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On Bazilevič schlicht functions

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I. E. Bazilevič has proved that functions

$$f(z) = \left\{\frac{m}{1+a^2}\int_0^z \big(p(s)-ai\big)\, s^{-\frac{mai}{1-a^2}-1} f_*(s)^{\frac{m}{1+a^2}}\, ds\right\}^{\frac{1+ai}{m}} \tag{1}$$

are a subclass of the class S of schlicht functions [1]. In the formula $p(z) = 1+a_1 z+\ldots$, $|z|<1$ belong to the class P of functions for which $\mathrm{re}\,p(z)>0$; $f_*(z)$ belongs to the class S_* of starlike schlicht functions; a is any real number and $m>0$. The powers appearing in the formula are meant as principal values.

Let B be the class of functions denoted by formula (1). We can easily see that the classes S_a of the spiral functions of Špaček [5] and the class L of linearly accessible (close-to-convex) functions (see M. Biernacki [2]) belong to the class B. In order to obtain the classes S_a we have to put $p(z) = zf'_*(z)/f_*(z)$ into formula (1) and for class L we have to put $a=0$, $m=1$. When we fix the numbers m and a in formula (1), we get some subclass $B_{m,a}$ of the class B. Now, let $f(z)$ belong to the class $B_{m,a}$. Of course we can write

$$f(z) = z+a_2 z^2+\ldots, \qquad |z|<1\,. \tag{2}$$

Let $a_k = x_k+iy_k$. We can regard as the n-th region of variability of coefficients of the functions of the class $B_{m,a}$ a set of all the points $(a_k = x_k+y_k,\ k=2,\ldots,n)$ of $n-1$ dimensional complex space, the coordinates of which can be the coefficients of the expansion (2) of any functions of the class $B_{m,a}$.

Let $E(a_2,\ldots,a_n)$ be a real function of the class C_1 depending on $2n-2$ real variables x_k, y_k defined in an open set comprising the n-th region of variability of the coefficients of functions of the class $B_{m,a}$. Moreover

$$\operatorname*{grad}_{x_k,y_k} E \neq 0\,. \tag{3}$$

The function E will be considered as a functional defined for the functions of the class $B_{m,a}$, namely

$$E(a_2,\ldots,a_n) = E(f)\,.$$

THEOREM 1. *Functions of the class $B_{m,a}$ for which any functional $E(f)$ obtains its extremal value always have the following form*

$$f(z) = \left\{\frac{m}{1+a^2}\int_0^z s^{\frac{m}{1+ai}-1}\left(1-ai+\sum_{k=1}^{n-1}\frac{\beta_k\sigma_k s}{1-\sigma_k s}\right)\prod_{k=1}^{n-1}(1-\tau_k s)^{-\delta_k\frac{m}{1+a^2}}\right\}^{\frac{1+ai}{m}}$$

where $|\sigma_k| = |\tau_k| = 1$, $\beta_k > 0$, $\delta_k > 0$, $\sum_{k=1}^{n-1}\beta_k = \sum_{k=1}^{n-1}\delta_k = 2$.

The proof of the theorem is based on some auxiliary definitions and lemmas. We denote by $G_{m,a}$ the class of the functions defined by the formula

$$(4)\qquad g(z) = \big(p(z)-ai\big)\exp\left\{\frac{m}{1+a^2}\int_0^z\frac{p_1(s)-1}{s}\,ds\right\} = b_0+b_1z+\ldots, \qquad |z|<1,$$

where $p(z) = 1+a_1z+1\ldots$ and $p_1(z) = 1+\gamma_1z+\ldots$ belong to the class P. Because $g(z)\neq 0$, we have from (4)

$$\log g(z) = \log\big(p(z)-ai\big) + \frac{m}{1+a^2}\int_0^z\frac{p_1(z)-1}{s}\,ds.$$

When differentiating, we have

$$\frac{g'(z)}{g(z)} = \frac{p'(z)}{p(z)-ai}+\frac{m}{1+a^2}\,\frac{p_1(z)-1}{z},$$

and hence

$$(5)\qquad \sum_{k=0}^{\infty}kb_kz^k\sum_{k=0}^{\infty}a_kz^k = \sum_{k=0}^{\infty}ka_kz^k\sum_{k=0}^{\infty}b_kz^k + \frac{m}{1+a^2}\sum_{k=0}^{\infty}a_kz^k\sum_{k=1}^{\infty}\gamma_kz^k\sum_{k=0}^{\infty}b_kz^k,$$

where $a_0 = 1-ai$. Making some further arrangements, we obtain from (5)

$$\sum_{q=1}^{\infty}z^q\sum_{k=1}^{q}kb_ka_{q-k} = \sum_{q=1}^{\infty}z^q\sum_{k=0}^{q-1}b_k\left\{(q-k)a_{q-k}+\frac{m}{1+a^2}\sum_{j=0}^{q-k-1}a_j\gamma_{2-k-j}\right\}.$$

Introducing the denotation

$$d_{q,k} = (q-k)a_{q-k}+\frac{m}{1+a^2}\sum_{j=0}^{q-k-1}a_j\gamma_{q-k-j},$$

and, regarding that from (4) $b_0 = 1 - ai$, we have

$$(6) \qquad b_q = \frac{1}{q!(1-ai)^{q-1}} \begin{vmatrix} d_{q,q-1} & d_{q,q-2} & \dots & d_{q,0} \\ (1-q)(1-ai) & d_{q-1,q-2} & \dots & d_{q-1,0} \\ 0 & (2-q)(1-ai) & \dots & d_{q-2,0} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_{1,0} \end{vmatrix}.$$

It follows from (6) that b_q is a polynomial of the coefficients a_k and γ_k, $k = 1, \dots, q$. Let $E_1(g) = E_1(b_1, \dots, b_n)$ be a functional defined for the class $G_{m,a}$.

LEMMA 1. *Functions of the class $G_{m,a}$ for which the functional $E_1(g) = E_1(b_1, \dots, b_n)$,* $\operatorname{grad} E \neq 0$, *obtains its extremal value always have the the following form*

$$g(z) = \left(1 - ai + \sum_{k=1}^{n} \frac{\beta_k \sigma_k z}{1 - \sigma_k z}\right) \prod_{k=1}^{n} (1 - \tau_k z)^{-\delta_k \frac{m}{1-a^2}}$$

where $|\sigma_k| = |\tau_k| = 1$, $\beta_k > 0$, $\delta_k > 0$, $\sum_{k=1}^{n} \beta_k = \sum_{k=1}^{n} \delta_k = 2$.

Proof. Let $g(z)$ be a function giving the extremal value to the functional $E_1(g)$ and let functions $p(z)$ and $p_1(z)$ be functions defining this function according to formula (4). From the theorem of Carathéodory [3] there are functions $r(z)$ and $r_1(z)$ of the class P which are of the form

$$(7) \qquad \begin{aligned} r(z) &= 1 + \sum_{k=1}^{n+1} \frac{\beta_k \sigma_k z}{1 - \sigma_k z}, \quad \sigma_k = e^{is_k}, \quad \beta_k > 0, \quad \sum_{k=1}^{n+1} \beta_k = 2, \\ r_1(z) &= 1 + \sum_{k=1}^{n+1} \frac{\delta_k \tau_k z}{1 - \tau_k z}, \quad \tau_k = e^{it_k}, \quad \delta_k > 0, \quad \sum_{k=1}^{n+1} \delta_k = 2 \end{aligned}$$

and have the first n coefficients equal to the first n coefficients of the functions $p(z)$ and $p_1(z)$. From (6) we infer that the first n coefficients of functions

$$g_1(z) = \big(r(z) - ai\big) \exp\left\{\frac{m}{1+a^2} \int_0^z \frac{r_1(s) - 1}{s}\, ds\right\}$$

are the same as the first n coefficients of function $g(z)$, and hence $E_1(g_1) = E_1(g)$. $E_1(g_1)$ is a function of $4n+4$ real variables

$$E_1(\beta_1, \dots, \beta_{n+1}; \delta_1, \dots, \delta_{n+1}; s_1, \dots, s_{n+1}; t_1, \dots, t_{n+1})$$

connected with conditions $\sum_{k=1}^{n+1}\beta_k = \sum_{k=1}^{n+1}\delta_k = 2$. Since we assume that the function $g(z)$ gives the extremum for the functional $E_1(g)$, the function $g_1(z)$ also gives the extremum, i.e.

$$\frac{\partial E_1}{\partial \beta_k}+\lambda = 0\,, \quad \frac{\partial E_1}{\partial \delta_k}+\mu = 0\,, \quad \frac{\partial E_1}{\partial s_k} = 0\,, \quad \frac{\partial E_1}{\partial t_k} = 0 \tag{8}$$

where λ and μ are Lagrange multipliers for the additional conditions. Let $b_k = u_k + iv_k$. Then

$$\frac{\partial E_1}{\partial \beta_k} = \sum_{j=1}^{n}\frac{\partial E_1}{\partial u_j}\cdot\frac{\partial u_j}{\partial \beta_k}+\sum_{j=1}^{n}\frac{\partial E_1}{\partial v_j}\cdot\frac{\partial v_j}{\partial \beta_k}\,; \tag{9}$$

similarly we calculate the derivatives of the remaining variables. We note further that

$$\frac{\partial g_1}{\partial \beta_k} = \sum_{j=1}^{\infty}\frac{\partial b_j}{\partial \beta_k}z^j = \frac{\sigma_k z}{1-\sigma_k z}\prod_{k=1}^{n+1}(1-\tau_k z)^{-\delta_k\frac{m}{1+a^2}}\,.$$

Let

$$\prod_{k=1}^{n+1}(1-\tau_k z)^{-\delta_k\frac{m}{1+a^2}} = 1 + c_1 z + \ldots\,,$$

where

$$c_k = \frac{1}{k!}\left(\frac{m}{1+a^2}\right)^2\begin{vmatrix} \gamma_1 & -\frac{1+a^2}{m} & 0 & \ldots & 0 \\ \gamma_2 & \gamma_1 & -2\frac{1+a^2}{m} & \ldots & 0 \\ \gamma_3 & \gamma_2 & \gamma_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \gamma_k & \gamma_{k-1} & \gamma_{k-2} & \ldots & \gamma_1 \end{vmatrix}\,, \qquad \gamma_j = \sum_{q=1}^{n+1}\delta_q\tau_q^j\,; \tag{10}$$

hence

$$\sum_{j=1}^{\infty}\frac{\partial b_j}{\partial \beta_k}z^j = \sum_{j=1}^{\infty}(z\sigma_k)^j\sum_{j=0}^{\infty}c_j z^j\,,$$

and thus

$$\frac{\partial b_j}{\partial \beta_k} = \sigma_k^j + \sigma_k^{j-1}c_1 + \ldots + \sigma_k c_{j-1}\,. \tag{11}$$

In an analogous way

$$\frac{\partial b_j}{\partial s_k} = i\beta_k\{j\sigma_k^j + (j-1)\sigma_k^{j-1}c_1 + \ldots + \sigma_k c_{j-1}\}\,, \tag{12}$$

$$\frac{\partial b_j}{\partial \delta_k} = \frac{m}{1+a^2}\left\{\frac{\tau_k^j}{j}b_0 + \frac{\tau_k^{j-1}}{j-1}b_1 + \ldots + \tau_k b_{j-1}\right\}\,, \tag{13}$$

$$\frac{\partial t_k}{\partial b_j} = \frac{m\delta_k i}{1+a^2}\{\tau_k^j b_0 + \tau_k^{j-1}b_1 + \ldots + \tau_k b_{j-1}\}\,. \tag{14}$$

Equations (8), after using formulas (9), (11), (12), (13), (14) and after simple modifications, can be written as follows:

$$
(15)\quad \begin{aligned}
&A_n\sigma_k^{2n}+A_{n-1}\sigma^{2n-1}+\ldots+A_1\sigma_k^{n+1}-2\lambda\sigma_k^n+\bar{A}_1\sigma_k^{n-1}+\ldots+\bar{A}_n=0\,,\\
&2nA_n\sigma_k^{2n-1}+(2n-1)A_{n-1}\sigma^{2n-2}+\ldots+(n+1)A_1\sigma_k^n-2\lambda n\sigma_k^{n-1}+\\
&\qquad\qquad+(n-1)\bar{A}_1\sigma_k^{n-2}+\ldots+\bar{A}_{n-1}=0\,,
\end{aligned}
$$

$$
(16)\quad \begin{aligned}
&B_n\tau_k^{2n}+B_{n-1}\tau_k^{2n-1}+\ldots+B_1\tau_k^{n+1}-2\mu\tau_k^n+\bar{B}_1\tau_k^{n-1}+\ldots+\bar{B}_n=0\,,\\
&2nB_n\tau_k^{2n-1}+(2n-1)B_{n-1}\tau_k^{2n-2}+\ldots+(n+1)B_1\tau_k^n-2\mu n\tau_k^{n-1}+\\
&\qquad\qquad+(n-1)\bar{B}_1\tau_k^{n-2}+\ldots+\bar{B}_{n-1}=0\,,
\end{aligned}
$$

$$
\left.\begin{aligned}
A_k&=\sum_{j=0}^{n-k}c_j\left(\frac{\partial E_1}{\partial u_{k+j}}-i\frac{\partial E_1}{\partial V_{k+j}}\right),\\
B_k&=\frac{m}{1+a^2}\cdot\frac{1}{k}\sum_{j=0}^{n-k}b_j\left(\frac{\partial E_1}{\partial U_{k+j}}-i\frac{\partial E_1}{\partial V_{k+j}}\right),
\end{aligned}\right.\qquad k=1,\ldots,n,
$$

where b_j and c_j are expressed by formulas (6) and (10), in which we put $a_k=\sum_{j=1}^{n+1}\beta_j\sigma_j^k$, $\gamma_k=\sum_{j=1}^{n+1}\delta_j\tau_j^k$.

Let us study the group of equations (15). It follows from them that the polynomial

$$A_n\sigma^{2n}+A_{n-1}\sigma^{2n-1}+\ldots+A_1\sigma^{n+1}-2\lambda\sigma^n+\bar{A}_1\sigma^{n-1}+\ldots+\bar{A}_n$$

has double roots σ_k (¹), and thus there can be n different numbers σ_k at most. We obtain an analogical result concerning the numbers τ_k from the group of equations (16). Hence we see that functions (7) are of the following form:

$$
(17)\qquad r(z)=1+\sum_{k=1}^{n}\frac{\beta_k\sigma_k z}{1-\sigma_k z}\,,\qquad r_1(z)=1+\sum_{k=1}^{n}\frac{\delta_k\tau_k z}{1-\tau_k z}\,.
$$

On the basis of the paper of Carathéodory [3], we infer that if the functions $p(z)$ and $p_1(z)$ have the first n coefficients identical to the first n coefficients of the functions $r(z)$ and $r_1(z)$, then by (17) $p(z)\equiv z(r)$ and $p_1(z)\equiv r_1(z)$. This completes the proof of the lemma.

LEMMA 2. *The n-th region of variability of the coefficient of functions of the class $B_{m,a}$ is a homeomorphic map of the $n-1$ region of variability of the coefficients of functions of the class $G_{m,a}$.*

Proof. We introduce the auxiliary functions

$$F(z)=\frac{m}{1+a^2}\int_0^z s^{\frac{m}{1+ai}-1}g(s)\,ds=z^{\frac{m}{1+ai}}(1+e_1z+\ldots)\,,$$

(¹) It can easily be proved that this polynomial cannot identically equal zero as we have from the assumption grad $E_1\neq 0$.

where

$$e_k = \frac{1+ai}{k+m+kai} b_k, \quad k = 1, \ldots \tag{18}$$

Formula (18) gives the continuous and one-one correspondence between the coefficients of the functions $F(z)$ and $g(z)$. Let us observe that

$$f(z) = \{F(z)\}^{(1+ai)/m} = z + a_2 z^2 + \ldots \tag{19}$$

Let us write

$$\sum_{k=j}^{\infty} e_k^{(j)} z^k = \Big(\sum_{k=1}^{\infty} e_k z^k\Big)^j, \quad \sum_{k=j}^{\infty} a_k^{(j)} z^k = \Big(\sum_{k=2}^{\infty} a_k z^{k-1}\Big)^j, \quad j = 1, \ldots$$

Hence and from (19) we have

$$a_k = \sum_{l=1}^{k-1} \binom{(1+ai)/m}{l} e_{k-1}^{(l)}, \tag{20}$$

$$e_k = \sum_{l=1}^{k} \binom{m/(1+ai)}{l} a_{k+1}^{(l)}. \tag{21}$$

Hence again we see that between the coefficients of the functions $f(z)$ and $F(z)$ there is a continuous and one-one correspondence, and this completes the proof of lemma 2.

Now, let us start on the proof of theorem 1. From the assumptions regarding the functional $E(f)$ we infer that this functional has its extremal value on the boundary of the n-th region of variability of the coefficients of functions of the class $B_{m,a}$. We infer from lemma 2 that there is a continuous and one-one correspondence between the boundary points of the n-th region of variability of the coefficients of functions of the class $B_{m,a}$ and the boundary points of the $n-1$ region of variability of the coefficients of functions of the class $B_{m,a}$. To the boundary points of the latter region correspond only functions of the form given in lemma 1 (we put $n-1$ instead of n). From the homeomorphism proved we obtain the theorem.

THEOREM 2. *Let $N = [1/m]$. Then for functions of class $B_{m,0}$ we have*

$$|a_n| \leqslant n \quad \textit{for} \quad n = 2, \ldots, N+2.$$

Proof. On the basis of theorem proved previously [6]: for the function

$$f_*^m(z) = z^m(1 + c_1 z + \ldots)$$

there is an estimation

$$|c_k| \leqslant \frac{1}{k!} \prod_{j=0}^{k-1} |2m+j|,$$

and the sign of equality occurs for the functions

$$z^m(1+\eta z)^{-2m}, \quad |\eta| = 1. \tag{22}$$

We can easily see that all the expansion coefficients of functions (22) are positive when we put $\eta = -1$. Now let us consider the function

$$F(z) = m\int_0^z p(s)s^{-1}f_*^m(s)\,ds = z^m(1+e_1z+\ldots).$$

Hence we get

$$e_k = \frac{m}{k+m}(c_k + a_1c_{k-1}+\ldots+a_k);$$

so e_k is positive and obtains its maximal value when all the coefficients c_k and a_k are positive and maximal. That is true for functions (22) with $\eta = -1$ and for

$$p(z) = \frac{1+z}{1-z}. \tag{23}$$

The function

$$f(z) = \{F(z)\}^{1/m} = z + a_2z^2+\ldots$$

has its coefficients expressed by formulas (20) with the coefficients $e_{k-1}^{(l)}$ (we have to put $a = 0$). Numbers $e_{k-1}^{(l)}$ obtain their maximal and positive values for the maximal and positive values of e_k, of which they are composed. Thus they obtain the greatest values after substituting functions (22), (23). From the assumption that $N = [1/m]$ we have

$$\binom{1/m}{l} > 0 \quad \text{for} \quad l = 1, \ldots, N+1$$

and hence

$$a_n = \sum_{l=1}^{n-1}\binom{1/m}{l}e_{n-1}^{(l)}, \quad n \leqslant N+2$$

obtains its maximal values for functions (22), (23). A simple calculation gives us the theorem. As a consequence of theorem 2 we have

THEOREM 3. *Let $N = 1/m$ be a natural number. Then for functions of class $B_{m,0}$ we have*

$$|a_n| \leqslant n.$$

In order to prove this theorem it is enough to note that now

$$\binom{1/m}{l} > 0 \quad \text{for} \quad l = 1, \ldots, N+1$$

and

$$\binom{1/m}{l} = 0 \quad \text{for} \quad l \geqslant N+2$$

and then repeat the proof of theorem 2. This theorem has been known for $m = 1$ (see [1], [4]).

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