ANNALES
POLONICI MATHEMATICI
XXV (1972)

Stability in the large of certain non-linear systems of differential equations of order three

by Stanisław Kasprzyk (Kraków)

In this paper we will consider the following four types of systems:

$$\dddot{x}+a\ddot{x}+b\dot{x}+f(x)=0, \tag{1}$$

$$\dddot{x}+f(\ddot{x})+b\dot{x}+cx=0; \tag{2}$$

$$\begin{aligned}\dot{x}_1 &= -cx_1+x_2-f(x_1),\\ \dot{x}_2 &= -x_1+x_3,\\ \dot{x}_3 &= -cx_1+bf(x_1);\end{aligned} \tag{3}$$

$$\begin{aligned}\dot{x}_1 &= x_2-f(x_1),\\ \dot{x}_2 &= -x_1+x_3,\\ \dot{x}_3 &= -ax_1,\end{aligned} \tag{4}$$

where a, b, c, are constants and f is a non-linear function such that $f(0)=0$, so that each of them admits the zero solution.

For each of those systems we give sufficient conditions for the global asymptotic stability of the trivial solution. The corresponding four theorems are stated in the next section and proved in the last section. They are all obtained as an application of a result due to Hartman and Olech [6]. The latter is recalled, for the convenience of the reader, in Section 2.

Note that each of systems (1)-(4) can be written in the form

$$\dot{x}=Ax+kf(\sigma), \qquad \sigma=\langle c, x\rangle, \tag{5}$$

where x, k, c are vectors, A is an $n\times n$ matrix and $\langle\ ,\ \rangle$ stands for the scalar product in R.

The stability of the trivial solution of (5) has been extensively studied under the assumption that

$$\big(f(\sigma)/\sigma\big)\in I, \tag{6}$$

where I is an interval. The famous Aizerman conjecture states that the trivial solution of (5) is globally asymptotically stable for each f satisfying (6) if I is equal to the so-called *Hurvitz interval* H for (5); this is the interval such that for each constant $h \in H$ the linear system obtained from (5) by substituting $f(\sigma) = h\sigma$ is asymptotically stable.

As is well known now, this conjecture has been proved to be false in general. An example was given first by Pliss [9] for an equation of the form (2).

On the other hand, Jakubowicz, Kalman, Lurie and Popov have proposed methods to determine I in (6) dependent on A, k and c, but not on f such that (5) is absolutely stable, i.e., asymptotically stable in the large for each f satisfying (6). However, in general the interval I of absolute stability obtained by those methods is properly contained in the Hurvitz interval H for (5).

In this paper we work with a stronger assumption on f, namely that

$$f'(u) \in I^*, \tag{7}$$

where I^* is again an interval. Manifestly, condition (7) implies (6) with $I = I^*$. However, the results we obtain are also stronger in the sense that in each of the cases considered the global stability is established under (7) for f (and some other condition) with I^* equal to the Hurvitz interval but containing (in some of the cases considered properly) the interval obtained by Popov's method.

Several authors have studied the stability properties of systems (1)-(4) (cf [2]-[4] and [9]-[11]). The systems (1)-(4) were studied in [7] and [8] but the proofs contain some mistakes. However, the results presented here subsume those given in [7] and [8].

In the remarks following the statements of the theorems we contrast our results with those previously obtained by other authors.

1. Statements of the results. We will now state four theorems corresponding to systems (1)-(4), respectively.

THEOREM 1. *Assume in* (1) *that* $a > 0$ *and* $b > 0$, *that* f *is of class* C^1, *and that*

$$f(0) = 0 \quad \text{and} \quad f(x) \neq 0 \quad \text{if } x \neq 0, \tag{8}$$

$$\int_0^\infty |f(s)|\,ds = +\infty, \tag{9}$$

$$0 < f'(0) < ab \tag{10}$$

and

$$|f'(x)| \leqslant ab \quad \text{for each } x. \tag{11}$$

Then $x(t) \equiv 0$ *is a globally asymptotically stable solution of* (1).

THEOREM 2. *Let* $b > 0$, $c > 0$ *and let* $f(0) = 0$, $f \in C^1$. *If*

$$f'(u) \geqslant c/b \quad \text{for each } u \tag{12}$$

and the inequality is strict for $u = 0$, *then* $x(t) \equiv 0$ *is a globally asymptotically stable solution of* (2).

THEOREM 3. *Assume in* (3) *that* $c > 0$, $b > 0$ *and* $c^2 > b$ *that* f *is of class* C^1, *and that*

$$f(0) = 0 \quad \text{and} \quad f(x) \neq \frac{c}{b}x \quad \text{if } x \neq 0, \tag{13}$$

$$0 < f'(0) < \frac{c}{b} \tag{14}$$

and

$$f'(x) \geqslant 0 \quad \text{for each } x. \tag{15}$$

Then $x(t) \equiv 0$ *is a globally asymptotically stable solution of* (3).

THEOREM 4. *Let* $a > 0$ *and let* $f(0) = 0$, $f \in C^1$. *If*

$$f'(v) \geqslant a \quad \text{for each } v \tag{16}$$

and the inequality is strict for $v = 0$, *then* $x(t) \equiv 0$ *is a globally asymptotically stable solution of* (4).

Remark 1. The Hurvitz interval for (1) is $H = (0, ab)$. Sędziwy [11] showed (using Popov's method [1]) that if f is continuous and satisfies (6) with $I = [\varepsilon, ab)$, where $\varepsilon > 0$ but arbitrary, then the conclusion of the theorem holds. From (8) and (11) it follows that f satifies (7) with $I^* = H$, but the class of functions f is more restrictive because of (11). Assumption (10) is to guarantee that $x(t) \equiv 0$ is locally asymptotically stable.

Remark 2. Note that $H = (c/b, \infty)$ is the Hurvitz interval for equation (2). If we assume only (6) with $I = (c/b, N)$ for N big enough, then, as shown by Pliss [9], equation (2) may admit a periodic solution, and thus $x(t) = 0$ cannot be globally asymptotically stable. If $I = (c/b, b^2/c + c/b - \varepsilon)$ in (6), then, as follows from [11], the conclusion of the theorem holds. In our case f satisfied (7) with $I^* = H$, but again we have the additional condition on f, namely inequality (12). Notice also that the strong inequality required in (12) for $x = 0$ means that local asymptotic stability takes place.

Remark 3. The Hurvitz interval for (3) is $H = (0, c/b)$. In paper [1] the stability in the large is shown (by using Popov's method) if f is continuous and satisfies (6) with $I = (0, 1/c)$ for $c^2 > b$. From (13) and (15) it follows that f satisfies (7) with $I^* = H$ if $c^2 > b$. But again we have a more restrictive assumption concerning the derivative of f. As in the previous cases, we assume the local asymptotic stability of the zero solution (assumption (14)).

Remark 4. In the case of system (4), $H = (a, \infty)$. In paper [5] it is proved that if f is of class C^1 and satisfies (7) with $I^* = (0, N)$ for N

big enough, then the conclusion of Theorem 4 holds. From the assumption $f(0) = 0$ and (16) it follows that f satisfies (7) with $I^* = [a, \infty)$.

2. Theorem of P. Hartman and Cz. Olech. The proofs of our theorems are based upon the following result (cf. [6]):

THEOREM A. *Consider a system of n real differential equations*

$$\dot{x} = f(x) \quad (x = (x_1, \ldots, x_n) \in E^n), \tag{i}$$

in which $f(x)$ is of class C^1 on E^n with values in E^n. Denote by $J(x)$ the Jacobian matrix of $f(x)$ and denote by $H(x) = \frac{1}{2}(J + J^*)$ the symmetric part of $J(x)$.

Assume the following conditions:

1° $f(0) = 0$, $f(x) \neq 0$ *if* $x \neq 0$,

2° $x = 0$ *is a locally asymptotically stable solution of* (i),

3° *the eigenvalues $\lambda_i(x)$ of $H(x)$ satisfy the inequalities $\lambda_i(x) + \lambda_j(x) \leqslant 0$, $1 \leqslant i < j \leqslant n$,*

4° $\int\limits_0^\infty [\min\limits_{\|x\|=r} \|f(x)\|]\, dr = +\infty$ $(\|x\| = (\sum\limits_{i=1}^n x_i^2)^{1/2})$.

Then $x(t) \equiv 0$ is a globally asymptotically stable solution of (i).

We shall also need certain simple algebraic lemmas, useful in checking assumption 3° [7].

LEMMA 1. *If a polynomial with a real coefficient*

$$\varphi(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

has n real roots $x_1, x_2, \ldots, x_n$ and if

$$a_i > 0 \quad \text{for } i = 1, 2, \ldots, (n-1), \quad a_n \geqslant 0,$$

then

$$x_i + x_j \leqslant 0 \quad \text{for } 0 \leqslant i < j \leqslant n.$$

LEMMA 2. *Let a polynomial of 3rd degree with a real coefficient*

$$\varphi(x) = x^3 + ax^2 + bx + c \tag{ii}$$

have real roots x_1, x_2, x_3.

The roots of (ii) *satisfy the inequalities*

$$x_i + x_j \leqslant 0 \quad \text{for } i = 1, 2, 3$$

if and only if either of the following conditions hold:

1. $a > 0$ *if* $b \geqslant 0$,

2. $a > 0$, $a^2 + b \geqslant 0$, $c \leqslant ab$ *if* $b \leqslant 0$.

LEMMA 3. *Let a polynomial of 4th degree with real coefficients*

$$\varphi(x) = x^4 + ax^3 + bx^2 + cx + d \tag{iii}$$

have real roots x_1, x_2, x_3, x_4.

The roots of (iii) *satisfy the inequalities*

$$x_i + x_j \leqslant 0 \quad \textit{for } 0 \leqslant i < j \leqslant 4$$

if and only if either of the following conditions hold:

1. $a > 0,\ b > 0$ *if* $c \geqslant 0$,

2. $a > 0,\ b > 0,\ a^3 + 4c \geqslant 0,\ (ab - 2c)^2 + ca^3 \geqslant 0,\ (c/a)(b - c/a) \geqslant d$ *if* $c \leqslant 0$.

3. Proofs of the results. Without any loss of generality we can put $b = 1$ in (1) and (2), because any other case can be reduced to this one by a linear change of coordinates.

Proof of Theorem 1. Replace (1) by the equivalent system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = -f(x_1) - x_2 - a x_3. \tag{17}$$

The change of variables

$$x_1 = y_1, \quad x_2 = \tfrac{1}{2}\sqrt{2}y_3, \quad x_3 = -(\tfrac{1}{2})y_1 + (\tfrac{1}{2})y_2 - \tfrac{1}{2}\sqrt{2}a y_3 \tag{18}$$

transforms (17) into the system

$$\dot{y}_1 = \tfrac{1}{2}\sqrt{2}y_3, \quad \dot{y}_2 = -2f(y_1) - \tfrac{1}{2}\sqrt{2}y_3, \quad \dot{y}_3 = -\tfrac{1}{2}\sqrt{2}y_1 + \tfrac{1}{2}\sqrt{2}y_2 - a y_3. \tag{19}$$

Since the transformation (18) is non-singular, to prove Theorem 1 it is enough to show that the trivial solution of (19) is globally asymptotically stable. For this purpose, we will demonstrate that the right-hand sides of (19) satisfy the assumptions 1°-4° of Theorem A.

For $|y_1|$ small enough

$$f(y_1) - f'(0)y_1 + o(y_1).$$

From (10) it follows that the origin is locally asymptotically stable. It is obvious that from (8) and (9) follow conditions 1° and 4°, respectively. The examination of assumption 3° requires more lines.

Denote by $J(y_1, y_2, y_3)$ the Jacobian matrix of the right-hand side of (19) and by J^* the transpose of J.

A simple calculation gives

$$H = \tfrac{1}{2}(J + J^*) = \begin{bmatrix} 0 & -f' & 0 \\ -f' & 0 & 0 \\ 0 & 0 & -a \end{bmatrix}$$

and the characteristic equation of H is

$$\lambda^3 + a\lambda^2 - (f')^2\lambda - a(f')^2 = 0. \tag{20}$$

It is obvious that the coefficients of (20) satisfy the assumption of Lemma 2 if

$$|f'(y)| \leqslant a.$$

Hence by (11) assumption 3° of Theorem A also holds, which ends the proof of Theorem 1.

Proof of Theorem 2. Instead of the equation of (2), consider the equivalent system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = -cx_1 - x_2 - f(x_3). \tag{21}$$

We will obtain the conclusion of Theorem 2 for the system

$$\dot{z}_1 = cz_1 - cz_2 + z_3, \quad \dot{z}_2 = cz_1 - cz_2, \quad \dot{z}_3 = -z_1 - f(z_3), \tag{22}$$

which is obtained from (24) by the non-singular linear transformation

$$x_1 = (1/c)z_2, \quad x_2 = z_1 - z_2, \quad x_3 = z_3.$$

From the assumptions $c > 0$, $f(0) = 0$, $f'(0) > c$ $(b = 1)$ using the same argument as that applied before, we conclude that (22) satisfies assumptions 1°, 2° and 4° of Theorem A. It remains to prove that 3° also holds. The Hermitian H of the right-hand side of (22) is a diagonal matrix

$$H = \mathrm{Diag}(c, -c, -f').$$

Since $f'(u) \geqslant c$, we see that (22) also satisfies 3°. Thus Theorem A ends the proof of Theorem 2.

Proof of Theorem 3. As previously, we will show the conclusion of Theorem 3 for the system

$$\begin{aligned} \dot{y}_1 &= -cy_1 + \sqrt{1+b}\,y_3 - f(y_1), \\ \dot{y}_2 &= -cy_2 - \sqrt{c^2-b}\,y_3, \\ \dot{y}_3 &= -\sqrt{1+b}\,y_1 + \sqrt{c^2-b}\,y_2 - cy_3, \end{aligned} \tag{23}$$

which is obtained from (3) by the non-singular linear transformation

$$x_1 = y_1, \quad x_2 = \sqrt{1+b}\,y_3, \quad x_3 = -by_1 + \sqrt{(1+b)(c^2-b)}\,y_2 + c\sqrt{1+b}\,y_3.$$

From the assumptions $c > 0$, $b > 0$, $c^2 > b$, (13) and (14) one can check, in the same way, as in the case of Theorem 1, that (23) satisfies assumptions 1°, 2° and 4° of Theorem A. Taking the symmetric part of the Jacobian matrix of (23), we again obtain a diagonal matrix

$$H = \mathrm{Diag}(-c-f', -c, -c).$$

From (15) and the form H it follows immediately that 3° is also satisfied, which completes the proof of Theorem 3.

Proof of Theorem 4. Applying the non-singular linear transformation

$$x_1 = w_1; \quad x_2 = w_2; \quad x_3 = aw_2 + aw_3$$

to the system (4) we obtain:

$$\dot{w}_1 = -f(w_1) + w_3, \quad \dot{w}_2 = -aw_2 - aw_3, \quad \dot{w}_3 = -w_1 + aw_2 + aw_3. \tag{24}$$

From the assumptions $a > 0$, $f(0) = 0$, $f'(0) > 0$ it follows directly that the right-hand side of (24) satisfies 1°, 2° and 4° of Theorem A.

The Jacobian matrix of (24) has the form

$$J = \begin{bmatrix} -f' & 0 & 1 \\ 0 & -a & -a \\ -1 & a & a \end{bmatrix}.$$

Hence the Hermitian of J is a diagonal matrix

$$H = \mathrm{Diag}(-f', -a, a)$$

and condition 3° of Theorem A follows from the assumed inequality $f'(u) \geqslant a$. Thus Theorem A ends the proof of Theorem 4.

References

[1] M. A. Ajzerman and F. P. Gantmacher, *Absolutnaja ustojcziwost reguliru-jemych sistem*, Moskwa 1963.

[2] E. A. Barbaszin, *Ob ustojcziwosti reszenia odnogo nieliniejnogo urawnienia tretiego poriadka*, PMM 16 (5) (1952), p. 629-632.

[3] J. O. C. Eezeilo, *On the boundedness of solutions of a certain differential equation of the third order*, Proc. London Math. Soc. 3 (9) (1959), p. 74-114.

[4] — *On the existence of periodic solutions of a certain third order differential equations*, Proc. Cambr. Phil. Soc. 56 (1960), p. 381-389.

[5] Ju. M. Filimonow, *Ob ustojcziwosti reszenij differencjalnych urawnienij tretiego poriadka*, Prikl. Mat. Mech. 25 (4) (1961), p. 785-790.

[6] P. Hartman and Cz. Olech, *On global asymptotic stability of solutions of differential equations*, Trans. Amer. Math. Soc. 104 (1962), p. 154-178.

[7] S. Kasprzyk, *Pewne zagadnienie z globalnej asymptotycznej stabilności*, Prace IV KKA 1 (1967), p. 197-203.

[8] — *Asymptotic behaviour of solutions of third order non-linear differential equations*, Zesz. Nauk. AGH, Autom. 3 (1969), p. 51-56.

[9] W. A. Pliss, *Issledowanie odnogo nieliniejnogo differencjalnogo urawnienia tretiego poriadka*, DAN 111 (6) (1956), p. 1078-1080.

[10] — *Ob ograniczennosti reszenij niekotorych nieliniejnych differencjalnych urawnienij tretiego poriadka*, DAN, 139 (2) (1961), p. 302-304.

[11] S. Sędziwy, *Asymptotic properties of solutions of non-linear differential equations of the higher order*, Zesz. Nauk. UJ, Prace Mat. 11 (1966), p. 69-80.

Reçu par la Rédaction le 13. 6. 1970