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A note on holomorphic mappings with two fixed points

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Abstract. Let X be a hyperbolic Riemann surface and let $a, b \in X$, $a \neq b$. We will prove that the set of all holomorphic mappings $f\colon X \to X$ with $f(a) = a$, $f(b) = b$ is a finite cyclic subgroup of the group of all holomorphic automorphisms of X.

Throughout the paper the following notation will be used:

$\boldsymbol{D} = \{z \in \boldsymbol{C}\colon |z| < 1\}$, $\boldsymbol{D}_* = \boldsymbol{D} \setminus \{0\}$,

$P(r, R) = \{z \in \boldsymbol{C}\colon r < |z| < R\}$, $0 \leqslant r < R \leqslant +\infty$;

X – a Riemann surface;

$\mathscr{G}(X; a, b)$ – the set of all holomorphic mappings $f\colon X \to X$ such that $f(a) = a$, $f(b) = b$ ($a, b \in X$, $a \neq b$);

$\mathrm{Aut}(X)$ – the group of all holomorphic automorphisms of X;

$f^{\langle n \rangle}$ – the n-th iterate of a mapping $f\colon X \to X$, i.e. $f^{\langle 0 \rangle} = \mathrm{id}_X$, $f^{\langle n \rangle} = f^{\langle n-1 \rangle} \circ f$, $n \in \boldsymbol{N}$.

The main result is the following theorem.

THEOREM 1. *Let X be a hyperbolic Riemann surface and let $a, b \in X$, $a \neq b$. Then $\mathscr{G}(X; a, b)$ is a finite cyclic subgroup of* $\mathrm{Aut}(X)$.

The proof will be based on the following elementary lemma.

LEMMA 1. *Let $\varnothing \neq A \subset \boldsymbol{D}_*$ be a set with no accumulation points in $\boldsymbol{D}$. Put $\mathscr{G}(A) := \{g \in \mathcal{O}(\boldsymbol{D}, \boldsymbol{D})\colon g(0) = 0,\ g(A) \subset A\}$. Then $\mathscr{G}(A)$ is a finite cyclic subgroup of the group* $\mathrm{Aut}_0(\boldsymbol{D})$ *of all rotations of* $\boldsymbol{D}$.

Proof. Observe that

$$\mathrm{id}_{\boldsymbol{D}} \in \mathscr{G}(A), \quad g_2 \circ g_1 \in \mathscr{G}(A), \quad g_1, g_2 \in \mathscr{G}(A). \tag{1}$$

Put $\varrho := \min\{|w|\colon w \in A\}$, $B := \{w \in A\colon |w| = \varrho\}$. It is clear that B is non-empty and finite. By the Schwarz lemma,

$$|g(w)| \leqslant |w|, \quad w \in \boldsymbol{D},\ g \in \mathscr{G}(A),$$

which implies that

$$\mathscr{G}(A) \subset \mathscr{G}(B). \tag{2}$$

Hence, in view of the definition of B (using again the Schwarz lemma), we conclude that $\mathscr{G}(B) \subset \mathrm{Aut}_0(\boldsymbol{D})$ and that $\mathscr{G}(B)$ is finite. In consequence, in view of (1) and (2), for every $g \in \mathscr{G}(A)$ there exists $n = n(g) \in \boldsymbol{N}$ such that $g^{-1} = g^{\langle n-1 \rangle}$. This shows that $\mathscr{G}(A)$ is a finite subgroup of $\mathrm{Aut}_0(\boldsymbol{D})$. In order to prove that $\mathscr{G}(A)$ is also cyclic it is enough to observe that $\mathscr{G}(A)$ is a subgroup of $\{\alpha \cdot \mathrm{id}_{\boldsymbol{D}}\colon \alpha \in \sqrt[n]{1}\}$, where $n := \prod_{g \in \mathscr{G}(A)} n(g)$. This completes the proof of the lemma.

Proof of Theorem 1. Let $p\colon \boldsymbol{D} \to X$ be a universal covering of X such that $p(0) = a$ (cf. [2], Chapter 3, §27). Put $A := p^{-1}(b)$. Note that A satisfies all the assumptions of Lemma 1. For $f \in \mathscr{G}(X;\, a, b)$ let $\tilde{f}\colon \boldsymbol{D} \to \boldsymbol{D}$ denote the lifting of f such that $\tilde{f}(0) = 0$. It is clear that $\tilde{f} \in \mathscr{G}(A)$. In particular, in view of Lemma 1, for every $f \in \mathscr{G}(X;\, a, b)$ there exists $n = n(f) \in \boldsymbol{N}$ such that $(f^{\langle n \rangle})^{\sim} = \tilde{f}^{\langle n \rangle} = \mathrm{id}_{\boldsymbol{D}}$. Consequently, $f^{\langle n \rangle} = \mathrm{id}_X$. This proves that $\mathscr{G}(X;\, a, b)$ is a subgroup of $\mathrm{Aut}(X)$ (note that if X is a bounded domain in $\boldsymbol{C}$, then the inclusion $\mathscr{G}(X;\, a, b) \subset \mathrm{Aut}(X)$ follows, for instance, from Satz 29, §6 in [1]). The mapping

$$\mathscr{G}(X;\, a, b) \ni f \to \tilde{f} \in \mathscr{G}(A)$$

may be now regarded as a group monomorphism and therefore (again by Lemma 1) we conclude that $\mathscr{G}(X;\, a, b)$ is a finite cyclic subgroup of $\mathrm{Aut}(X)$. The proof is finished.

Theorem 1 will be illustrated by examples.

Example 1. If X is a simply connected hyperbolic Riemann surface, then $\mathscr{G}(X;\, a, b) = \{\mathrm{id}_X\}$, $a, b \in X$, $a \neq b$.

Proof. In this case p is biholomorphic and consequently $\# A = 1$.

Example 2. Let $0 < r < R < +\infty$, $X = P = P(r, R)$, $a, b \in P$, $a \neq b$. Then

$$\mathscr{G}(P;\, a, b) = \begin{cases} \{\mathrm{id}_P,\ z \to a^2/z\} & \text{iff } |a| = \sqrt{rR} \text{ and } b = -a, \\ \{\mathrm{id}_P\} & \text{otherwise.} \end{cases}$$

Proof. By standard arguments one can reduce the proof to the case where $R > 1$, $r = 1/R$ and $1/R < a < R$. Define

$$p(z) = \exp(F_1^{-1} \circ F_2^{-1}(z)), \qquad z \in \boldsymbol{D},$$

where

$$F_1(z) := \tan(\mu z), \qquad -\log R < \operatorname{Re} z < \log R, \qquad \mu := \pi/(4 \log R),$$

$$F_2(z) := (z - \delta)/(1 - \delta z), \qquad z \in \boldsymbol{D},\ \delta := \tan(\mu \log a).$$

It is easily seen that $p\colon \boldsymbol{D} \to X$ is a universal covering of P and that $p(0) = a$.

Let $A := p^{-1}(b)$ and let B be as in the proof of Lemma 1. One can prove that $\# B \leqslant 2$. Hence, either $\mathcal{G}(A) = \{\mathrm{id}_D\}$ (and so $\mathcal{G}(P;\ a,\ b) = \{\mathrm{id}_P\}$) or $\mathcal{G}(A) = \{\mathrm{id}_D,\ -\mathrm{id}_D\}$. In the second case we get $\delta = 0$ and $\mathcal{G}(P;\ a,\ b) = \{\mathrm{id}_P,\ z \to 1/z\}$; consequently $a = 1,\ b = -1$.

EXAMPLE 3. For every $n \in N$ there exist a hyperbolic Riemann surface X and points $a,\ b \in X,\ a \neq b$, such that $\# \mathcal{G}(X;\ a,\ b) = n$.

Proof. In view of Examples 1, 2, we may assume that $n \geqslant 3$. Let $X := \hat{C} \setminus \sqrt[n]{1},\ a = 0,\ b = \infty$. Then, in virtue of the Picard theorem, $\mathcal{G}(X; 0, \infty) \subset \mathcal{G}(\hat{C}; 0, \infty)$ and therefore, in view of Theorem 1, $\mathcal{G}(X; 0, \infty) = \{\alpha \mathrm{id}_X\colon \alpha \in \sqrt[n]{1}\}$.

Remark 1. (a) If $P = P(0, R),\ 0 < R < +\infty$, or $P = P(r, +\infty),\ 0 < r +\infty$, then $\mathcal{G}(P;\ a,\ b) = \{\mathrm{id}_P\},\ a,\ b \in P,\ a \neq b$.

(b) If $P = P(0,\ +\infty)$ (P is a parabolic space) then $\{z \to z^{2k+1},\ k \in \mathbf{Z}\} \subset \mathcal{G}(P; 1, -1)$ and so neither $\mathcal{G}(P; 1, -1)$ is finite nor $\mathcal{G}(P; 1, -1) \subset \mathrm{Aut}(P)$.

Remark 2. Example 2 permits us to give an alternative method of the proof of the following well-known theorem on biholomorphisms between annuli (cf. [3], Theorem 14.22).

Let $0 < r_j < R_j < +\infty,\ P_j := P(r_j,\ R_j),\ j = 1, 2$. Then $P_1,\ P_2$ are biholomorphically equivalent iff $r_1/R_1 = r_2/R_2$. Moreover, every biholomorphism $F\colon P_1 \to P_2$ is either of the form $z \to \alpha r_2 z/r_1$ or $z \to \alpha r_1 R_2/z,\ z \in P_1,\ |\alpha| = 1$.

Proof. We may assume that

$$r_1/R_1 \leqslant r_2/R_2. \tag{3}$$

Fix a biholomorphism $F\colon P_1 \to P_2$ and let $a := \sqrt{r_1 R_1},\ b := -a$. The mapping

$$G(P_1;\ a,\ b) \ni f \to F \circ f \circ F^{-1} \in G(P_2;\ F(a),\ F(b))$$

is an isomorphism. Hence, by Example 2,

$$|F(a)| = \sqrt{r_2 R_2}, \qquad F(b) = -F(a). \tag{4}$$

Put

$$f(z) := aF(z)/F(a), \qquad z \in P_1.$$

Observe that, in view of (3), (4),

$$f(P_1) = P(\sqrt{r_1 r_2 R_1/R_2},\ \sqrt{r_1 R_1 R_2/r_2}) \subset P_1, \qquad f(a) = a,\ f(b) = b.$$

Thus $f \in \mathcal{G}(P_1;\ a,\ b)$ and therefore, by Example 2, $f \in \mathrm{Aut}(P_1)$ (in particular, $f(P_1) = P_1$ and so $r_1/R_1 = r_2/R_2$) and either $f = \mathrm{id}_{P_1}$ ($F(z) = \alpha r_2 z/r_1$) or $f(z) = a^2/z$ ($F(z) = \alpha r_1 R_2/z$).

References

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