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Asymptotic relationships between the solutions of two systems of differential equations

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Abstract. This paper is devoted to the study of the system (1) $dx/dt = A(t)x + f(t, x)$ on an interval $I = [t_0, \infty)$, where $A(t)$ is an $n \times n$ matrix and $f(t, x)$ is an n-vector whose components f_i, $i = 1, \ldots, n$ depend only on $t, x_{i_1}, \ldots, x_{i_q}$, $1 \leqslant i_1 < \ldots < i_q \leqslant n$. Conditions are found leading to an equivalence between certain components of solutions of the system (1) and certain components of solutions of the linear system $dy/dt = A(t)y$. The main result is proved with use of the Schauder–Tychonoff fixed point theorem; its application to the differential equation $x'' = a(t)x + f(t, x)$ yields a result announced by T. G. Hallam in Ann. Polon. Math. 24 (1971), p. 195–300.

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$$\frac{dy}{dt} = A(t)y. \tag{2}$$

The basic assumptions that will be made about (1) are as follows:

(i) *$A(t)$ is continuous on I.*

Under this assumption there exists to any constant vector c and $t_0 \in I$ the unique solution of (2) defined on I and satisfying the initial condition $y(t_0) = c$. Let $Y(t) = (y_{ij}(t))$ be a fundamental matrix of (2), $Y^{-1}(t) = (y^{ji}(t))$. Here $y^{ji}(t) = \det[Y(t)]^{-1} Y_{ji}(t)$, where $Y_{ji}(t)$ is the cofactor of $y_{ji}(t)$.

Let $z_{ij}(t), z^{ij}(t)$ be non-negative functions defined on I satisfying here the inequalities

$$|y_{ij}(t)| \leqslant z_{ij}(t), \quad |y^{ij}(t)| \leqslant z^{ij}(t); \quad i, j = 1, \ldots, n.$$

Let p be an integer $0 \leqslant p \leqslant n-q$ and choose p integers $i_{q+1}, \ldots, i_{q+p}$ between 1 and n different to $i_1, \ldots, i_q$ and denote

$$\mu_i(t) = \max_{j=1,\ldots,n} z_{ij}(t), \qquad i = i_1, \ldots, i_{q+p}$$

and by $m(t)$ the $(q+p)$-vector with the components $\mu_{i_1}(t), \ldots, \mu_{i_{p+q}}(t)$.

(ii) *The components f_i of f are continuous on $I \times R_q$ and satisfy the inequality*

$$|f_i(t, x_{i_1}, \ldots, x_{i_q})| \leqslant \omega_i(t, |x_{i_1}|, \ldots, |x_{i_q}|),$$

where $\omega_i(t, r_1, \ldots, r_q)$ are continuous functions for $t \in I$, $r_j \geqslant 0$, $j = 1, \ldots, q$ and $\omega_i(t, r_1, \ldots, r_q)$ are non-decreasing for each fixed $t \in I$.

Let $A = \{i_1, \ldots, i_q\}$ and $B = \{1, \ldots, n\} \setminus A$. Suppose that there exists a constant $\varkappa \geqslant 1$ such that

(iii) $\int_{t_0}^{\infty} z^{jk}(s)\,\omega_j\big(s, \varkappa m(s)\big)\,ds < \infty$ *for* $k \in A$, $j = 1, \ldots, n$,

(iv) $\lim_{t\to\infty} \dfrac{z_{ik}(t)}{\mu_i(t)} \int_{t_0}^{t} z^{jk}(s)\,\omega_j\big(s, \varkappa m(s)\big)\,ds = 0$ *for* $k \in B$, $j = 1, \ldots, n$, $i = i_1, \ldots, i_{q+p}$.

THEOREM. *Let conditions* (i)–(iv) *be satisfied. Then, to any constant vector $c = (\gamma_1, \ldots, \gamma_n)$ such that $\sum_{j=1}^{n} |\gamma_j| < \varkappa$, there exists a solution $x = x(t)$ of* (1) *whose i_k-th component is defined on I and satisfies*

$$\Big|x_{i_k}(t) - \sum_{j=1}^{n} y_{i_k,j}(t)\gamma_j\Big| = o\big(\mu_{i_k}(t)\big), \qquad k = 1, \ldots, q+p. \tag{3}$$

This theorem will be proved with use of the Schauder–Tychonoff fixed point theorem, which we are going to introduce in the following form (see [1], p. 9):

SCHAUDER–TYCHONOFF THEOREM. *Let F be the subset formed by those n-vectors $x(t) \in C(I)$ such that $|x(t)| \leqslant \mu(t)$ for all $t \in I$, where $\mu(t)$ is a fixed positive continuous function.*

Let T be a mapping of F into itself with the properties:

(i) *T is continuous in the sense that if $x_n \in F$ $(n = 1, 2, \ldots)$ and $x_n \to x$ uniformly on every compact subinterval of I, then $Tx_n \to Tx$ uniformly on every compact subinterval of I;*

(ii) *the functions in the image set TF are equicontinuous and bounded at every point of I.*

Then the mapping T has at least one fixed point in F.

Proof of Theorem. Without loss of generality we may suppose $i_1 = 1, \ldots, i_{q+p} = q+p$. Consider the subset $F \subset D_{q+p}(I)$, the set of continuous $(q+p)$-vectors $\tilde{x}(t)$ whose components $\xi_i(t)$, $i = 1, \ldots, q+p$ satisfy the conditions

$$|\xi_i(t)| \leqslant \varkappa\mu_i(t), \qquad t \in I.$$

For $\tilde{x} \in F$ define $T\tilde{x} = \big((T\tilde{x})_1, \ldots, (T\tilde{x})_{q+p}\big)$ by the equations

$$(4) \qquad \big(T\tilde{x}(t)\big)_i = \sum_{j=1}^{n} y_{ij}(t)\gamma_j + \int_{t_0}^{t} \sum_{\substack{j=1\\ k\in B}}^{n} y_{ik}(t) y^{jk}(s) f_j\big(s, \tilde{x}(s)\big)ds - \\ - \int_{t}^{\infty} \sum_{\substack{j=1\\ k\in A}}^{n} y_{ik}(t) y^{jk}(s) f_j\big(s, \tilde{x}(s)\big)ds, \qquad i = 1, \ldots, q+p.$$

By virtue of (iii) and (iv) we can assume that t_0 is sufficiently large so that

$$\int_{t}^{\infty} z^{jk}(s)\,\omega_j\big(s, \varkappa m(s)\big)ds < \varepsilon, \qquad \frac{z_{ik}(t)}{\mu_i(t)} \int_{t_0}^{t} z^{jk}(s)\,\omega_j\big(s, \varkappa m(s)\big)ds < \varepsilon \quad \text{for } t \geqslant t_0,$$

where

$$\varepsilon < \tfrac{1}{2} n^{-2}\Big(\varkappa - \sum_{j=1}^{n} |\gamma_j|\Big).$$

Then we have

$$\big|\big(T\tilde{x}(t)\big)_i\big| \leqslant \sum_{j=1}^{n} z_{ij}(t)\,|\gamma_j| + \int_{t_0}^{t} \sum_{\substack{j=1\\ k\in B}}^{n} z_{ik}(t) z^{jk}(s)\,\omega_j\big(s, \varkappa m(s)\big)ds + \\ + \int_{t}^{\infty} \sum_{\substack{j=1\\ k\in A}}^{n} z_{ik}(t) z^{jk}(s)\,\omega_j\big(s, \varkappa m(s)\big)ds < \mu_i(t) \sum_{j=1}^{n} |\gamma_j| + \mu_i(t) n^2 \varepsilon + \mu_i(t) n^2 \varepsilon \\ = \mu_i(t)\Big[\sum_{j=1}^{n} |\gamma_j| + 2n^2\varepsilon\Big] < \varkappa\mu_i(t).$$

This inequality shows that $TF \subset F$.

Next we verify that the transformation T is continuous.

Let $\tilde{x}_n$ $(n = 1, 2, \ldots)$ and $\tilde{x}$ be in F with $\tilde{x}_n$ converging uniformly to $\tilde{x}$ on every compact subinterval of I. Consider any interval of the form $[t_0, T]$. Let $\varepsilon > 0$ be given and choose $t_1 \geqslant T$ such that

$$(5) \qquad \int_{t_1}^{\infty} \sum_{\substack{j=1\\ k\in A}}^{n} z^{jk}(s)\,\omega_j\big(s, \varkappa m(s)\big)ds < \frac{\varepsilon}{4n^2K},$$

where

$$K = \max_{k\in A} \max_{t\in[t_0, T]} z_{ik}(t), \qquad i = 1, \ldots, q+p.$$

Since f is continuous and the sequence $\{\tilde{x}_n\}$ converges uniformly to $\tilde{x}$ on $[t_0, t_1]$, there exists a positive constant N such that if $n \geqslant N$, then

$$z^{jk}(s)\left|f_j\big(s, \tilde{x}_n(s)\big) - f_j\big(s, \tilde{x}(s)\big)\right| < \frac{\varepsilon}{4n^2 K |t_1 - t_0|}. \tag{6}$$

Using (5) and (6) we obtain

$$\left|\big(T\tilde{x}_n(t) - T\tilde{x}(t)\big)_i\right| \leqslant \int_{t_0}^{t} \sum_{\substack{j=1\\ k\in B}}^{n} z_{ik}(t) z^{jk}(s)\left|f_j\big(s, \tilde{x}_n(s)\big) - f_j\big(s, \tilde{x}(s)\big)\right| ds +$$

$$+ \int_{t}^{t_1} \sum_{\substack{j=1\\ k\in A}}^{n} z_{ik}(t) z^{jk}(s)\left|f_j\big(s, \tilde{x}_n(s)\big) - f_j\big(s, \tilde{x}(s)\big)\right| ds + 2\int_{t_1}^{\infty} \sum_{\substack{j=1\\ k\in A}}^{n} z_{ik}(t) z^{jk}(s)\,\omega_j\big(s, \varkappa m(s)\big)\, ds$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} = \varepsilon$$

for $n \geqslant N$, $t \in [t_0, T]$.

Therefore, the mapping T is continuous on F. From (4) we obtain

$$\big(T\tilde{x}(t)\big)_i' = \sum_{j=1}^{n} y_{ij}'(t)\gamma_j + \sum_{\substack{j=1\\ k\in B}}^{n} y_{ik}'(t) \int_{t_0}^{t} y^{jk}(s) f_j\big(s, \tilde{x}(s)\big) ds +$$

$$+ \sum_{\substack{j=1\\ k\in B}}^{n} y_{ik}(t) y^{jk}(t) f_j\big(t, \tilde{x}(t)\big) - \sum_{\substack{j=1\\ k\in A}}^{n} y_{ik}'(t) \int_{t}^{\infty} y^{jk}(s) f_j\big(s, \tilde{x}(s)\big) ds +$$

$$+ \sum_{\substack{j=1\\ k\in A}}^{n} y_{ik}(t) y^{jk}(t) f_j\big(t, \tilde{x}(t)\big)$$

so that in view of $\sum\limits_{j,k=1}^{n} y_{ik}(t) y^{jk}(t) f_j\big(t, \tilde{x}(t)\big) = f_i\big(t, \tilde{x}(t)\big)$

$$\left|\big(T\tilde{x}(t)\big)_i'\right| \leqslant \sum_{j=1}^{n} |y_{ij}'(t)|\,|\gamma_j| + \sum_{\substack{j=1\\ k\in B}}^{n} |y_{ik}'(t)| \int_{t_0}^{t} z^{jk}(s)\,\omega_j\big(s, \varkappa m(s)\big) ds +$$

$$+ \sum_{\substack{j=1\\ k\in A}}^{n} |y_{ik}'(t)| \int_{t}^{\infty} z^{jk}(s)\,\omega_j\big(s, \varkappa m(s)\big) ds + \omega_i\big(t, \varkappa m(t)\big).$$

For t in any compact subinterval of I, the right-hand side of the above inequality is bounded by a constant indenpendent of $\tilde{x} \in F$.

Thus TF is equicontinuous on every finite subinterval of I.

The Schauder–Tychonoff fixed point theorem implies that the transformation T has a fixed point $\tilde{x}_0 = \tilde{x}_0(t)$ in F.

Putting $x(t) = (\tilde{x}_0(t), \tilde{y}(t))$, where

$$\tilde{y}(t) = (\eta_{q+p+1}(t), \ldots, \eta_n(t)),$$

$$\eta_i(t) = \sum_{j=1}^{n} y_{ij}(t)\gamma_j + \int_{t_0}^{t} \sum_{\substack{j=1\\k\in B}}^{n} y_{ik}(t) y^{jk}(s) f_j(s, \tilde{x}_0(s))\,ds -$$

$$- \int_{t}^{\infty} \sum_{\substack{j=1\\k\in A}}^{n} y_{ik}(t) y^{jk}(s) f_j(s, \tilde{x}_0(s))\,ds, \qquad i = q+p+1, \ldots, n$$

it is easy to see from (4) that $x(t)$ is a solution of (1).

To complete the proof of the theorem, relation (3) must be verified. Using (4) with $Tx = x$ we obtain

$$\Big|x_i(t) - \sum_{j=1}^{n} y_{ij}(t)\gamma_j\Big| \leqslant \int_{t_0}^{t} \sum_{\substack{j=1\\k\in B}}^{n} z_{ik}(t) z^{jk}(s)\,\omega_j[s, \varkappa m(s)]\,ds +$$

$$+ \int_{t}^{\infty} \sum_{\substack{j=1\\k\in A}}^{n} z_{ik}(t) z^{jk}(s)\,\omega_j[s, \varkappa m(s)]\,ds, \qquad i = 1, \ldots, q+p.$$

Since the right-hand side of this inequality is $o(\mu_{ik}(t))$ by virtue of (iii) and (iv), the proof is complete.

Remark 1. Suppose $z_{ik}(t) > 0$ on I for $k \in B$, $i = i_1, \ldots, i_{q+p}$ and write $z_{ik}(t)/\mu_i(t) = \gamma_{ik}(t)$. Then (iv) can be written in the form

$$\lim_{t\to\infty} \int_{t_0}^{t} \gamma_{ik}(t)\gamma_{ik}^{-1}(s) \frac{z_{ik}(s) z^{jk}(s)\,\omega_j[s, \varkappa m(s)]}{\mu_i(s)}\,ds = 0 \tag{7}$$

$$\text{for } k \in B;\ j = 1, \ldots, n;\ i = i_1, \ldots, i_{q+p}.$$

Sufficient conditions for the validity of relation (7) may be found in [2] or in [4].

Remark 2. The application of the above theorem to the differential equation

$$x'' = a(t)x + f(t, x) \tag{8}$$

yields a result deduced by T. G. Hallam [3]:

Let $y_1 = y_1(t)$ and $y_2 = y_2(t)$ be solutions of the equation

$$y'' = a(t)y \qquad (t \geqslant t_0) \tag{9}$$

with the Wronskian of y_1 and y_2 equal to -1. Suppose that there exist positive continuous functions $y_i^* = y_i^*(t)$, $i = 1, 2$, which satisfy the inequality $|y_i(t)| \leqslant y_i^*(t)$ $(i = 1, 2;\ t \geqslant t_0)$. In equation (8) we will assume that $f = f(t, x)$ is continuous on $I \times R$. Furthermore, let f satisfy the inequality

$$|f(t, x)| \leqslant \omega(t, |x|),$$

where $\omega(t, r)$ is a continuous function defined for $t \geqslant t_0$, $r \geqslant 0$ that is non-decreasing in r for each fixed $t \in I$. Suppose that

$$\int_{t_0}^{\infty} y_1^*(s)\,\omega\big(s, ky_2^*(s)\big)ds < \infty$$

for some constant $k > 1$; and that

$$\gamma(t)\int_{t_0}^{t} \gamma^{-1}(s)\,y_1^*(s)\,\omega\big(s; ky_2^*(s)\big)ds = o(1) \qquad (t \to \infty),$$

where $\gamma(t) \equiv y_1^*(t)[y_2^*(t)]^{-1}$. Then there exists a solution $x = x(t)$ of equation (8) *such that*

$$|x(t) - y_2(t)| = o[y_2^*(t)].$$

References

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