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Some remarks on an operator equation in a Banach space

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Abstract. In this paper we give theorems on the existence of solutions of the equation $x(\cdot) = (\mathscr{F}x)(\cdot)$ in which $\mathscr{F}$ is an operator on the space of continuous functions from an interval I to an infinitely dimensional Banach space E. Assuming that $\mathscr{F}$ admits an approximation by a sequence of operators $\mathscr{F}_i$ satisfying some regularity conditions with respect to the so-called *measure of non-compactness*, we prove that if the approximation is "good enough", then equation $(+)$ has a solution.

The results of this paper extend the results of papers [11], [12], [14], and [15].

Let E be an arbitrary Banach space with a norm $\|\cdot\|$ and let $I = [0, a]$. Denote by $C(I, E)$ the space of all continuous functions from I into E with the usual supremum norm $|||\cdot|||$.

In the present note we are concerned with the equation

$$x(\cdot) = (\mathscr{F}x)(\cdot), \tag{+}$$

where $x(\cdot)$ is an unknown function and $\mathscr{F}\colon C(I, E) \to C(I, E)$ is known.

1. We shall deal with equation $(+)$ using the method developed by Ambrosetti [1] and Rzymowski [15] for the existence of a solution of Cauchy problem for an ordinary differential equation in a Banach space. This method is based on the properties of a set function α which can be considered as a kind of "measure of non-compactness".

DEFINITION (K. Kuratowski [9]; [10], Vol. I, p. 318). For any bounded subset V of a space E we denote by $\alpha(V)$ the infimum of all $\varepsilon > 0$ such that there exists a finite covering of V by sets of diameter $\leqslant \varepsilon$.

The number $\alpha(V)$ is called the *measure of non-compactness* of the set V. For properties of the function α see [3], [1], [4], [5] and [10]. In particular, for arbitrary bounded subsets A, B, A_n $(n = 1, 2, \ldots)$ of a Banach space E, we have(1)

(a) if $A \subset B$, then $\alpha(A) \leqslant \alpha(B)$;

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(1) $\bar{A}$ denotes the closure of A, and $\operatorname{conv}(A)$ the smallest convex set containing A.

(b) $\alpha(A+B) \leqslant \alpha(A)+\alpha(B)$;

(c) $\alpha(q\cdot A) = |q|\cdot\alpha(A)$ for real q;

(d) $\alpha(\bar{A}) = \alpha(A)$;

(e) $\alpha(\operatorname{conv} A) = \alpha(A)$;

(f) $\alpha(A) = 0$ if and only if A is precompact;

(g) if $A_{n+1} \subset A_n$ for every $n \geqslant 1$ and if $\lim_{n\to\infty} \alpha(A_n) = 0$, then $\bigcap_{n=1}^{\infty} \bar{A}_n$ is a non-empty precompact subset of E.

The set $X \subset C(I, E)$ is said to be *regular* if it is bounded and all functions belonging to X are equicontinuous.

For $X \subset C(I, E)$, $V \subset E$ and $t \in I$ we write

$$X(t) = \{x(t)\colon x \in X\}, \qquad \int_0^t X(s)\,ds = \Big\{\int_0^t x(s)\,ds\colon x \in X\Big\},$$

$$\tilde{V} = \{x \in C(I, E)\colon x[I] \subset V\},$$

where $x[I]$ denotes the image of I by the function x.

Let $X \subset C(I, E)$ be a regular set. We shall repeatedly make use of the following generalization of the Ascoli–Arzela Theorem, due to Ambrosetti [1]:

$$\alpha(X) = \alpha\big(\bigcup\{X(t)\colon t \in I\}\big) = \sup\big\{\alpha\big(X(t)\big)\colon t \in I\big\}.$$

Moreover, the following inequalities holds (Goebel and Rzymowski [5]):

$$\alpha\Big(\int_0^t X(s)\,ds\Big) \leqslant \int_0^t \alpha\big(X(s)\big)\,ds \qquad \text{for } t \in I,$$

$$\big|\alpha\big(X(t)\big) - \alpha\big(X(s)\big)\big| \leqslant W(X, |t-s|) \qquad \text{for } t, s \in I,$$

where $W(X, \cdot)$ denotes the common modulus of continuity for $x \in X$.

Let $X_n \subset C(I, E)$, $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and let X_1 be a regular set. Then the following theorem holds:

1° $\lim_{n\to\infty} \alpha\big(X_n(t)\big) = g(t)$ *uniformly on* I, *where* g *is a continuous function and* $\lim_{n\to\infty} \alpha(X_n) = \sup\{g(t)\colon t \in I\}$.

2° *If* $\lim_{n\to\infty} \alpha\big(X_n(t)\big) = 0$ *for each* $t \in I$, *then* $\bigcap_{n=1}^{\infty} \bar{X}_n$ *is a non-empty compact subset of* $C(I, E)$.

The proof of these theorems is similar to that given in [15]; it results easily by the Goebel–Rzymowski inequality, Ambrosetti's theorem and properties (a)–(g).

2. Throughout this section we assume that $\mathscr{X}_0$ is a non-empty regular convex subset of $C(I, E)$, $\mathscr{F}\colon \mathscr{X}_0 \to \mathscr{X}_0$ is a continuous operator and $\mathscr{F}_i\colon \mathscr{X}_0 \to \mathscr{X}_0$ for $i = 1, 2, \ldots$

We shall now prove that, if the approximation of $\mathscr{F}$ by the operators $\mathscr{F}_i$ is "good enough", then equation $(+)$ has a solution.

We introduce the following

DEFINITION. A continuous function $x\colon I \to E$ is said to be a *solution of equation* $(+)$ if $x \in \mathscr{X}_0$ and $x(t) = (\mathscr{F}x)(t)$ for every $t \in I$.

Let us denote

$$\mathscr{X}_{n+1} = \operatorname{conv}(\mathscr{F}[\mathscr{X}_n]) \quad \text{for } n = 0, 1, 2, \ldots$$

By the theorem from Section 1 and by Schauder's fixed point theorem, we obtain

THEOREM. *If* $\lim_{n\to\infty} a(\mathscr{X}_n) = 0$, *then equation* $(+)$ *has at least one solution.*

This theorem is a modified version of Rzymowski's result [15], Theorem 1 (cf. [2], [18]).

We denote by $C^+(I)$ the space of all continuous functions mapping the interval I into $[0, \infty)$ with the usual norm and natural partial order $\prec$. Let us write for $t \in I$

$$g_n(t) = a(\mathscr{X}_n(t)), \qquad g(t) = \lim_{n\to\infty} g_n(t)$$

and let

$$a_i = \sup\{a((\mathscr{F}-\mathscr{F}_i)[\mathscr{X}_0](t))\colon t \in I\},$$
$$x_1^{(i)}(t) = g(t) \quad \text{for } t \in I,$$
$$x_{k+1}^{(i)}(t) = a_i + (\Phi_i x_k^{(i)})(t) \quad \text{for } t \in I \text{ and } k \geqslant 2,$$

where Φ_i is an operator from the following

ASSUMPTION. Suppose that there exist operators $\Phi_i\colon C^+(I) \to C^+(I)$ $(i = 1, 2, \ldots)$ such that

1° for an arbitrary non-empty subset $\mathscr{X}$ of $\mathscr{X}_0$ we have $a(\mathscr{F}_i[\mathscr{X}](t)) \leqslant (\Phi_i y)(t)$ for each $t \in I$, where $y(t) = a(\mathscr{X}(t))$;

2° if (y_n), $y_n \in C^+(I)$ is a non-increasing sequence convergent to y_0, then the sequence $(\Phi_i y_n)$ converges to $\Phi_i y_0$ as $n \to \infty$;

3° if $g(t) \leqslant a_i + (\Phi_i g)(t)$ for $t \in I$, then $g \prec x_n^{(i)}$ for $n \geqslant 1$;

4° there exists $\lim_{n\to\infty} x_n^{(i)}(t)$ for $t \in I$.

The following theorem holds:

THEOREM 1. *Let conditions* 1°–4° *be satisfied and let*

$$\inf_{i\geqslant 1}\sup_{t\in I}\lim_{n\to\infty} x_n^{(i)}(t) = 0.$$

Then equation $(+)$ *has at least one solution.*

Proof. It is easy to verify that

$$\mathscr{F}[\mathscr{X}_n] \subset \mathscr{F}_i[\mathscr{X}_n]+(\mathscr{F}-\mathscr{F}_i)[\mathscr{X}_0]$$

and therefore

$$\begin{aligned} a\big(\mathscr{F}[\mathscr{X}_n](t)\big) &\leqslant a\big(\mathscr{F}_i[\mathscr{X}_n](t)\big)+a\big((\mathscr{F}-\mathscr{F}_i)[\mathscr{X}_0](t)\big) \\ &\leqslant a_i+(\Phi_i g_n)(t) \end{aligned}$$

for $t \in I$. Hence $g_{n+1}(t) \leqslant g_n(t) \leqslant a_i+(\Phi_i g_n)(t)$ for $t \in I$.

We have

$$\lim_{n\to\infty} g_n(t) = g(t) \quad \text{uniformly on } I$$

and, consequently, $g(t) \leqslant a_i+(\Phi_i g)(t)$ for $t \in I$. Hence $g \prec x_n^{(i)}$ *for* $i, n \geqslant 1$. This implies

$$g(t) \leqslant \sup_{t\in I} \lim_{n\to\infty} x_n^{(i)}(t) \quad \text{for } t \in I \text{ and } i \geqslant 1,$$

which means that $g(t) = 0$ for each $t \in I$, and this proves our theorem.

It is easy to verify that if Φ_i preserves the partial order and $g(t) \leqslant a_i + (\Phi_i g)(t)$ for $t \in I$, then $g = x_1^{(i)} \prec x_2^{(i)} \prec \ldots$ Therefore, from the proof of Theorem 1 it follows:

PROPOSITION. *If conditions* 1° *and* 2° *are satisfied and if* Φ_i *preserves the partial order, then* $g(t) \leqslant x_n^{(i)}(t)$, $\big(x_n^{(i)}(t)\big)$ *is a non-decreasing sequence and* $\lim_{n\to\infty} x_n^{(i)}(t) = \sup_{n\geqslant 1} x_n^{(i)}(t)$ *for each* $t \in I$.

THEOREM 2. *Assume that there exist constants* $0 \leqslant K_i < 1$ $(i = 1, 2, \ldots)$ *such that* $a(\mathscr{F}_i[\mathscr{X}]) \leqslant K_i \cdot a(\mathscr{X})$ *for every subset* $\mathscr{X}$ *of* $\mathscr{X}_0$. *If*

$$\inf\{(1-K_i)^{-1}\cdot a_i:\ i = 1, 2, \ldots\} = 0,$$

then equation (+) *has at least one solution.*

Proof. Let us put $(\Phi_i x)(t) = K_i \cdot \sup\{x(t):\ t \in I\}$ for $x \in C^+(I)$. Obviously, Φ_i acts on $C^+(I)$, is monotone and satisfies condition 2°. For $\mathscr{X} \subset \mathscr{X}_0$ and $t \in I$, using the Ambrosetti theorem, we have

$$a\big(\mathscr{F}_i[\mathscr{X}](t)\big) \leqslant a(\mathscr{F}_i[\mathscr{X}]) \leqslant K_i \cdot \sup\big\{a\big(\mathscr{X}(t)\big):\ t \in I\big\}.$$

Consequently, conditions 1°, 3° and 4° are satisfied.

It is easy to verify that

$$\begin{aligned} x_2^{(i)}(t) &= a_i+K_i\cdot\sup\{g(t):\ t \in I\}, \\ x_n^{(i)}(t) &= a_i+a_i\cdot\sum_{r=1}^{n-2} K_i^r + K_i^{n-1}\cdot\sup\{g(t):\ t\in I\} \qquad \text{for } n \geqslant 2, \end{aligned}$$

where $\lim_{n\to\infty} x_n^{(i)}(t) = (1-K_i)^{-1}\cdot a_i$ for $t \in I$ and $i \geqslant 1$. An application of Theorem 1 completes the proof.

THEOREM 3. *Let condition* 1° *be satisfied, let the operators* Φ_i $(i = 1, 2, \ldots)$ *be linear and suppose that the spectral radius of each operator* *is less than* 1. *If*

$$\inf_{i\geqslant 1}\sup_{t\in I} G_i(t) = 0,$$

where $G_i \in C^+(I)$ *is the solution of the equation*

$$(++) \qquad x(t) = a_i + (\Phi_i x)(t),$$

then equation $(+)$ *has at least one solution.*

Proof. The operators Φ_i are monotone and satisfy condition 2°. Then conditions 3° and 4° are also satisfied. Applying Theorem I.2.2 from [7], p. 26, one can see that $\lim_{n\to\infty} x_n^{(i)} = G_i$, where G_i is the solution of equation $(++)$. An application of Theorem 1 completes the proof.

3. Now we are going to give some example of application of Theorems 1 and 2 to the theory of differential equations. The results of this section extend those of the previous works [11], [12] and [14].

Let $B = \{x \in E\colon \|x-x_0\| \leqslant r\}$. By (PC) we shall denote the problem of finding the solution of the equation

$$x'(t) = (Fx)(t)$$

satisfying the condition

$$x(0) = x_0,$$

F being an operator from $\tilde{B}$ to $C(I, E)$, and the derivative being understood in the strong sense(²).

In the case when $(Fx)(\cdot) = f(\cdot, x(\cdot))$, where $f\colon I\times B \to E$ is known, problem (PC) gets the form

$$(*) \qquad x'(t) = f(t, x(t)), \quad x(0) = x_0.$$

As regards theorems on the existence of solutions and properties of the set of solutions for problem $(*)$, see [1], [2], [5], [15], [16], [17], [18] and [19]; for problem (PC), see [11], [12], [13] and [14].

Suppose that $F\colon \tilde{B} \to C(I, E)$ is a bounded continuous operator. Assume, moreover, that the operators $F_i\colon \tilde{B} \to C(I, E)$ $(i = 1, 2, \ldots)$ are continuous and bounded with the same constant K and let $L = \max(M, K)$, $a\cdot L \leqslant r$, where $M = \sup\{|||Fx|||\colon x \in \tilde{B}\}$.

Let $\mathscr{X}_0$ be the set of all $x \in \tilde{B}$ such that $x(0) = x_0$ and $\|x(t)-x(s)\| \leqslant L|t-s|$ for $t, s \in I$. In this case the set $\mathscr{X}_0$ is convex and regular. The question of existence of solution for (PC) is equivalent to that of the existence of a fixed point of the transformations

$$(\mathscr{F}x)(t) = x_0 + \int_0^t (Fx)(s)\,ds$$

(²) A function $x\colon I \to E$ is said to be a *solution of problem* (PC), if it is a differentiable function on I such that $x(0) = x_0$, $x(t) \in B$ for $t \in I$ and $x'(t) = (Fx)(t)$ for $t \in I$.

in the set $\mathscr{X}_0$. Now, in the set $\mathscr{X}_0$ we define transformations $\mathscr{F}_i$ $(i = 1, 2, \ldots)$ by the formula

$$(\mathscr{F}_i x)(t) = x_0 + \int_0^t (F_i x)(s)\,ds.$$

Obviously, each $\mathscr{F}_i$ maps $\mathscr{X}_0$ into itself.

We have the following:

COROLLARY 1. *Let there exist a constant $k \geqslant 0$ such that*

$$a\big(\bigcup \{(F_i x)[I]:\ x \in \tilde{V}\}\big) \leqslant k \cdot a(V) \qquad (i = 1, 2, \ldots)$$

for every subset V of B and let $a \cdot k < 1$. If

$$\inf_{i \geqslant 1} \sup_{t \in I} a\Big(\int_0^t (F - F_i)[\mathscr{X}_0](s)\,ds\Big) = 0,$$

then there exists a solution of problem (PC).

Proof. It suffices to verify the assumptions of Theorem 2.

Let $\mathfrak{X} \subset \mathfrak{X}_0$. For vector-valued functions the integral mean-value theorem may be stated as follows:

$$\int_0^t x(s)\,ds \in t \cdot \overline{\text{conv}}\big(\{x(s):\ 0 \leqslant s \leqslant t\}\big).$$

Therefore,

$$a\Big(\int_0^t F_i[\mathscr{X}](s)\,ds\Big) \leqslant t \cdot a\big(\bigcup \{(F_i x)[I]:\ x \in \mathscr{X}\}\big) \leqslant a \cdot k \cdot a\big(\bigcup \{x[I]:\ x \in \mathscr{X}\}\big)$$
$$= a \cdot k \cdot \sup\{a(\mathscr{X}(t)):\ t \in I\} = a \cdot k \cdot a(\mathscr{X})$$

and consequently $a(\mathscr{F}_i[\mathscr{X}]) \leqslant K_i \cdot a(\mathscr{X})$, where $K_i = a \cdot k$ for each $i \geqslant 1$. This ends the proof.

COROLLARY 2. *Let the operators F_i $(i = 1, 2, \ldots)$ map every regular subset of $\tilde{B}$ into a regular set and suppose that there exist integrable functions p_i: $I \to [0, \infty)$ such that $a(F_i[\mathscr{X}](t)) \leqslant p_i(t) \cdot a(\mathscr{X}(t))$ for each $t \in I$ and for any regular subset $\mathscr{X}$ of $\tilde{B}$. If*

$$\inf_{i \geqslant 1} \exp\Big(\int_0^a p_i(s)\,ds\Big) \cdot \sup_{t \in I} a\Big(\int_0^t (F - F_i)[\mathscr{X}_0](s)\,ds\Big) = 0,$$

then there exists a solution of problem (PC).

Proof. Let us put $(\Phi_i x)(t) = \int_0^t p_i(s) x(s)\,ds$ for $x \in C^+(I)$. Obviously, Φ_i acts on $C^+(I)$ and has the spectral radius equal zero ([8], p. 143).

If $\mathscr{X} \subset \mathscr{X}_0$, then

$$a\Big(\int_0^t F_i[\mathscr{X}](s)\,ds\Big) \leqslant \int_0^t a\big(F_i[\mathscr{X}](s)\big)\,ds \leqslant \int_0^t p_i(s) \cdot a\big(\mathscr{X}(s)\big)\,ds$$

for $t \in I$. By virtue of Theorem 3 it is enough to prove only that $\inf_{i \geq 1} \sup_{t \in I} G_i(t) = 0$, where G_i is the solution of the equation

$$x(t) = \sup_{s \in I} a\Big(\int_0^s (F - F_i)[\mathscr{X}_0](r)\,dr\Big) + (\Phi_i x)(t).$$

We have

$$G_i(t) = a_i \cdot \exp\Big(\int_0^t p_i(s)\,ds\Big) \leqslant a_i \cdot \exp\Big(\int_0^a p_i(s)\,ds\Big)$$

for $t \in I$ and $i \geqslant 1$, where $a_i = \sup_{s \in I} a\big(\int_0^s (F - F_i)[\mathscr{X}_0](r)\,dr\big)$. This completes the proof.

Remark. Corollary 2 generalizes Rzymowski's result [15], Theorem 2, which follows from it by putting $(Fx)(\cdot) = f(\cdot, x(\cdot))$ and $(F_i x)(\cdot) = f_i(\cdot, x(\cdot))$, where $f\colon I \times B \to E$ is a bounded continuous function and $f_i\colon I \times B \to E$ $(i = 1, 2, \ldots)$ are commonly bounded uniformly continuous functions such that $a(f_i(t, V)) \leqslant p_i(t) \cdot a(V)$ for $t \in I$ and for any subset V of B, p_i being integrable functions.

The result of [15] can be obtained by the well-known Gronwall Lemma [6]. By the proof of Theorem 3, we obtain the following version of Gronwall Lemma:

Denote by $\leqslant$ the partial order in E generated by a cone $S \subset E$. Let a linear continuous operator Φ act in S and let the spectral radius of Φ be less than 1. *Assume that there exist elements c and y in E such that $y \leqslant c + \Phi y$. Then $y \leqslant z_0$, where z_0 is the solution of the equation $\Phi z + c = z$.*

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