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# Two-sided $L^1$-estimates for Szegö kernels on classical domains

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This paper is dedicated to the memory of Loo-Keng Hua. The first author presented it at the "Symposium for Hua", Beijing, PRC, August 1–7, 1988.

**Abstract.** Let $D$ be a bounded symmetric domain in $C^N (N > 1)$ with Bergman–Shilov boundary $b$ and $S(z, t)$ $(z \in D,\ t \in b)$ the Szegö kernel of $D$. The order of the integral $\int_b |S(rv, t)|\, ds_t$ $(v \in b,\ 0 \leqslant r < 1)$ is found for the classical domains $R_{IV}$ (the hyperbolic space of Lie spheres) and the matrix spaces $R_I(n, n)$, $R_{II}$ and $R_{III}$ ($n$ even) (using Hua's notation in [6]). An upper bound is obtained for $R_I (m < n)$ and $R_{III}$ ($n$ odd). The results are applied to the family of operators $\int_b S(z, t) f(t)\, ds_t \cdot R_{\gamma,\delta}(r)$ $(z \in D_r = \{rw\colon w \in D\}$ where $R_{\gamma,\delta}(r)$ is a function of the Szegö kernel $S(rv, rv)$ $(v \in b))$ and a related operator to obtain necessary and sufficient conditions on $\gamma$, $\delta$ for the operators to map $L^1(b)$ into $H^1(D)$. Using the Harish-Chandra realization of an irreducible bounded symmetric domain, Faraut and Korányi obtain the order of $\int_b |S(z, t)|^{1+q} ds_t$ for $q > q_0 > 0$ [1]. This gives a mapping theorem for a related operator to map $L^p(b)$ into $L^p(D)$ for $p \geqslant 1$.

## 1. Introduction

1. We begin with the family of operators

$$(\mathfrak{F} f)(z) = \int_b S(z, t) f(t)\, ds_t. \tag{1}$$

Here $b$ is the Bergman–Shilov (B–S) boundary of a classical domain $D$ in $C^N (N > 1)$, $S(z, w)$ $(z \in D,\ w \in$ the closure $D^-$ of $D)$ is the Szegö kernel of $D$([6], p. 88). Note that $S(z, w)$ is called the Cauchy kernel in [6]. Additional properties of the domains $D$ are described in Part 2.

The idea for this paper comes from the fact that the operator (1) does not map $L^1(b)$ into $L^1(b)$ independently of $r$. However, there is a well-known necessary and sufficient condtition for the operator $T_r$ given by

$$(T_r f)(z) = (\mathfrak{F} f)(z) \cdot \boldsymbol{R}(r) \tag{2}$$

$(z \in D_r = \{rw\colon w \in D\}$, $\boldsymbol{R}(r)$ a function of $r)$ to be a bounded operator on

$L^1(b)$ ([10], p. 377). Here the necessary and sufficient condition is given in the lemma. Our method of proof is to obtain two-sided estimates in terms of $r$ for the integral

$$(3) \qquad \int_b |S(rv, t)|\, ds_t \qquad (v \in b).$$

The exact order of (3) is in terms of the function

$$(4) \qquad R_{\alpha,\beta}(r) = (1-r)^{\alpha}/\log^{\beta}\frac{1}{1-r}$$

($\alpha \geqslant 0, \beta \geqslant 0$), where $\alpha$, $\beta$ depend on the classical domain $D$ and its dimension. (See Theorems 1 and 2.) These precise estimates are used to prove mapping theorems from $L^1(b)$ to $H^1(D)$ for the family of operators (2), which are independent of $r$. Necessary and sufficient conditions are obtained for the family (2) to map $L^1(b)$ into $H^1(D)$ and for a related family to map $L^1(b)$ into $L^1(b)$. (See Theorems 3 and 4.) A mapping theorem is also proved for $p \geqslant 1$. (See Theorem 5.) The estimates and mapping theorems are generalizations to $C^N$ $(N > 1)$ of the corresponding results for the unit disk.

For $N = 1$ an example of (2) is given by the family

$$F_r(\varphi) = \log^{-1}\frac{1}{1-r}(H_r f)(\varphi),$$

where $(H_r f)(\varphi) = \int_{1-r \leqslant |\theta| \leqslant 1} \theta^{-1} f(\theta - \varphi)\, d\theta$ is the cut-off Hilbert transform. (See [14], Chapter V, or [16].) In order to obtain $L^1$ into $L^1$ mapping properties of the family $\{H_r f\colon 0 \leqslant r < 1\}$ with bound independent of $r$ we must use the operators $F_r(\varphi)$. This is so because

$$\sup_{\|f\|_1 \neq 0} \frac{\|H_r f\|_1}{\|f\|_1} = O\left(\log\frac{1}{1-r}\right) \equiv c(r),$$

which is a well-known result. (See also lemma in Section 4.1.) Since $f \in L^1$, $\lim_{r\to 1-} (H_r f)(\varphi)$ exists for almost all $\varphi$ ([14], p. 132, Theorem 100). Hence $\lim_{r\to 1-} F_r(\varphi) = 0$ for almost all $\varphi$. This last fact was pointed out by the reviewer in [9] for operators similar to (2).

2. The domains $D$ considered are bounded symmetric domains in the complex vector space $C^N (N > 1)$ with $0 \in D$. They possess the following properties, which are used either explicitly or implicitly in the analysis. The domain $D$ has a group of holomorphic automorphisms $G$, which is transitive on $D$ and extends continuously to the topological boundary of $D$, $D$ has a (B–S) boundary $b$, which is a compact real-analytic submanifold of $C^N$. The domain $D$ is circular and star-shaped with respect to 0 and $b$ is circular; also, $b$ is

invariant under $G$, and the isotropy group $G_0 = \{g \in G\colon g(0) = 0\}$ is transitive on $b$ and can be represented by unitary matrices. The boundary $b$ has a unique normalized $G_0$-invariant measure $\mu$ given by $d\mu_t = (1/V)ds_t$, $V$ the Euclidean volume of $b$ and $ds_t$ the Euclidean volume element at $t \in b$. If $D$ is irreducible, it can be realized as either one of the classical domains $R_j$ ($j = I, II, III, IV$), which are generalizations of the unit disk in $C^1$, or one of the special domains with $N = 16$ or 27. The groups $G$ of the domains $R_j$ are classical semi-simple Lie groups. (See [4], [5], [6], [7].)

Any bounded symmetric domain has a Szegö kernel, $S(z, w)$, $z \in D$, $w \in D^-$, which is holomorphic in $(z, \bar{w})$ on $D \times D$, continuous on $D \times D^-$ and has a singularity on $b$ at $z = w$. Let $0 \leqslant r < 1$. Also the slice function $S_r$ defined by $S_r(z, w) = S(rz, w)$ is hermitian symmetric, that is $S_r(z, w) = \bar{S}_r(w, z)$. If $t \in b$, then

$$S(rt, rt) = \frac{1}{V(1-r^2)^N}$$

[6]. If $z \in D_r$, then

$$\int_b |S(z, t)|^p \, ds_t = O\big((1-r)^{-N(p-1)}\big) \quad \text{for } p \geqslant 2$$

[10]. For $R_I$ and $R_{II}$ the lower bound for $p$ is sharpened in Theorem 2 of [10]. Note that the function $R_{\alpha,\beta}$ of (4) can be expressed in terms of the Szegö kernel $S(rv, rv)$ $(v \in b)$. In [13] (pp. 17–19) the order of $\int_b |S(z, t)|^p ds_t$ is obtained for the complex unit ball. Also see [7]. Using the Harish-Chandra realization of irreducible bounded symmetric domains, Faraut and Korányi generalize these inequalities to bounded symmetric domains in Theorem 4.1 of [1]. For the Harish-Chandra realization and its connection with E. Cartan's classification of globally symmetric spaces, which includes the classical domains, see ([5], pp. 311–327, 281, 354 and [8]). Since the Szegö kernel is unique, the Szegö kernel of a bounded symmetric domain $D$, obtained from the Harish-Chandra realization is the same as the Szegö kernel of $D$ obtained from the E. Cartan classification.

In the remainder of the paper $A$, $B$, $C$,... are constants, depending on certain parameters but independent of $r$ and $f$. The constants are not necessarily the same at each occurrence. Also any complex powers are taken in the principal value sense.

## 2. The hyperbolic space $R_{IV}$ of Lie spheres

**1. Preliminaries.** The hyperbolic space of Lie spheres is given by

$$R_{IV} = \{z\colon |zz'|^2 + 1 - 2z\bar{z}' > 0,\ |zz'| < 1\} \quad (N \geqslant 2)$$

($\bar{z}'$ = conjugate transpose of $z$) with B–S boundary

$$b_{IV} = \{t\colon t = e^{i\theta}x,\ 0 \leqslant \theta \leqslant \pi,\ x \text{ a real vector with } xx' = 1\}.$$

The complex dimension of $R_{IV}$ is $N$ and the real dimension of $b_{IV}$ is $N$. Its Szegö kernel at the point $e = (1, 0, \ldots, 0) \in b_{IV}$ is

$$S(re, t) = (1/V)[1 + r^2 e^{-2i\theta} - 2rx_1 e^{-i\theta}]^{-N/2}.$$

We find upper and lower bounds for

$$I(re) = \int_b |S(re, t)|\, ds_t.$$

Set

$$g(r, x_1) = (1/V)\int_0^\pi |1 + r^2 e^{-2i\theta} - 2rx_1 e^{-i\theta}|^{-N/2}\, d\theta.$$

Using ([7], pp. 1040–1041),

$$I(re) = 2\pi \int_{xx'<1} g(r, x_1)\dot{x} \quad (\dot{x} \text{ the volume element of } xx' = 1)$$

$$= 2\pi \int_{x_1^2<1} g(r, x_1)\, dx_1 \int_{\tilde{x}\tilde{x}'<(1-x_1^2)} \dot{\tilde{x}} \quad (\tilde{x} = (x_2, \ldots, x_{N-2}))$$

$$= C \int_{x_1^2 \leqslant 1} g(r, x_1)(1 - x_1^2)^{\frac{1}{2}(N-3)}\, dx_1.$$

Following ([3], p. 527) set $x_1 = \cos\varphi$. Then

$$1 + r^2 e^{-2i\theta} - 2rx_1 e^{-i\theta} = (1 - re^{-i(\theta+\varphi)})(1 - re^{-i(\theta-\varphi)}) \quad \text{and}$$

$$|1 - re^{-i(\theta\pm\varphi)}|^2 = (1-r)^2 + 2r(1 - \cos(\theta \pm \varphi)) = [\ldots].$$

Thus

$$I(re) = C\int_0^\pi \sin^{N-2}\varphi\, d\varphi \int_0^\pi [\ldots]^{-N/4}[\ldots]^{-N/4}\, d\theta. \tag{1}$$

The order of (1) is given in

THEOREM 1. *Let $N$ be a positive integer $\geqslant 2$ and $0 \leqslant r < 1$. For the domain $D = R_{IV}(N)$,*

$$B\mathbf{R}_{\alpha,\beta}^{-1}(r) \leqslant \int_b |S(rv, t)|\, ds_t \leqslant A\mathbf{R}_{\alpha,\beta}^{-1}(r) \tag{2}$$

*for $v \in b$ and $r$ sufficiently close to 1, where $\alpha = (N/2) - 1$, $\beta = 0$ for $N > 2$, $\alpha = 0$, $\beta = 2$ for $N = 2$ and $\mathbf{R}_{\alpha,\beta}$ is given by (1.4).*

To prove the theorem we need the estimates

$$C_1 t^2 \leqslant 1 - \cos t \leqslant C_2 t^2 \quad \text{for } 0 \leqslant t \leqslant (11/6)\pi, \tag{3a}$$

$$C_3(2\pi - t)^2 \leqslant 1 - \cos t \leqslant C_4(2\pi - t)^2 \quad \text{for } \tfrac{1}{4}\pi \leqslant t \leqslant 2\pi, \tag{3b}$$

(3c) $$0 \leqslant \sin t \leqslant t \quad \text{if} \quad 0 \leqslant t \leqslant \pi,$$

(3d) $$0 \leqslant \sin t \leqslant (\pi - t) \quad \text{if} \quad \tfrac{1}{2}\pi \leqslant t \leqslant \pi,$$

and for $\frac{1}{2} \leqslant r < 1, N > 2$ the estimates

(4a) $$\int_{1-r}^{2\pi} x^{N/2-2}\,dx \int_0^{1-r} (1-r)^{-N/2}\,dt \leqslant C(1-r)^{-(N/2-1)},$$

(4b) $$\int_{1-r}^{2\pi} x^{N/2-2}\,dx \int_{1-r}^{x} t^{-N/2}\,dt \leqslant C(1-r)^{-(N/2-1)},$$

(5) $$\int_0^{\pi/2} x^{N-2}[(1-r)^2+x^2]^{-N/4}\,dx \int_0^{\pi-x} [(1-r)^2+t^2]^{-N/4}\,dt \leqslant C(1-r)^{-(N/2-1)}.$$

If $N = 2$ (4a) is bounded by $\log[1/(1-r)]$ and (4b) and (5) by $\log^2[1/(1-r)]$. In (4b) note that

$$\int_{1-r}^{x} t^{-N/2}\,dt \leqslant C(1-r)^{-(N/2-1)} \quad \text{and} \quad \int_{1-r}^{2\pi} x^{-N/2-2}\,dx \leqslant C$$

if $N > 2$ so that (4b) holds. The estimate is clear for $N = 2$. The proofs of (4a) and (5) are similar.

The method of proof to obtain (2) for $N > 2$ used here is to subdivide the integral $I(re)$ in (1) into convenient pieces and to use certain estimates to get the order. Another method to find the order of $I(re)$, that has been suggested, is to use the residue theorem to estimate the cases $N = 4M (M = 1, 2, 3, \ldots)$; then Hölder's inequality gives the intermediate cases $4M < N < 4(M+1)$. The cases $N = 2$ and $2 < N < 4$ are evaluated separately.

**2. Proof of the upper bounds in Theorem 1 for** $N > 2$. Since $I(re)$, given by (1), is bounded for $0 \leqslant r \leqslant 1/2$, we need only consider $1/2 < r < 1$. We break $I(re)$ into:

$$I = \Big\{ \int_0^{1-r} \int_0^{\varphi+(1-r)} + \int_{\pi-(1-r)}^{\pi} \int_{\varphi-(1-r)}^{\pi} \Big\} \mathscr{A}_r(\varphi, \theta) \equiv I_1 + I_2,$$

$$II = \int_{1-r}^{\pi-(1-r)} \int_{\varphi-(1-r)}^{\varphi+(1-r)} \mathscr{A}_r(\varphi, \theta),$$

$$III = \int_{1-r}^{\pi} \int_0^{\varphi-(1-r)} \mathscr{A}_r(\varphi, \theta),$$

$$IV = \Big\{ \int_0^{1-r} \int_{\varphi+(1-r)}^{\pi} + \int_{1-r}^{\pi/4} \int_{\varphi+(1-r)}^{\pi} + \int_{\pi/4}^{\pi-(1-r)} \int_{\varphi+(1-r)}^{\pi} \Big\} \mathscr{A}_r(\varphi, \theta)$$
$$\equiv IV_1 + IV_2 + IV_3,$$

where

$$\mathscr{A}_r(\varphi, \theta) = C\frac{(\sin\varphi)^{N-2}\,d\varphi\,d\theta}{[\ldots]^{N/4}[\ldots]^{N/4}}.$$

Estimates for $I$–$IV$. We show that $I$ is bounded and that $II$–$IV$ have the upper bound $O((1-r)^{-N/2+1})$.

(a) By (3a), (3b) and (3d)

$$IV_3 \leqslant C \int_{\pi/4}^{\pi-(1-r)} (\pi-\varphi)^{N-2}\, d\varphi \int_{\varphi+(1-r)}^{\pi} [(1-r)^2+(\theta-\varphi)^2]^{-N/4} \times [(1-r)^2+(2\pi-(\theta+\varphi))^2]^{-N/4}\, d\theta.$$

After making the change of variable $u=\theta-\varphi, v=\pi-\varphi$ the estimate follows by (4b). By similar arguments estimates for $IV_1$ and $IV_2$ are obtained.

(b) By (3c) and (3d) we have

$$III \leqslant C\{\int_{1-r}^{\pi/2} \varphi^{N-2}\, d\varphi + \int_{\pi/2}^{\pi} (\pi-\varphi)^{N-2}\, d\varphi\} \times \int_{0}^{\varphi-(1-r)} [\ldots]^{-N/4}[\ldots]^{-N/4}\, d\theta \equiv III_1 + III_2.$$

(i) By (3a) and (3b)

$$III_2 \leqslant C \int_{\pi/2}^{\pi} (\pi-\varphi)^{N-2}\, d\varphi \int_{0}^{\varphi-(1-r)} [(1-r)^2+(\theta-\varphi)^2]^{-N/4} \times [(1-r)^2+(2\pi-(\theta+\varphi))^2]^{-N/4}\, d\theta.$$

Set $u=\varphi-\theta$ and $\psi=\pi-\varphi$ and the estimate follows from (4b) and (5).

(ii) By (3a)

$$III_1 \leqslant C \int_{1-r}^{\pi/2} \varphi^{N-2}\, d\varphi \int_{0}^{\varphi-(1-r)} (\varphi-\theta)^{-N/2}(\varphi+\theta)^{-N/2}\, d\theta \leqslant C \int_{1-r}^{\pi/2} \varphi^{N/2-2}\, d\varphi \int_{1-r}^{\varphi} u^{-N/2}\, du,$$

were $u=\varphi-\theta$ and the result follows by (4b).

(c) For $II$ we have

$$II \leqslant C(1-r)^{-N/2}\{\int_{1-r}^{(7/8)\pi} + \int_{(7/8)\pi}^{\pi-(1-r)}\} \sin^{N-2}\varphi\, d\varphi \times \int_{\varphi-(1-r)}^{\varphi+(1-r)} [(1-r)^2+2r(1-\cos(\theta+\varphi))]^{-N/4}\, d\theta \equiv II_1 + II_2.$$

(i) By (3b) and (3d)

$$II_2 \leqslant C(1-r)^{-N/2} \int_{(7/8)\pi}^{\pi-(1-r)} (\pi-\varphi)^{N-2}\, d\varphi \int_{\varphi-(1-r)}^{\varphi+(1-r)} [2\pi-(\theta+\varphi)]^{-N/2}\, d\theta.$$

Setting $u = 2\pi-(\theta+\varphi)$ and then $s = 2(\pi-\varphi)-(1-r)$ gives

$$II_2 \leqslant C(1-r)^{-N/2} \int_{1-r}^{\pi/4-(1-r)} s^{N-2}\,ds \int_{s}^{s+2(1-r)} u^{-N/2}\,du,$$

from which the result follows by (4a).

(ii) By (3a) and (3c)

$$II_1 \leqslant C(1-r)^{-N/2} \int_{1-r}^{(7/8)\pi} \varphi^{N-2}\,d\varphi \int_{\varphi-(1-r)}^{\varphi+(1-r)} (\theta+\varphi)^{-N/2}\,d\theta$$
$$\leqslant C(1-r)^{1-N/2} \int_{1-r}^{(7/8)\pi} \varphi^{N-2}\,d\varphi \leqslant C(1-r)^{1-N/2}.$$

(d) In $I_2$ set $\psi = \pi-\varphi$, $\tau = \pi-\theta$. Then $I_2 = I_1$ and

$$I_1 \leqslant \int_0^{1-r} \varphi^{N-2}\,d\varphi \int_0^{\varphi+(1-r)} \left[(1-r)^2+2r(1-\cos(\varphi+\theta))\right]^{-N/4}\left[(1-r)^2\right.$$
$$\left.+2r(1-\cos(\varphi-\theta))\right]^{-N/4}\,d\theta$$
$$\leqslant C(1-r)^{-2}\int_0^{1-r} d\varphi \int_0^{2(1-r)} d\theta \leqslant C.$$

This completes the proof of the upper bound in (2).

**3. Proof of the lower bound in Theorem 1.** The integrands of $I$, $II$, $III_1$, $III_2$, $IV$ and intervals of integration are all non-negative for $r \geqslant 1/2$. Hence

$$I(re) \geqslant III \geqslant III_1 = C\int_{1-r}^{\pi/2}\int_0^{\varphi-(1-r)} \mathscr{A}_r(\varphi,\theta).$$

We show that the lower bound of $III_1$ for $N>2$ is $C(1-r)^{-N/2+1}$ and for $N=2$ it is $C\log^2[1/(1-r)]$.

For $III_1$ since $1-r \leqslant \varphi\pm\theta$, $1-\cos(\varphi\pm\theta) \leqslant C(\varphi\pm\theta)^2$, $\sin\varphi \geqslant \frac{2}{\pi}\varphi$ in $\left[0,\frac{\pi}{2}\right]$ and $\varphi+\theta \leqslant 2\varphi$, we have

$$I(re) \geqslant C\int_{1-r}^{\pi/2} \varphi^{N-2}\,d\varphi \int_0^{\varphi-(1-r)} (\varphi-\theta)^{-N/2}(\varphi+\theta)^{-N/2}\,d\theta \tag{6}$$
$$\geqslant C\int_{1-r}^{\pi/2} \varphi^{N/2-2}\,d\varphi \int_0^{\varphi-(1-r)} (\varphi-\theta)^{-N/2}\,d\theta.$$

Set $u = \varphi-\theta$ in the inner integral and integrate. We get if $N>2$

$$I(re) \geqslant C\int_{1-r}^{\pi/2}\left(\frac{\varphi^{N/2-2}}{(1-r)^{N/2-1}}-\varphi^{-1}\right)d\varphi \geqslant C\left(\frac{1}{s}-1-\log\frac{1}{s}\right),$$

where $s = [2(1-r)/\pi]^{N/2-1}$. But $1/2s \geqslant 1+\log(1/s)$ for $1-r$ sufficiently small. Thus $I(re) \geqslant C/(1-r)^{N/2-1}$. If $N = 2$ the inner integral in (6) is

$$\int_{1-r}^{\varphi} u^{-1}\,du = \log\varphi + \log\frac{1}{1-r} \geqslant \log\varphi, \quad \text{and} \quad I(re) \geqslant C\log^2\frac{1}{1-r}.$$

This completes the proof of Theorem 1.

Remark. Theorem 1 is likely true for any $p$, $0 < p < \infty$, but the details of the proof would be more complicated.

## 3. The matrix spaces $R_j (j = I, II, III)$

**1. Preliminaries.** The classical domains $R_j\,(j = I, II, III)$ are defined by

$$D = \{z\colon\ I - zz^* > 0\},$$

where $z$ is a matrix of complex numbers, $z^*$ its conjugate transpose and $I$ an identity matrix. If $D = R_I(m, n)$ $(m \leqslant n)$, $z$ is of order $m \times n$ and $I$ of order $m$; $R_I(1, n)$ is the complex unit ball in $C^n$; if $D = R_{II}(n)$, $z$ is a symmetric matrix of order $n$; if $D = R_{III}(n)$, $z$ is a skew-symmetric matrix of order $n$. The B–S boundary is given by

$$b = \{z\colon\ zz^* = I]$$

where $z$ is an $m \times n$ matrix for $R_I(m, n)$, a symmetric unitary matrix of order $n$ for $R_{II}(n)$, a skew-symmetric unitary matrix of order $n$ for $R_{III}(n)$, $n$ even. For $n$ odd

$$b = \{UDU'\colon\ U \text{ is unitary and } D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus \ldots \dotplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus 0\}.$$

The complex dimension of the domains $R_j$ is $mn$, $(n/2)(n+1)$, $(n/2)(n-1)$ respectively, while the real dimension of their B–S boundaries is $m(2n-m)$, $(n/2)(n+1)$, $[n+(1+(-1)^{n-1})]\,(n-1)/2$ respectively [6].

The Szegö kernel is

$$S(z, w) = \frac{1}{V\det^{\alpha}(I - zw^*)} \qquad (z \in D,\ w \in D^-),$$

where $V$ is the volume of the domain $b$, $\alpha = n$ for $R_I(m, n)$, $(n+1)/2$ for $R_{II}(n)$, $(n-1)/2$ for $R_{III}(n)$, $n$ even, and $\frac{1}{2}n$ for $R_{III}(n)$, $n$ odd.

**2. Upper and lower bounds for the integral**

$$I(rv) = \int_b |S(rv, t)|\,ds_t \qquad (v \in b,\ 0 \leqslant r < 1)$$

are given in

THEOREM 2. *For matrix spaces $R_I(n, n)$, $R_{II}(n)$, and $R_{III}$ $(n\ \text{even})$*

$$BR_{\alpha,\beta}^{-1}(r) \leqslant \int_b |S(rv,t)|\,ds_t \leqslant AR_{\alpha,\beta}^{-1}(r), \tag{1}$$

*for all $v \in b$ and $1-r$ sufficiently small. If $n$ is even for $R_I$ and $R_{II}$, then $\beta = 0$ and $\alpha = n^2/4$ for $R_I$ and $n^2/8$ for $R_{II}$; if $n$ is odd $\beta = 1$ and $\alpha = \frac{1}{4}(n^2-1)$ for $R_I$ and $\frac{1}{8}(n^2-1)$ for $R_{II}$. For $R_{III}$ if $n = 4u$, $u$ a positive integer, then $\beta = 0$ and $\alpha = n^2/8$, if $n = 4u-2$ then $\beta = 1$ and $\alpha = (n-4)^2/8$. Upper bounds for $R_I$ $(m < n)$ and $R_{III}$ ($n$ odd) are given by* (30) *and* (27).

Proof of Theorem 2. The upper bounds in (1) were obtained for the spaces $R_I(n,n)$ and $R_{II}(n)$ in [12].

**3. Lower bound for the space** $R_I(n) \equiv R_I(n,n)$. By formula (2.1) of [12]

$$I(rv) = C\prod_{k=1}^{n}\int_0^{2\pi}|1-re^{i\theta_k}|^{-n}\,d\theta_k \prod_{1\leqslant j<k\leqslant n}|e^{i\theta_j}-e^{i\theta_k}|^2$$

$$\geqslant C\prod_{k=1}^{\sigma}\int_0^{\pi/2}|1-re^{i\theta_k}|^{-n}\,d\theta_k \prod_{k=\sigma+1}^{n}\int_0^{3\pi/2}|1-re^{i\theta_k}|^{-n}\,d\theta_k \prod_{1\leqslant j<k\leqslant n}|e^{i\theta_j}-e^{i\theta_k}|^2,$$

where $\sigma$ is a positive integer to be chosen later. It can be shown that

$$I(rv) \geqslant C\prod_{k=1}^{\sigma}\int_0^{\pi/2^k}|1-re^{i\theta_k}|^{-n}d\theta_k \prod_{1\leqslant j<k\leqslant\sigma}|e^{i\theta_j}-e^{i\theta_k}|^2 \equiv C\mathfrak{A}$$

and

$$\mathfrak{A} \geqslant \int_{1-r}^{\pi/2^\sigma} d\theta_\sigma \int_{2\theta_\sigma}^{\pi/2^{\sigma-1}} d\theta_{\sigma-1}\dots\int_{2\theta_2}^{\pi/2} d\theta_1 \prod_{k=1}^{\sigma}|1-re^{i\theta_k}|^{-n}\prod_{1\leqslant j<k\leqslant\sigma}|e^{i\theta_j}-e^{i\theta_k}|^2. \tag{2}$$

From the inequality $|a-b| \geqslant |a|-|b|$ follows for $0 < r \leqslant 1$

$$|e^{i\theta_j}-e^{i\theta_k}|^2 \geqslant (|1-re^{i\theta_j}|-|1-re^{i\theta_k}|)^2, \tag{3}$$

also, there exists a constant $b$, $0 < b < 1$, independent of $r$, $\theta_j$ and $\theta_k$ such that if $\theta_j \geqslant 2\theta_k$ for $j < k$, $1-r < \theta_j < \pi$ and $r \geqslant \frac{1}{2}$

$$|1-re^{i\theta_k}| \leqslant b\,|1-re^{i\theta_j}|. \tag{4}$$

Under these restrictions on $\theta_j$, $\theta_k$ and $r$ by (3) and (4)

$$|e^{i\theta_j}-e^{i\theta_k}|^\alpha \geqslant B\,|1-re^{i\theta_j}|^\alpha \quad (B = 1-b) \tag{5}$$

for any $\alpha > 0$. Inequality (5) implies that

$$\prod_{k=j+1}^{\sigma}|e^{i\theta_j}-e^{i\theta_k}|^\alpha \geqslant B\,|1-re^{i\theta_j}|^{\alpha(\sigma-j)} \quad (1 \leqslant j < \sigma). \tag{6}$$

Also since $\theta_k \geqslant 1-r$ and $r \geqslant \frac{1}{2}$

(7) $$\theta_k^2 \geqslant \tfrac{1}{2}(\theta_k^2 + r\theta_k^2) \geqslant \tfrac{1}{2}[(1-r)^2 + r\theta_k^2] \geqslant \tfrac{1}{2}|1-re^{i\theta_k}|^2.$$

Use (6) with $\alpha = 2$ in (2). This gives

(8) $$\mathfrak{A} \geqslant C \int_{1-r}^{\pi/2^\sigma} d\theta_\sigma \int_{2\theta_\sigma}^{\pi/2^{\sigma-1}} d\theta_{\sigma-1} \cdots \int_{2\theta_2}^{\pi/2} d\theta_1 \prod_{k=1}^{\sigma} |1-re^{i\theta_k}|^{2(\sigma-k)-n}.$$

If $n$ is odd, take $\sigma = \frac{1}{2}(n+1)$. Then $2(\sigma-k)-n = 1-2k$ and by (7)

$$\mathfrak{A} \geqslant C \int_{1-r}^{\pi/2^\sigma} d\theta_\sigma \int_{2\theta_\sigma}^{\pi/2^{\sigma-1}} d\theta_{\sigma-1} \cdots \int_{2\theta_2}^{\pi/2} d\theta_1 \prod_{k=1}^{\sigma} \theta_k^{1-2k}.$$

In the following assume that $1-r$ is so small that the lower bounds are all positive. Note that $\theta_k < 1$ for $2 \leqslant k \leqslant \sigma$ so that $\theta_k < \theta_k^{1/2}$. Also, the set

(9) $$\{\theta: 2\theta_k \leqslant \theta \leqslant A\} \supset \{\theta:\ 2\theta_k \leqslant \theta \leqslant 2\theta_k^{1/2},\ \theta_k \leqslant (A/2)^2\},$$

since $2\theta_k^{1/2} \leqslant A$. Here $A$ takes on the values $\pi/2^k$. When $A = \pi/2$

$$\mathfrak{A} \geqslant C \int_{1-r}^{\pi/2^\sigma} d\theta_\sigma \int_{2\theta_\sigma}^{\pi/2^{\sigma-1}} d\theta_{\sigma-1} \cdots \int_{2\theta_3}^{\pi/2^2} d\theta_2 \prod_{k=2}^{\sigma} \theta_k^{1-2k} \int_{2\theta_2}^{2\theta_2^{1/2}} \theta_1^{-1}\, d\theta_1.$$

The inner integral equals $\frac{1}{2}\log(1/\theta_2)$. Repeating with $A_1 = (A/2)^2$

$$\mathfrak{A} \geqslant C \int_{1-r}^{\frac{\pi}{2^{\sigma-3}}\left(\frac{A_1}{2}\right)^2} d\theta_\sigma \int_{2\theta_\sigma}^{\frac{\pi}{2^{\sigma-4}}\left(\frac{A_1}{2}\right)^2} d\theta_{\sigma-1} \cdots \int_{2\theta_4}^{\left(\frac{A_1}{2}\right)^2} d\theta_3 \prod_{k=3}^{\sigma} \theta_k^{1-2k} \int_{2\theta_3}^{2\theta_3^{1/2}} \theta_2^{-3} \log(1/\theta_2)\, d\theta_2.$$

The inner integral $\geqslant B\log(1/2^2\theta_3)\,\theta_3^{-2}$ so that

$$\mathfrak{A} \geqslant C \int_{1-r}^{\frac{\pi}{2^{\sigma-3}}\left(\frac{A_1}{2}\right)^2} d\theta_\sigma \int_{2\theta_\sigma}^{\frac{\pi}{2^{\sigma-4}}\left(\frac{A_1}{2}\right)^2} d\theta_{\sigma-1} \cdots \int_{2\theta_4}^{\left(\frac{A_1}{2}\right)^2} d\theta_3 \prod_{k=3}^{\sigma} \theta_k^{1-2k}\, \theta_3^{-2} \log(1/2^2\theta_3)\, d\theta_2.$$

Notice that upon integrating the power of $\theta_2$ is decreased by 1 and the subscript in the denominator increased by 1, that is, $\theta_2^3 \to \theta_3^2$. Repeating the above argument the exponent decreases by 1 for each $\theta_k$, $2 \leqslant k \leqslant \sigma-1$, while the subscript $k$ increases by 1. Thus we get

$$\mathfrak{A} \geqslant C \int_{1-r}^{(A_{\sigma-2}/2)^2} d\theta_\sigma\, \theta_\sigma^{-n} \log\frac{1}{c\theta_\sigma} \prod_{k=3}^{\sigma-1} \theta_\sigma^{2-2k} \geqslant C \int_{1-r}^{\sqrt{1-r}} \log\frac{1}{c\theta}\, \theta^{-(\sigma^2-\sigma+1)}\, d\theta,$$

where $c = 2^\alpha$, $\alpha > 1$. Since $(c\theta)^{-1} \geqslant \left(c^2(1-r)\right)^{-1/2}$ and $\log(1/c\theta) \geqslant$ $\geqslant B\log\left(1/(1-r)\right)$ for $1-r$ sufficiently small

(10) $$I(rv) \geqslant C\log\frac{1}{1-r} \int_{1-r}^{\sqrt{1-r}} \theta^{-(\sigma^2-\sigma+1)}\, d\theta = C(1-r)^{-(n^2-1)/4} \log\frac{1}{1-r}.$$

If $n$ is even take $\sigma = n/2$. Repeating the above procedure, we get

(11) $$I(rv) \geqslant C(1-r)^{-n^2/4}.$$

The lower bounds (10) and (11) for $I(rv)$ for the domain $R_I(n)$ have the same order as the upper bounds in [12].

**4. Lower bound of $I(rv)$ for the space $R_{II}(n)$.** We proceed as for $R_I(n)$. By formula (2.2) of [12]

$$I(rv) = C\prod_{k=1}^{n}\int_{0}^{2\pi} d\theta_k|1-re^{i\theta_k}|^{-(n+1)/2}\prod_{1\leqslant j<k\leqslant n}|e^{i\theta_j}-e^{i\theta_k}|.$$

Similarly as for (8), by (6) with $\alpha = 1$

$$(12)\qquad I(rv) \geqslant C\int_{1-r}^{\pi/2^\sigma} d\theta_\sigma \int_{2\theta_\sigma}^{\pi/2^{\sigma-1}} d\theta_{\sigma-1}\dots\int_{2\theta_2}^{\pi/2} d\theta_1\prod_{k=1}^{\sigma}|1-re^{i\theta_k}|^{\sigma-k-(n+1)/2},$$

If $n$ is odd, take $\sigma = (n+1)/2$ which gives by (7)

$$(13)\qquad I(rv) \geqslant C\int_{1-r}^{\pi/2^\sigma} d\theta_\sigma \int_{2\theta_\sigma}^{\pi/2^{\sigma-1}} d\theta_{\sigma-1}\dots\int_{2\theta_2}^{\pi/2} d\theta_1\prod_{k=1}^{\sigma}\theta_k^{-k}$$

$$\geqslant C\log\frac{1}{1-r}\int_{1-r}^{\pi/2^\sigma} d\theta\,\theta^{-(\sigma^2-\sigma+2)/2} \geqslant C(1-r)^{-(n^2-1)/8}\log\frac{1}{1-r}.$$

If $n$ is even take $\sigma = (n+2)/2$, which gives

$$(14)\qquad I(rv) \geqslant C(1-r)^{-n^2/8}.$$

The lower bounds for $I(rv)$ for $R_{II}(n)$ given by (13) and (14) have the same order as the upper bounds in [12].

**5. Bounds for $I(rv)$ for the space $R_{III}(n)$.** For $R_{III}(n)$

(15) $I(rk_0) = \dfrac{1}{V}\int_b|\det(I+rk_0\bar{t})|^{-\alpha}\,ds_t$, $\alpha = \dfrac{n-1}{2}$ for $n$ even and $\alpha = \frac{1}{2}n$ for $n$ odd

where $k_0$ is a skew-symmetric unitary matrix and $0 \leqslant r < 1$. Let $n$ be even. Without loss of generality take

$$k_0 = D_1 = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} \dotplus \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} \dotplus \dots \dotplus \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\qquad (m = \tfrac{1}{2}n \text{ times}).$$

By ([11], p. 385)

$$|\det(I+rD_1\bar{t})| = |\det(I-rtD_1')| = \prod_{k=1}^{m}|1-re^{i\varphi_k}|^2$$

since $tD_1'$ is unitary and hence unitarily equivalent to a diagonal matrix $d = [e^{i\theta_1},\dots,e^{i\theta_n}]$, where $\theta_{2k-1} = \theta_{2k} = \varphi_k$ $(1 \leqslant k \leqslant m)$. Under the transformation $v = tD_1'$, $b \to b' = \{v\colon v = tD_1' = Ud\bar{U}',\ t\in b,\ U \text{ unitary}\}$. By a calculation following ([6], p. 56) and using ([15], p. 60) the volume element of $b'$ is

$$ds_v = C_n^{1/2}\prod_{1\leqslant j<k\leqslant m}|e^{i\varphi_j}-e^{i\varphi_k}|^4\,d\varphi_1\dots d\varphi_m\,\dot{U}.$$

Then (15) with $k_0 = D_1$ equals

$$(16)\qquad I(rD_1) = C \prod_{k=1}^{m} \int_0^{2\pi} |1-re^{i\varphi_k}|^{-(n-1)} d\varphi_k \prod_{1 \leqslant j<k \leqslant m} |e^{i\varphi_j}-e^{i\varphi_k}|^4.$$

Upper bounds for the integral (16). The lemma in ([12], p. 374) with $a_j = |1-re^{i\varphi_j}|$, $\alpha = 2$, $\beta = n-1$, gives

$$I(rD_1) \leqslant \prod_{k=1}^{m} \int_0^{2\pi} |1-re^{i\varphi_k}|^{n+1-4k} d\varphi_k.$$

It is well known that $\int_0^{2\pi} |1-re^{i\varphi_k}|^{-q} d\varphi_k = O\big((1-r)^{-(q-1)}\big)$ if $q>1$, $= O\big(\log(1/(1-r))\big)$ if $q=1$ and $= O(1)$ if $q<1$.

If $n = 4u$, $u$ a positive integer, $4k-n-1 < 1$ for $1 \leqslant k \leqslant u$ and $>1$ for $u+1 \leqslant k \leqslant m = 2u$. This gives

$$(17)\qquad I(rD_1) \leqslant C \prod_{k=u+1}^{2u} \int_0^{2\pi} |1-re^{i\varphi_k}|^{n+1-4k} d\varphi_k \leqslant C(1-r)^{-n^2/8}.$$

If $n = 4u-2$, $4k-n-1 = 1$ for $k = u$, $<1$ for $1 \leqslant k \leqslant u-1$ and $>1$ for $u+1 \leqslant k \leqslant 2u-1 = m$. This gives

$$(18)\qquad I(rD_1) \leqslant C(1-r)^{-(n^2-4)/8} \log\frac{1}{1-r}.$$

Lower bounds for the integral (16). As for $R_I(n)$ and $R_{II}(n)$ by (6) with $\alpha = 4$ (see (8) and (12))

$$(19)\quad I(rD_1) \geqslant C \int_{1-r}^{\pi/2^\sigma} d\varphi_\sigma \int_{2\varphi_\sigma}^{\pi/2^{\sigma-1}} d\varphi_{\sigma-1} \cdots \int_{2\varphi_2}^{\pi/2} d\varphi_1 \prod_{k=1}^{\sigma} |1-re^{i\varphi_k}|^{4(\sigma-k)-(n-1)}.$$

Choose $\sigma$ as the largest integer $<(n+3)/4$. If $n/4$ is an integer, $\sigma = n/4$. If $(n+2)/4$ is an integer, $\sigma = (n+2)/4$. The integrands in (19) are $\prod_{k=1}^{\sigma} |1-re^{i\varphi_k}|^{-(4k-1)}$ if $\sigma = n/4$ and $\prod_{k=1}^{\sigma} |1-re^{i\varphi_k}|^{-(4k-3)}$ if $\sigma = (n+2)/4$. By (7) these integrals are greater than or equal to

$$(20)\qquad C\prod_{k=1}^{\sigma} \varphi_k^{-(4k-1)} \quad (\sigma = n/4), \qquad C\prod_{k=1}^{\sigma} \varphi_k^{-(4k-3)} \quad (\sigma = (n+2)/4)$$

respectively. Integrate (20) with respect to $\varphi_k (1 \leqslant k \leqslant \sigma-1)$. By the same reasoning as in Part 3 we get

$$(21)\qquad I(rD_1) \geqslant C(1-r)^{-n^2/8} \quad \text{if} \quad u = n/4$$

and

$$(22)\qquad I(rD_1) \geqslant C(1-r)^{-(n^2-4)/8} \log\frac{1}{1-r} \quad \text{if} \quad u = (n+2)/4.$$

The order of the lower bounds given in (21) and (22) for $R_{III}$ ($n$ even) agrees with the order of the upper bounds (17) and (18) respectively.

**6a. The case $R_{III}$ ($n$ odd).** By ([7], p. 1073) the closure of $R_{III}(n)$ can be embedded into that of $R_{III}(n+1)$ and $b_n = b_{III}(n) \subset b_{n+1} = b_{III}(n+1)$.

The Szegö kernel of $R_{III}(n)$ is

$$(23) \qquad S(z,t) = 1/V_n \det(I+z\bar{t})^{n/2} \quad (z \in R_{III}(n),\ t \in b_n \text{ and } V_n = V(b_n)).$$

By ([7], p. 1073) any $t_1 \in b_{n+1}$ can be written in the form $t_1 = \begin{pmatrix} t & U'h' \\ -hU & 0 \end{pmatrix}$ $t = U'DU$, $U$ an arbitrary unitary matrix and $h = (0,\ldots,0,e^{i\theta})$, $0 \leqslant \theta \leqslant 2\pi$. Thus $t \in b_n$. Also

$$(24) \qquad V_n = (1/2\pi)\, V_{n+1}.$$

Let $z_1 = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$. It is easy to check that $z_1 \in R_{III}(n+1)$. Also

$$(25) \qquad \det(I^{n+1} + z_1\bar{t}_1) = \det(I^n + z\bar{t}).$$

Hence, by (25), $\det(I^{n+1}+z_1\bar{t}_1)$ is independent of $\bar{U}'\bar{h}'$ and by (24) we get the formula

$$(26) \qquad \frac{1}{V_n}\int_{b_n} \frac{ds_t}{|\det(I^n+z\bar{t})|^{n/2}} = \frac{1}{V_{n+1}}\int_{b_{n+1}} \frac{ds_{t_1}}{|\det(I^{n+1}+z_1\bar{t}_1)|^{n/2}}$$

Now set $z = rv$, where $v \in b_n$, $0 \leqslant r < 1$ and $v_1 = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}$; $v_1 \notin b_{n+1}$, $v_1 \in \bar{R}_{III}(n+1) - R_{III}(n+1) - b(n+1)$. Let $v_0$ be an arbitrary point of $b_{n+1}$. Then

$$(27) \qquad \sup_{v \in b_n} \frac{1}{V_n}\int_{b_n} \frac{ds_t}{|\det(I^n+rv\bar{t})|^{n/2}} = \sup_{v_1 \in b_{n+1}} \frac{1}{V_{n+1}}\int_{b_n} \frac{ds_{t_1}}{|\det(I^{n+1}+rv_1\bar{t}_1)|^{n/2}}$$
$$\leqslant \sup_{v_0 \in b_{n+1}} \frac{1}{V_{n+1}}\int_{b_{n+1}} \frac{ds_{t_1}}{|\det(I^{n+1}+rv_0\bar{t}_1)|^{n/2}}$$

by the maximum principle.

The upper bound for $R_{III}$ ($n$ odd) follows by (26) and (27) from the upper bound for $R_{III}(n+1)$.

**6b. The case $R_I\,(m<n)$.** Here we wish to find the order of the integral

$$(28) \qquad \int_{b_{mn}} |S(rv,t)|\, ds_t,$$

where $b_{mn}$ is the B–S boundary of $R_I\,(m<n)$ and $v \in b_{mn}$.

Upper bound for (28). We obtain a formula for (28) over the B-S boundary $b_n$ of $R_I(n)$. Then use the upper bound given in Theorem 2. Let $z_1 = \begin{pmatrix} z \\ 0 \end{pmatrix}$, $t_1 = \begin{pmatrix} t \\ u \end{pmatrix}$. The point $z_1 \in R_I(n)$ since $I^n - z_1 z_1^* = \begin{pmatrix} I^m - zz^* & 0 \\ 0 & I^{n-m} \end{pmatrix}$ is positive definite. However, note that the point $v_1 = \begin{pmatrix} v \\ 0 \end{pmatrix} (v \in b_{mn}) \notin b_n$ since $v_1 v_1^* = \begin{pmatrix} I^m & 0 \\ 0 & 0 \end{pmatrix} \neq I^n$.

Now $\det(I^n - z_1 t_1^*) = \det(I^m - zt^*)$ so that by ([6], p. 94)

$$\int_{b_{mn}} |S(z,t)|\, ds_t = \frac{1}{V_{mn}} \int_{b_{mn}} \frac{ds_t}{|\det(I^m - zt^*)|^n} = \frac{1}{V_n} \int_{b_n} \frac{ds_{t_1}}{|\det(I^n - z_1 t_1^*)|^n}. \tag{29}$$

Set $z = rv$ $(v \in b_{mn})$, $z_1 = rv_1 = r\begin{pmatrix} v \\ 0 \end{pmatrix}$. By (29), the maximum principle and Theorem 2

$$\begin{aligned} \int_{b_{mn}} |S(rv,t)|\, ds_t &\leqslant \sup_{u_1 \in b_n} \int_{b_n} |S(ru_1, t_1)|\, ds_{t_1} \\ &\leqslant A(1-r)^{-n^2/4} && \text{if } n \text{ is even,} \\ &\leqslant A(1-r)^{-(n^2-1)/4} \log\frac{1}{1-r} && \text{if } n \text{ is odd.} \end{aligned} \tag{30}$$

($A$ independent of $r$.)

Lower bound for (28). Since there is no minimum principle, it is not possible to get a lower bound by this method. A determinantal inequality due to Hua (Scientia Sinica, Notes 1, vol. XIV, No. 5 (1964)), viz, for $I - ww^* > 0$, $I - zz^* > 0$,

$$\det(I - ww^*)\det(I - zz^*) \leqslant |\det(I - wz)|^2 \leqslant \det(I + ww^*)\det(I + zz^*),$$

yields

$$\frac{1}{|\det(I^n - rv_1 t_1^*)|^2} \geqslant \frac{1}{(1+r)^{2n}},$$

so that by (29)

$$\int_{b_{mn}} |S(rv,t)|\, ds_t \geqslant \frac{1}{V_n(1+r)^{n^2}}. \tag{31}$$

This completes the proof of Theorem 2.

## 4. Mapping theorems

1. Let $\{U_r\}$ be a family of linear operators defined on a measurable set $b$ and depending on a parameter $r$, $0 \leqslant r < 1$. We say that the family $\{U_r;\ 0 \leqslant r < 1\}$ *maps $L^1(b)$ into $L^1(b)$ uniformly* if

$$\|U_r f\|_1 \leqslant C\|f\|_1,$$

where $C$ is independent of $f$ and $r$.

For example, the family $\{F_r(\varphi)\colon 0 \leqslant r < 1\}$ of Section 1.1 maps $L^1([-\pi, \pi])$ into $L^1([-\pi, \pi])$ uniformly.

Let $D$ be a bounded circular domain in $C^N$ with $0 \in D$, which is star-shaped with respect to 0 and has a measurable B–S boundary $b$. Let $K(z, w)$ be a measurable function defined on $D \times D^-$ with the properties:

(i) the slice function $K_r$ is hermitian symmetric on $b \times b$;

(ii) for each $r$, $0 \leqslant r < 1$, and $v \in b, \int_b |K_r(v, t)|\, ds_t \leqslant B_r < \infty$, where $B_r$ is independent of $v$.

Let $T_r$ be the operator defined as in Section 1 by

$$(T_r f)(z) = \int_b K(z, t) f(t)\, ds_t \qquad (z \in D_r, f \in L^1(b)). \tag{1}$$

We give a necessary and sufficient condition that the family of operators $\{T_r\colon 0 \leqslant r < 1\}$ maps $L^1(b)$ into $L^1(b)$ uniformly.

LEMMA. *A necessary and sufficient condition that the family of operators $\{T_r\colon 0 \leqslant r < 1\}$, given by* (1), *maps uniformly from $L^1(b)$ to $L^1(b)$ is that*

$$\sup_{0 \leqslant r < 1} \operatorname*{ess\,sup}_{v \in b} \int_b |K(rv, t)|\, ds_t \leqslant C < \infty. \tag{2}$$

Proof. Condition (2) is sufficient. The inequality $\|T_r f\|_1 \leqslant C\|f\|_1$ follows immediately from Fubini's theorem, the hermitian symmetry of $K_r(v, t)$ and (2).

Condition (2) is necessary. Assume for arbitrary $f \in L^1(b)$ that

$$\|T_r f\|_1 \leqslant C\|f\|_1, \tag{3}$$

where $C$ is independent of $f$ and $r$, and prove that (2) holds.

If $g \in L^\infty(b)$, then by Fubini's theorem, the hermitian symmetry of $K_r(v, t)$ and (3)

$$\begin{aligned} \Big|\int_b f(v)\, ds_v \int_b K_r(v, t) g(t)\, ds_t\Big| &\leqslant \|g\|_\infty \int_b ds_t \Big|\int_b K_r(t, v) \bar{f}(v)\, ds_v\Big| \\ &\leqslant C\|g\|_\infty \|f\|_1, \end{aligned} \tag{4}$$

where $C$ independent of $f$, $r$ and $g$. The linear functional given by

$$S_g(f) = \int_b f(v)\, ds_v \int_b K_r(v, t) g(t)\, ds_t \qquad (f \in L^1(b),\ g \in L^\infty(b))$$

is bounded, since by (4)

$$\|S_g\| = \sup_{\|f\|_1 \neq 0} \frac{|S_g(f)|}{\|f\|_1} \leqslant C\|g\|_\infty,$$

where $C$ is independent of $g$ and $r$. The rest of the proof of condition (2) is standard and the necessity of the lemma follows.

**2. First mapping theorem.** Apply the lemma to the domains $R_j$. Let

$$K_{\gamma,\delta}(z,w) = S(z,w)\,R_{\gamma,\delta}(r) \quad (z \in D_r,\ w \in D^-),$$

where $R_{\gamma,\delta}$ is defined by (1.4) with values $\gamma$, $\delta$ related to $\alpha$, $\beta$ in Theorems 1 and 2. $K_{\gamma,\delta}$ satisfies properties (i) and (ii) of Section 1; (ii) follows since $K_{\gamma,\delta}$ is continuous. By (1)

$$(5) \qquad (T^{(r)}_{\gamma,\delta} f)(z) = \int_b S(z,t) f(t)\, ds_t\, R_{\gamma,\delta}(r) \quad (z \in D_r).$$

Then:

THEOREM 3. *Let $D$ be one of the classical domains $R_I(n,n)$, $R_{II}$, $R_{III}$ ($n$ even) or $R_{IV}$ with* B–S *boundary $b$. The family of operators $\mathfrak{F}_{\gamma,\delta} = \{T^{(r)}_{\gamma,\delta}\colon 0 \leqslant r < 1\}$, $T^{(r)}_{\gamma,\delta} f$ given by (5) maps uniformly from $L^1(b)$ to $H^1(D)$ if and only if either $\gamma > \alpha$, $\delta$ arbitrary or $\gamma = \alpha$, $\delta \geqslant \beta$. For $R_I(m<n)$ and $R_{III}$($n$ odd) the condition is sufficient for the family $\mathfrak{F}_{\gamma,\delta}$ to map uniformly from $L^1(b)$ to $H^1(D)$.*

Proof. Assume that either $\gamma > \alpha$, $\delta$ arbitrary or $\gamma = \alpha$, $\delta \geqslant \beta$ and prove that $T^{(r)}_{\gamma,\delta}$ maps $L^1(b)$ uniformly into $H^1(D)$.

By Theorems 1 and 2 and the hermitian symmetry of $S_r(v,t)$

$$B \leqslant \int_b |S(rv,t)|\, ds_v\, R_{\alpha,\beta}(r) \leqslant A.$$

Multiply by $R_{\gamma-\alpha,\delta-\beta}(r)$. Since $(1-r)^a/\log^b[1/(1-r)]$ is bounded on $0 \leqslant r < 1$ if $a > 0$, $b$ arbitrary or $a = 0$, $b \geqslant 0$, we have

$$(6) \qquad BR_{\gamma-\alpha,\delta-\beta}(r) \leqslant \int_b |S(rv,t)|\, ds_v R_{\gamma,\delta}(r) \leqslant AR_{\gamma-\alpha,\delta-\beta}(r) \leqslant A.$$

Since $T^{(r)}_{\gamma,\delta} f$ is holomorphic in $z$ on $D$, we use the $H^1$ metric

$$\|T^{(r)}_{\gamma,\delta} f\|_{H^1} = \sup_{0 \leqslant \varrho < 1} \int_b |(T^{(r)}_{\gamma,\delta} f)(\varrho r v)|\, ds_v.$$

By (5), Fubini's theorem and the monotonicity of the mean $\int_b |S(\varrho rv,t)|\, ds_v$ in $\varrho$ ([4], p. 523)

$$\|T^{(r)}_{\gamma,\delta} f\|_{H^1} \leqslant \int_b |f(t)|\, ds_t \int_b |S(rv,t)| ds_v R_{\gamma,\delta}(r) \leqslant A\, \|f\|_1$$

by (6), where $A$ is independent of $f$ and $r$. Thus the family $\mathfrak{F}_{\gamma,\delta}$ maps $L^1(b)$ uniformly into $H^1(D)$ if $\gamma > \alpha$, $\delta$ arbitrary or $\gamma = \alpha$, $\delta \geqslant \beta$.

Conversely assume that the family $\mathfrak{F}_{\gamma,\delta}$ maps $L^1(b)$ uniformly into $H^1(D)$ for the domains $R_I(n,n)$, $R_{II}$, $R_{III}$ ($n$ even) and $R_{IV}$ and prove that either $\gamma > \alpha$, $\delta$ arbitrary or $\gamma = \alpha$, $\delta \geqslant \beta$. We give a proof by contradiction. Suppose first that $\gamma < \alpha$ and $\delta$ is arbitrary. In (6) take sup with respect to $t$, giving

$$BR_{\gamma-\alpha,\delta-\beta}(r) \leqslant \sup_{t\in b} \int_b |S(rv,t)|\, ds_v\, R_{\gamma,\delta}(r) \leqslant A\, R_{\gamma-\alpha,\delta-\beta}(r).$$

Hence

$$\sup_{0\leqslant r<1} \sup_{v\in b} \int_b |S(rv,t)|\, ds_t\, R_{\gamma,\delta}(r) = \infty \tag{7}$$

and by the Lemma the family $\mathfrak{F}_{\gamma,\delta}$ does not map $L^1(b)$ uniformly into $H^1(D)$. Thus $\gamma \geqslant \alpha$. Similarly if $\alpha = \gamma$ and $\delta < \beta$ (7) holds and $\mathfrak{F}_{\gamma,\delta}$ does not map $L^1(b)$ uniformly into $H^1(D)$.

**3. Second mapping theorem.** Let $D = R_I$, $R_{II}$, $R_{III}$ or $R_{IV}$. Take $\varepsilon$, $0 < \varepsilon < 1$, and define the operator $L_\varepsilon^{\gamma,\delta}$ by

$$(L_\varepsilon^{\gamma,\delta} f)(v) = \inf_{1-\varepsilon\leqslant r<1} \int_b |S(rv,t)|\, |f(t)|\, ds_t\, R_{\gamma,\delta}(r)$$

$(v\in b,\ f\in L^1(b))$. The function $L_\varepsilon^{\gamma,\delta} f$ is non-decreasing in $\varepsilon$ as $\varepsilon\to 0$. Hence the limit exists as an extended real-valued function. Let

$$L^{\gamma,\delta} f = \lim_{\varepsilon\to 0} L_\varepsilon^{\gamma,\delta} f. \tag{8}$$

Then:

THEOREM 4. *Let $D$ be one of the classical domains $R_I(n,n)$, $R_{II}$, $R_{III}$ ($n$ even) or $R_{IV}$ with* B–S *boundary $b$ and $f\in L^1(b)$. Then $L^{\gamma,\delta} f$ given by* (8) *is a bounded operator from $L^1(b)$ to $L^1(b)$ if and only if either $\gamma > \alpha$, $\delta$ arbitrary or $\gamma = \alpha$, $\delta \geqslant \beta$. The condition is sufficient for $R_I$ ($m < n$) and $R_{III}$ ($n$ odd).*

Proof. Assume that either $\gamma > \alpha$, $\delta$ arbitrary or $\gamma = \alpha$, $\delta \geqslant \beta$ and prove that $L^{\gamma,\delta} f$ maps $L^1(b)$ into. By a property of inf, Fubini's theorem and Theorems 1 and 2 we have

$$\int_b (L_\varepsilon^{\gamma,\delta} f)(v)\, ds_v \leqslant A\, \|f\|_1\, R_{\gamma-\alpha,\delta-\beta}(r) \leqslant A\, \|f\|_1$$

($A$ independent of $\varepsilon$), since $R_{\gamma-\alpha,\delta-\beta}(r)$ is bounded. Hence by the monotone convergence theorem

$$\|L^{\gamma,\delta} f\|_1 = \int_b (L^{\gamma,\delta} f)(v)\, ds_v = \lim_{\varepsilon\to 0} \int_b (L_\varepsilon^{\gamma,\delta} f)(v)\, ds_v \leqslant A\, \|f\|_1$$

if $\gamma > \alpha, \delta$ arbitrary or $\gamma = \alpha, \delta \geqslant \beta$ and $A$ independent of $r$ and $\varepsilon$.

For the necessity of Theorem 4 assume that $L^{\gamma,\delta}$ maps $L^1(b)$ into $L^1(b)$ and prove that either $\gamma > \alpha$, $\delta$ arbitrary or $\gamma = \alpha$, $\delta \geqslant \beta$. Assume that $\gamma < \alpha$, $\delta$ arbitrary. Take $f = 1$. Then $f\in L^1(b)$ and by Theorems 1 and 2

$$(L_\varepsilon^{\gamma,\delta} 1)(v) = \inf_{1-\varepsilon\leqslant r<1} \int_b |S(rv,t)|\, ds_t R_{\gamma,\delta}(r) \geqslant B \inf_{1-\varepsilon<r<1} R_{\gamma-\alpha,\delta-\beta}(r),$$

where $B$ is independent of $f$, $r$ and $\varepsilon$. Since $1/(1-r) \geqslant 1/\varepsilon$

$$(L_\varepsilon^{\gamma,\delta} 1)(v) \geqslant \inf_{1-\varepsilon\leqslant r<1} \log^{\beta-\delta}\frac{1}{1-r}\Big/(1-r)^{\alpha-\gamma} \quad (\gamma < \alpha,\ \delta \text{ arbitrary}) \tag{9}$$

$$\geqslant (\log 1/\varepsilon)^{\beta-\delta}/\varepsilon^{\alpha-\gamma}$$

for $1-r$ sufficiently small. By the monotonicity of $L_\varepsilon^{\gamma,\delta}1$ in $\varepsilon$ as $\varepsilon \to 0$

$$\int_b (L^{\gamma,\delta}1)(v)\,ds_v \geqslant \int_b (L_\varepsilon^{\gamma,\delta}1)(v)\,ds_v \geqslant B\log^{\beta-\delta}\frac{1}{\varepsilon}\Big/\varepsilon^{\alpha-\gamma} \tag{10}$$

by (9). Thus if $\gamma < \alpha$, $\delta$ arbitrary the right side of (10)$\to\infty$ as $\varepsilon\to 0^+$ so that $\int_b (L^{\gamma,\delta}f)(v)\,ds_v$ is not a bounded operator from $L^1(b)$ to $L^1(b)$ for all $f\in L^1(b)$; similarly if $\gamma=\alpha$, $\delta<\beta$. Hence either $\gamma>\alpha$, $\delta$ arbitrary or $\gamma=\alpha$, $\delta\geqslant\beta$ and the necessity is proved.

**4. Mapping theorem for $p \geqslant 1$.** Let $f\in L^p(b)$, $p\geqslant 1$. Define the operator $T_\sigma$ by

$$(T_\sigma f)(z) = \int_b K_\sigma(z,t)f(t)\,ds_t \qquad (z\in D),$$

where $\sigma > \sigma_0 > 0$ and

$$K_\sigma(z,t) = \frac{S(z,t)^{1+\sigma/N}}{S(z,z)^{\sigma/N}}. \tag{11}$$

The constant $\sigma_0$ depends on the complex dimension of the domain $D$ and constants, which come from the underlying Lie group theory in the Harish-Chandra realization of an irreducible bounded symmetric domain.

Fix $r$ in $(0,1)$ and set $z=rv$ for $v\in b$. Then $T_\sigma$ depends on $r$ and we call the operator $T_\sigma^{(r)}$:

$$(T_\sigma^{(r)} f)(v) = \int_b K_\sigma(rv,t)\,f(t)\,ds_t. \tag{12}$$

THEOREM 5. *For the classical domains the operator $T_\sigma^{(r)}$ $(\sigma>\sigma_0)$ given by* (12) *is a bounded linear operator from $L^p(b)$ to $H^p(D)$ for $p\geqslant 1$ and*

$$\|T_\sigma^{(r)} f\|_p \leqslant A\|f\|_p \qquad (A \textit{ independent of } r). \tag{13}$$

Proof. Fix $r$ in $[0,1)$ and note that

$$\int_b |K_\sigma(rv,t)|\,ds_t = \int_b \frac{|S(rv,t)|^{1+\sigma/N}}{|S(rv,rv)|^{\sigma/N}}\,ds_t = C(1-r^2)^\sigma\int_b |S(rv,t)|^{1+\sigma/N}\,ds_t,$$

since as noted in Section 1 for the classical domains $S(rv,rv) = 1/V(1-r^2)^N$ for all $v\in b$ [6].

The bounds in Theorem 4.1 of [1] using our notation, are

$$B\,S(z,z)^{\sigma/N} \leqslant \int_b |S(z,t)|^{1+\sigma/N}\,ds_t \leqslant A\,S(z,z)^{\sigma/N} \tag{14}$$

for $\sigma>\sigma_0$. If $z=rv\,(v\in b)$ (14) becomes

$$B(1-r^2)^{-\sigma} \leqslant \int_b |S(rv,t)|^{1+\sigma/N}\,ds_t \leqslant A(1-r^2)^{-\sigma}.$$

Thus

$$\int_b |K_\sigma(rv, t)|\, ds_t \leqslant A(1-r^2)^\sigma \cdot (1-r^2)^{-\sigma} \leqslant A, \tag{15a}$$

where $A$ is independent of $r$. By the hermitian symmetry and homogeneity of $K_\sigma$, also,

$$\int_b |K_\sigma(rv, t)|\, ds_v \leqslant A. \tag{15b}$$

By (15a) and (15b) and standard arguments we get our result.

The same theorem holds for the operator $\tilde{T}_\sigma^{(\varrho)}$ given by

$$(\tilde{T}_\sigma^{(\varrho)} f)(z) = \int_b \frac{S(z, t)^{1+\sigma/N}}{S(\varrho t, \varrho t)^{\sigma/N}} f(t) ds_t \qquad (0 \leqslant \varrho < 1),$$

where $\|\tilde{T}_\sigma^{(\varrho)} f\|_p \leqslant C\|f\|_p$, and $\lim\limits_{\varrho \to 1} \tilde{T}_\sigma^{(\varrho)} f$ exists in $L^p(b)$.

In this case $(\tilde{T}_\sigma^{(\varrho)} f)(z)$ is a holomorphic function of $z$ and the mapping is from $L^p(b)$ into $H^p(D)$.

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