z^1(x), \ldots, z^m(x)) \in C_m(S_t)$, $\tilde{z} = (\tilde{z}^1(x), \ldots, \tilde{z}^m(x)) \in C_m(S_t)$ the inequality $z \leqslant \tilde{z}$ means that

$$z^j(x) \leqslant \tilde{z}^j(x), \quad x \in S_t \quad (j = 1, \ldots, m).$$

Let $f^i(t, x, u, q, r, z)$ $(i = 1, \ldots, m)$, where $q = (q_1, \ldots, q_n)$ and $r = (r_{jk})$ is an $n \times n$ real symmetric matrix, be defined for $(t, x) \in D$, u, q, r arbitrary and $z \in C_m(S_t)$.

A function $u(t, x)$ is called *regular* in D if u is continuous in $\bar{D}$ and u_t, u_x, u_{xx} are continuous in D.

A regular function $u(t, x)$ in D is called a *regular solution* of (0.1) in D if it satisfies (0.1) for every $(t, x) \in D$.

According to the definition given by P. Besala in [1], given a regular function $u(t, x)$ in D the function $f^i(t, x, u, q, r, z)$ is said to be *uniformly parabolic* with respect to $u(t, x)$ in $U \subset D$ if there is a constant $\varkappa > 0$, such that for any two real symmetric matrices $r = (r_{jk})$ and $\tilde{r} = (\tilde{r}_{jk})$ we have

$$(1.1) \quad r \leqslant \tilde{r} \Rightarrow f^i(t, x, u(t, x), u_x^i(t, x), \tilde{r}, u(t, \cdot)) - f^i(t, x, u(t, x), u_x^i(t, x), r, u(t, \cdot)) \geqslant \varkappa \sum_{j=1}^{n} (\tilde{r}_{jj} - r_{jj}) \quad \text{for } (t, x) \in U.$$

The inequality $r \leqslant \tilde{r}$ means that $\sum_{j,k=1}^{n} (\tilde{r}_{jk} - r_{jk}) \lambda_j \lambda_k \geqslant 0$.

2. THEOREM 1 [1]. *Assume that*

1° *$f^i(t, x, u, q, r, z)$ $(i = 1, \ldots, m)$ are defined for $(t, x) \in D$, where D is of type C, for u, q, r arbitrary and $z \in C_m(S_t)$; the function f^i is increasing with respect to $u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^m, z$,*

2° *for $u \leqslant \tilde{u}$ and $t > 0$ we have*

$$(2.1) \quad f^i(t, x, \tilde{u}, q, r, \tilde{z}) - f^i(t, x, u, q, r, z) \leqslant \sigma\big(t, \max_j(\tilde{u}^j - u^j, \|\tilde{z} - z\|)\big) \quad (i = 1, \ldots, m),$$

where $\sigma(t, y)$ is continuous and non-negative for $t > 0$, $y \geqslant 0$, and such that $y(t) = 0$ is the unique solution of the ordinary differential equation $dy/dt = \sigma(t, y)$, satisfying the condition $\lim_{t\to 0} y(t) = 0$;

3° *$u(t, x) = (u^1(t, x), \ldots, u^m(t, x))$ is a regular solution of (0.1) in D and $v(t, x) = (v^1(t, x), \ldots, v^m(t, x))$ is a regular solution in D of the system*

$$(2.2) \quad v_t^i(t, x) \geqslant f^i(t, x, v(t, x), v_x^i(t, x), v_{xx}^i(t, x), v(t, \cdot)) \quad (i = 1, \ldots, m);$$

---

[1] Theorem 1 is in some direction more general than that in [4].

4° (2.3) $$u(0, x) \leqslant v(0, x), \quad x \in S_0,$$

(2.4) $$u(t, x) \leqslant v(t, x), \quad (t, x) \in \Sigma;$$

5° $f^i(t, x, u, q, r, z)$ $(i = 1, \ldots, m)$ *are parabolic with respect to* $u(t, x)$.

Under these assumptions we have

$$u(t, x) \leqslant v(t, x) \quad \text{for } (t, x) \in \bar{D}. \tag{2.5}$$

Proof. Put

$$M^i(t) = \max\{u^i(t, x) - v^i(t, x) \colon x \in S_t\}, \qquad M(t) = \max_i M^i(t).$$

Inequality (2.5) is equivalent with

$$M(t) \leqslant 0, \quad t \in \langle 0, T\rangle. \tag{2.6}$$

To prove (2.6) observe first that, D being of type C, we have (see [3], Theorem 33.1):

I. $M(t)$ is continuous in $\langle 0, T\rangle$,

and by (2.3)

II. $M(0) \leqslant 0$.

Now, we will show that

III. $D_-M(t) \leqslant \sigma\big(t, M(t)\big)$ for $t \in E = \{t \in (0, T) : M(t) > 0\}$.

Indeed, let $t_0 \in E$. There is an index j and a point $x_0 \in S_t$ such that

$$M(t_0) = M^j(t_0) = u^j(t_0, x_0) - v^j(t_0, x_0) > 0, \tag{2.7}$$

$$D_-M(t_0) \leqslant D^-M^j(t_0). \tag{2.8}$$

By (2.4) and (2.7) we see that (t_0, x_0) is an interior point of D, whence it follows by the definition of $M^j(t)$, that

$$u^j_x(t_0, x_0) = v^j_x(t_0, x_0), \tag{2.9}$$

$$u^j_{xx}(t_0, x_0) \leqslant v^j_{xx}(t_0, x_0). \tag{2.10}$$

Finally we get (see [3], Theorem 33.1)

$$D^-M^j(t_0) \leqslant u^j_t(t_0, x_0) - v^j_t(t_0, x_0),$$

whence, by (0.1), (2.2) and (2.8),

$$D_-M(t_0) \leqslant f^j\big(t_0, x_0, u(t_0, x_0), u^j_x(t_0, x_0) u^j_{xx}(t_0, x_0), u(t_0, \cdot)\big) - \\ - f^j\big(t_0, x_0, v(t_0, x_0), v^j_x(t_0, x_0) v^j_{xx}(t_0, x_0), v(t_0, \cdot)\big).$$

By (2.9), the last inequality can be written in the following form:

$$D_-M(t_0) \leqslant \big[f^j\big(t_0, x_0, u(t_0, x_0), u^j_x(t_0, x_0), u^j_{xx}(t_0, x_0), u(t_0, \cdot)\big) - \\ - f^j\big(t_0, x_0, u(t_0, x_0), u^j_x(t_0, x_0), v^j_{xx}(t_0, x_0), u(t_0, \cdot)\big)\big] + \\ + \big[f^j\big(t_0, x_0, u(t_0, x_0), v^j_x(t_0, x_0), v^j_{xx}(t_0, x_0), u(t_0, \cdot)\big) - \\ - f^j\big(t_0, x_0, v(t_0, x_0), v^j_x(t_0, x_0), v^j_{xx}(t_0, x_0), v(t_0, \cdot)\big)\big].$$

From 5° and (2.10) it follows that the first difference in brackets is non-positive, whence

(2.11) $$D_-M(t_0) \leqslant f^j\big(t_0, x_0, u(t_0, x_0), v^j_x(t_0, x_0), v^j_{xx}(t_0, x_0), u(t_0, \cdot)\big) - \\ - f^j\big(t_0, x_0, v(t_0, x_0), v^j_x(t_0, x_0), v^j_{xx}(t_0, x_0), v(t_0, \cdot)\big).$$

According to the definition of $M(t)$ and by (2.7) we get

$$u^j(t_0, x_0) = v^j(t_0, x_0) + M(t_0),$$
$$u^i(t_0, x) \leqslant v^i(t_0, x) + M(t_0), \quad x \in S_{t_0} \quad (i = 1, \ldots, m).$$

By the monotonicity property of f^j (see 1°) it follows from (2.11), by the last two formulas, that putting

$$\tilde{M}(t) = \big(M(t), \ldots, M(t)\big),$$

we have

(2.12) $$D_-M(t_0) \leqslant f^j\big(t_0, x_0, v(t_0, x_0) + \tilde{M}(t_0), v^j_x(t_0, x_0), v^j_{xx}(t_0, x_0), v(t_0, \cdot) + \\ + \tilde{M}(t_0)\big) - f^j\big(t_0, x_0, v(t_0, x_0), v^j_x(t_0, x_0), v^j_{xx}(t_0, x_0), v(t_0, \cdot)\big).$$

Since $M(t_0) > 0$, we have for the constant function $\tilde{M}(t_0)$

$$\|\tilde{M}(t_0)\| = M(t_0)$$

and finally, by (2.1) and (2.12).

$$D_-M(t_0) \leqslant \sigma\big(t_0, M(t_0)\big).$$

Thus we have proved III, and from I, II, III follows (see [3], § 14, Second Comparison Theorem) inequality (2.6).

Remark 1. If the functions f^i do not depend on the functional argument z, then Theorem 1 reduces to the known theorem on parabolic differential inequalities (see [3], Theorem 64.2)).

As a corollary to Theorem 1 we obtain

Maximum principle. *Let the right-hand sides of system* (0.1) *satisfy assumptons* 1° *and* 2° *of Theorem* 1. *Suppose* $u(t, x)$ *is a regular solution of* (0.1) *in* D, *satisfying inequalities*

(2.13) $$u(0, x) \leqslant A, \quad x \in S_{t_0},$$

(2.14) $$u(t, x) \leqslant A, \quad (t, x) \in \Sigma,$$

where $A = (A^1, \ldots, A^m) = \text{const}$. *Assume finally that hypothesis* 5° *of Theorem* 1 *holds true and that*

(2.15) $$f^i(t, x, A, 0, 0, A) \leqslant 0 \quad (i = 1, \ldots, m).$$

Under these assumptions we have

$$u(t, x) \leqslant A, \quad (t, x) \in \bar{D}.$$

3. THEOREM 2. *Let assumptions* $1^\circ, 3^\circ, 4^\circ$ *and* 5° *of Theorem* 1 *hold true. Replace* 2° *by a stronger assumption*:

6° *if* $r_{jk} = \tilde{r}_{jk}$ *for* $j \neq k$, *then we have*

$$|f^i(t, x, \tilde{u}, \tilde{q}, \tilde{r}, \tilde{z}) - f^i(t, x, u, q, r, z)| \leqslant a(t) \max_k (|\tilde{u}^k - u^k|, |\tilde{q}_k - q_k|, |\tilde{r}_{kk} - r_{kk}|, \|\tilde{z} - z\|),$$

where $a(t) \geqslant 0$ *is continuous for* $t \geqslant 0$.

Suppose finally that

7° $f^i(t, x, u, q, r, z)$ $(i = 1, \ldots, m)$ *are uniformly parabolic with respect to* $v(t, x)$ *in any compact subset of* D.

Under these assumptions we have inequalities (2.5) *and, moreover, if for some point* $(\tilde{t}, \tilde{x}) \in D$ *and some index* j *the equality*

$$u^j(\tilde{t}, \tilde{x}) = v^j(\tilde{t}, \tilde{x})$$

holds true, then

$$u^j(t, x) = v^j(t, x), \quad (t, x) \in S^-(\tilde{t}, \tilde{x}). \tag{3.1}$$

Proof. Inequality (2.5) follows by Theorem 1. Now suppose that the second assertion of Theorem 2 is not true; then there exist two points $(t^*, x^*) \in S^-(\tilde{t}, \tilde{x})$ and $(t^{**}, x^{**}) \in S^-(\tilde{t}, \tilde{x})$, $t^* < t^{**}$, such that the segment joining them is contained in D and

$$u^j(t^{**}, x^{**}) = v^j(t^{**}, x^{**}), \tag{3.2}$$

$$u^j(t^*, x^*) < v^j(t^*, x^*). \tag{3.3}$$

Denoting by $|x|$ the Euclidean norm of x and putting

$$\xi = (x^{**} - x^*)/(t^{**} - t^*)$$

we introduce the functions

$$\psi = \delta^2 - |x - x^* - \xi(t - t^*)|^2, \quad \varphi = \psi^2 e^{-at},$$

δ and a being positive constants to be determined later. Consider the oblique cylinder

$$\mathscr{C}\colon \psi > 0, \; t^* < t < t^{**},$$

with (t^*, x^*) and (t^{**}, x^{**}) as the centres of the bases. We choose δ so small that $\overline{\mathscr{C}} \subset D$ and that on the lower base B of $\mathscr{C}$ the inequality

$$u^j(t^*, x) < v^j(t^*, x), \quad (t^*, x) \in B \tag{3.4}$$

is satisfied. Put

$$f(t, x, w, q, r) = f^j\big(t, x, u^1(t, x), \ldots, u^{j-1}(t, x), w, u^{j+1}(t, x), \ldots, u^m(t, x), q, r, u(t, \cdot)\big).$$

By (0.1) we have obviously

$$u_t^j(t,x) \leqslant f\big(t,x,u^j(t,x),u_x^j(t,x),u_{xx}^j(t,x)\big), \quad (t,x)\in D. \tag{3.5}$$

Now, setting

$$w(t,x) = v^j(t,x) - \varepsilon\varphi(t,x), \quad \varepsilon > 0,$$

we will prove that

$$w_t(t,x) \geqslant f\big(t,x,w(t,x),w_x(t,x),w_{xx}(t,x)\big), \quad (t,x)\in\mathscr{C}, \tag{3.6}$$

provided that a be chosen large enough. To this effect, we substitute $v^j = w + \varepsilon\varphi$ into (2.2) and obtain

$$w_t \geqslant f^j\big(t,x,v(t,x),v_x^j(t,x),w_{xx}+\varepsilon\varphi_{xx},v(t,\cdot)\big) - \varepsilon\varphi_t. \tag{3.7}$$

A direct computation gives

$$\varphi_t = -ae^{-at}\psi^2 + 4e^{-at}\psi\xi\cdot\big(x - x^* - \xi(t-t^*)\big), \tag{3.8}$$

$$\varphi_x = -4e^{-at}\psi\big(x - x^* - \xi(t-t^*)\big), \tag{3.9}$$

$$\varphi_{xx} = -4e^{-at}\psi I + r, \tag{3.10}$$

where I is the identity matrix and r is the matrix with the elements

$$8e^{-at}\big(x_k - x_k^* - \xi_k(t-t^*)\big)\big(x_l - x_l^* - \xi_l(t-t^*)\big).$$

Since $\varepsilon r \geqslant 0$, it follows from (3.7) and (3.10), by hypothesis 7° (see (1.1)), that for $(t,x)\in\mathscr{C}$

$$w_t(t,x) \geqslant f^j\big(t,x,v(t,x),v_x^j(t,x),w_{xx} - 4e^{-at}\varepsilon\psi I, v(t,\cdot)\big) + \\ + 8\varepsilon\varkappa|x - x^* - \xi(t-t^*)|^2 e^{-at} - \varepsilon\varphi_t.$$

The last inequality and the monotonicity property 1° of f^j together with (2.5) imply that

$$w_t(t,x) \\ \geqslant f^j\big(t,x,u^1(t,x),\ldots,u^{j-1}(t,x),w(t,x)+\varepsilon\varphi,u^{j+1}(t,x),\ldots,u^m(t,x), \\ w_x(t,x)+\varepsilon\varphi_x, w_{xx}(t,x) - 4e^{-at}\varepsilon\psi I, u(t,\cdot)\big) + 8\varepsilon\varkappa|x - x^* - \xi(t-t^*)|^2 e^{-at} - \varepsilon\varphi_t.$$

Now, applying 6°, (3.8), (3.9) and the definition of $f(t,x,w,q,r)$, we get from the preceding inequality

$$w_t(t,x) \geqslant f\big(t,x,w(t,x),w_x(t,x),w_{xx}(t,x)\big) + \varepsilon g(t,x), \quad (t,x)\in\mathscr{C}, \tag{3.11}$$

where

$$g(t,x) = e^{-at}a\psi^2 + \big[8\varkappa|x - x^* - \xi(t-t^*)|^2 - 4\psi\xi\cdot\big(x - x^* - \xi(t-t^*)\big) - \\ - a(t)\max_j\big(\psi^2, 4\psi|x_j - x_j^* - \xi_j(t-t^*)|, 4\psi\big)\big]e^{-at}.$$

The expression in square brackets tends uniformly in $\mathscr{C}$ to $8\varkappa\delta^2 > 0$ as $\psi(t, x)$ tends to 0. Hence, there is a θ, such that $\delta > \theta > 0$ and $g(t, x) > 0$ for $(t, x)\epsilon\mathscr{C}$ with $0 < \psi(t, x) < \theta$. On the other hand, for $(t, x)\epsilon\mathscr{C}$ such that $\delta \geqslant \psi(t, x) \geqslant \theta > 0$, we can choose $a > 0$ so large as to make $a\psi^2$ larger than the absolute value of the expression in square brackets which is bounded independently of a. In this way, a being chosen sufficiently large we can make $g(t, x) > 0$ in $\mathscr{C}$, and thus, by (3.11), we get (3.6). In virtue of (3.4) we now choose an ε, $1 \geqslant \varepsilon > 0$, so small that

$$(3.12) \qquad u^j(t^*, x) \leqslant w(t^*, x), \qquad (t^*, x)\epsilon B.$$

Finally, observe that by (2.5) on the side surface of $\mathscr{C}$ we have

$$(3.13) \qquad u^j(t, x) \leqslant w(t, x),$$

since $w(t, x) = v^j(t, x)$ there.

According to assumption 5° we verify that the function $f(t, x, w, q, r)$ is parabolic with respect to $u^j(t, x)$. This last remark together with 6°, (3.5), (3.6), (3.12) and (3.13) shows that for $u^j(t, x)$, $w(t, x)$ and $f(t, x, w, q, r)$ all assumptions of the theorem on parabolic differential inequalities (see Remark 1) are satisfied in $\mathscr{C}$ with $m = 1$ and $\sigma(t, y) = a(t)y$. Therefore

$$u^j(t, x) \leqslant w(t, x) = v^j(t, x) - \varepsilon\varphi(t, x)$$

in $\bar{\mathscr{C}}$ and in particular, since $\varphi(t^{**}, x^{**}) > 0$,

$$u^j(t^{**}, x^{**}) < v^j(t^{**}, x^{**}),$$

what contradicts (3.2).

Remark 2. It is clear that in Theorem 2 assumption 4° and that on D to be of type C were used only to get inequality (2.5), whereas they were not explicitly used in the proof of the second assertion of Theorem 2. Hence, the following theorem is true: D being open (bounded or not) let assumptions 1°, 3°, 5°, 6° in any compact subset G of D (with $a(t)$ possibly depending on G) and 7° be satisfied. Suppose that inequality (2.5) holds true. Then, the second assertion of Theorem 2 is valid.

As a corollary to Theorem 2 we obtain

STRONG MAXIMUM PRINCIPLE. *Let the right-hand sides of system* (0.1) *satisfy assumptions* 1° *and* 6° *of Theorem* 2. *Suppose that* $u(t, x)$ *is a regular solution of* (0.1) *in* D, *satisfying inequalities* (2.13) *and* (2.14). *Assume finally that hypothesis* 5° *holds true and that inequalities* (2.15) *are satisfied, the functions* f^i *being uniformly parabolic with respect to the constant function* A.

Under these assumptions we have

$$u(t, x) \leqslant A, \quad (t, x) \in \bar{D},$$

and if, moreover, for some index j and some point $(\tilde{t}, \tilde{x}) \in D$ the equality $u^j(\tilde{t}, \tilde{x}) = A^j$ holds true, then

$$u^j(t, x) = A^j, \quad (t, x) \in S^-(\tilde{t}, \tilde{x}).$$

Remark 3. All theorems of the paper remain true if on some parts of Σ the boundary inequalities (2.4) are replaced by the following ones

$$\varphi^i\big(t, x, u^i(t, x)\big) - \varphi^i\big(t, x, v^i(t, x)\big) \leqslant a^i(t, x) d[u^i(t, x) - v^i(t, x)]/dl^i \quad (i = 1, \ldots, m),$$

where $a^i(t, x) \geqslant 0$, $\varphi^i(t, x, u)$ are strictly increasing with respect to u and the directions l^i are orthogonal to the t-axis.

References

[1] P. Besala, *An extension of the strong maximum principle for parabolic equations*, Bull. Acad. Polon. Sci. Sér. sci. math., astr. et phys. 19 (1971), p. 1003–1006.

[2] L. Nirenberg, *A strong maximum principle for parabolic equations*, Comm. Pure Appl. Math. 6 (1953), p. 167–177.

[3] J. Szarski, *Differential inequalities*, Warszawa 1967.

[4] — *Sur un système non linéaire d'inégalités différentielles paraboliques contenant des fonctionnelles*, Colloq. Math. 16 (1967), p. 141–145.

[5] W. Walter, *Differential- und Integral-Ungleichungen*, Springer – Verlag, 1964.

INSTYTUT MATEMATYKI UNIWERSYTETU JAGIELLOŃSKIEGO
MATHEMATICAL INSTITUTE OF THE JAGELLONIAN UNIVERSITY

Reçu par la Rédaction le 6. 1. 1973