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# A Jacobian condition for injectivity of differentiable plane maps

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*Zdzisław Opial in memoriam*

**Abstract.** In this paper we present a new condition on the Jacobian matrix of a differentiable plane map $T$ which entails the injectivity of $T$. This condition roughly speaking means that $T$ is monotone in two linearly independent directions.

Consider a class $C^1$ mapping $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ of the real plane $\mathbf{R}^2$,

$$T\colon (x, y) \to (P(x, y), Q(x, y)). \tag{1}$$

In this paper we present (in Theorem 1) a new condition on the derivative $T'$ of $T$, i.e. on the Jacobian matrix

$$T' = \begin{bmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{bmatrix}, \tag{2}$$

which entails the injectivity (univalence, or global one-to-one-ness) of the mapping $T$. Such conditions have many applications (see e.g. [5], [8]–[12]) and raise several unsolved problems (see especially [4], [6], [7], [9], [10] and the references cited there). Many authors, including the present ones, have previously written on this subject. See, e.g., [1]–[11] and the references cited there.

**Notation.** We think of vectors in $\mathbf{R}^2$ as column vectors and will frequently write $\mathbf{x} = (x, y)^t$ so that $\mathbf{x}_1 = (x_1, y_1)^t$ and $\mathbf{x}_2 = (x_2, y_2)^t$ denote two (possibly distinct) vectors in $\mathbf{R}^2$.

Theorem 1 (Main result). *A class $C^1$ mapping $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ of the real plane $\mathbf{R}^2$ into itself is globally one-to-one provided that*

(i) *for each $\mathbf{x}$ in $\mathbf{R}^2$*, $\det T'(x) \neq 0$, *and*

(ii) *there exist linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2)$ in $\mathbf{R}^2$ such that $\mathbf{0} \notin$ convex hull $\{T'(\mathbf{x})\mathbf{v}_i\colon \mathbf{x} \in \mathbf{R}^2\}$, $i = 1, 2$.*

Before proving this theorem we note that it has the following consequence.

COROLLARY 1. *The $C^1$ mapping* (1) *with Jacobian matrix* (2) *is globally one-to-one provided that* $\det T'$ *and at least one of the four products* $P_1P_2$, $P_1Q_2$, $Q_1P_2$, $Q_1Q_2$ *never vanishes on* $\boldsymbol{R}^2$.

Proof of Corollary 1. Choose $\boldsymbol{v}_1 = \boldsymbol{e}_1 = (1, 0)^t$ and $\boldsymbol{v}_2 = \boldsymbol{e}_2 = (0, 1)^t$, the standard basis vectors of $\boldsymbol{R}^2$. Then the convex sums (for $j = 1, 2$)

$$\sum_i \alpha_i T'(\boldsymbol{x}_i)\boldsymbol{e}_j = \Big(\sum_i \alpha_i P_j(\boldsymbol{x}_i), \sum_i \alpha_i Q_j(\boldsymbol{x}_i)\Big)^t$$

never vanish on $\boldsymbol{R}^2$ provided that at least one of the four products $P_1P_2$, $P_1Q_2$, $Q_1P_2$, $Q_1Q_2$ never vanishes on $\boldsymbol{R}^2$. It then follows that condition (ii) of Theorem 1 is satisfied. Since we were assuming condition (i), it now follows from Theorem 1 that $T$ is globally one-to-one. ∎

It is now obvious that Theorem 4 in [8] is contained in our Corollary 1 since it is assumed in [8] that $\det T' > 0$ on $\boldsymbol{R}^2$, trace $T' < 0$ on $\boldsymbol{R}^2$, and (at least) one of the two products $P_1Q_2$ or $P_2Q_1$ never vanishes on $\boldsymbol{R}^2$.

For the sake of comparison we state here another (different) theorem we recently obtained in [6].

THEOREM 2 (see [6]). *If* $T$: $\boldsymbol{R}^2 \to \boldsymbol{R}^2$ *is a polynomial mapping satisfying* $\det T' > 0$ *and trace* $T' < 0$ *on* $\boldsymbol{R}^2$, *then* $T$ *is globally one-to-one.*

In our proof (see [6]) of Theorem 2 the hypothesis that $T$ is a *polynomial* mapping plays an essential role, while in our proof (below) of Theorem 1, we require only that $T$ is of class $C^1$.

Proof of Theorem 1. Suppose that $T$ is not globally one-to-one. Then there exist points $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ in $\boldsymbol{R}^2$ forming a parallelogram $\mathscr{P}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$,

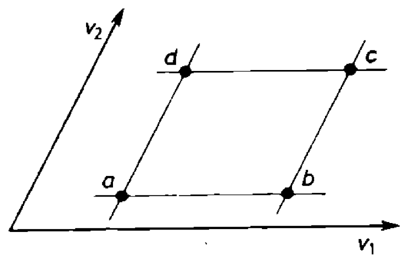


Fig. 1

where

$$\boldsymbol{b} = \boldsymbol{a} + k_1\boldsymbol{v}_1, \quad \boldsymbol{c} = \boldsymbol{b} + k_2\boldsymbol{v}_2,$$
$$\boldsymbol{d} = \boldsymbol{c} - k_1\boldsymbol{v}_1, \quad \boldsymbol{a} = \boldsymbol{d} - k_2\boldsymbol{v}_2,$$

with each $k_i > 0$ and having the following property:

(3) $T$ is one-to-one on the interior of $\mathscr{P}$, but $T(\boldsymbol{a}) = T(\boldsymbol{c})$ or $T(\boldsymbol{b}) = T(\boldsymbol{d})$.

Suppose $T(\boldsymbol{a}) = T(\boldsymbol{c})$. From (3) and hypothesis (i), it follows that the image of the two segments $\boldsymbol{ab}$ and $\boldsymbol{bc}$ (as well as $\boldsymbol{cd}$ and $\boldsymbol{da}$) form closed curves and one has to be inside the other. See Fig. 2.

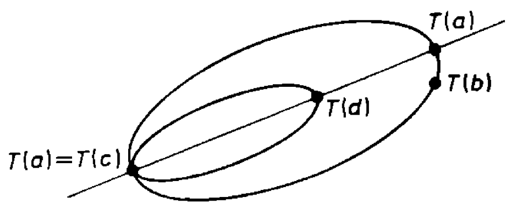


Fig. 2

The straight line joining $T(\boldsymbol{a})$ with $T(\boldsymbol{d})$ had to cross the outside boundary of $T(\mathscr{P})$ at some point $\boldsymbol{p} = T(\boldsymbol{x})$. Hence there is a point (vector) $\boldsymbol{x}$ in the segment $[\boldsymbol{a}, \boldsymbol{b}]$, or in the segment $[\boldsymbol{b}, \boldsymbol{c}]$, such that

$$T(\boldsymbol{x}) - T(\boldsymbol{a}) = s\big(T(\boldsymbol{d}) - T(\boldsymbol{a})\big) \quad \text{for some (real) } s > 1. \tag{4}$$

Suppose $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Then for some $\lambda > 0$, $\boldsymbol{x} - \boldsymbol{a} = \lambda \boldsymbol{v}_1$. We then apply to both sides of (4) the (vector) mean value theorem of McLeod [2] to obtain

$$T(\boldsymbol{x}) - T(\boldsymbol{a}) = \big(\alpha T'(\boldsymbol{x}_1) + \beta T'(\boldsymbol{x}_2)\big)\lambda \boldsymbol{v}_1$$

and

$$\begin{aligned} s\big(T(\boldsymbol{d}) - T(\boldsymbol{a})\big) = s\big(T(\boldsymbol{d}) - T(\boldsymbol{c})\big) &= s\big(\bar\alpha T'(\bar{\boldsymbol{x}}_1) + \bar\beta T'(\bar{\boldsymbol{x}}_2)\big)(\boldsymbol{d} - \boldsymbol{c}) \\ &= s\big(\bar\alpha T'(\bar{\boldsymbol{x}}_1) + \bar\beta T'(\bar{\boldsymbol{x}}_2)\big)(-k_1 \boldsymbol{v}_1). \end{aligned}$$

Hence

$$\lambda\big(\alpha T'(\boldsymbol{x}_1) + \beta T'(\boldsymbol{x}_2)\big)\boldsymbol{v}_1 + k_1 s\big(\bar\alpha T'(\bar{\boldsymbol{x}}_1) + \bar\beta T'(\bar{\boldsymbol{x}}_2)\big)\boldsymbol{v}_1 = 0,$$

which contradicts our hypothesis (ii). ∎

Our main result (Theorem 1 above) is true only in dimension two as the following example borrowed from Ravindran [9] shows. He defines the $C^1$-mapping of $\boldsymbol{R}^3$ into itself as

$$G(x) = A^{-1}F(Ax),$$

where

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

and

$$F(u, v, w) = \big(f_1(u, v), f_2(u, v), f_3(u, v, w)\big),$$

where

$$\begin{gathered} f_1(u, v) = e^{2u} - v^2 + 3, \quad f_2(u, v) = 4ve^{2u} - v^3, \\ f_3(u, v, w) = (10 + e^{2u})(e^v + e^{-v})(e^{100w} - e^{-100w}). \end{gathered}$$

Then $F(0, 2, 0) = F(0, -2, 0) = (0, 0, 0)$, so that $G$ is not injective, even though its Jacobian determinant is non-vanishing and each entry of the Jacobian matrix $G'$ is positive everywhere. See also [2].)

Indeed,

(A) the determinant of the Jacobian matrix $F'$ is positive,

(B) the third coordinate of $F'W$ is positive for the vector $W = (\alpha, \beta, 1)$ with $\alpha, \beta \in [0, 1]$,

(C) 0 does not belong to the convex hull of $\{F'(u, v, w)W\colon (u, v, w) \in \boldsymbol{R}^3\}$,

(D) there exist linearly independent vectors $W_i$, $i = 1, 2, 3$, in $\boldsymbol{R}^3$ such that, for each $i$, 0 is not in the convex hull of $\{F'(u, v, w)W_i\colon (u, v, w) \in \boldsymbol{R}^3\}$. For example, $W_i = (\alpha_i, \beta_i, 1)$ with $\alpha_i, \beta_i \in [0, 1]$, $i = 1, 2, 3$. Specifically, one could choose $(1, 0, 1)$, $(0, 1, 1)$, and $(0, 0, 1)$.

It follows from this example that our main result (Theorem 1) is not true in dimension three.

However, for arbitrary dimension $n$, we can state the following result (weaker than our Theorem 1 when the dimension is two), which is also based on McLeod's Mean Value Theorem [3].

THEOREM 3. *Let $T\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ denote a class $C^1$ mapping of n-dimensional real (euclidean) space $\boldsymbol{R}^n$ into itself. Then a sufficient condition for $T$ to be globally one-to-one is the following*:

(iii) *for every pair of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ in $\boldsymbol{R}^n$ with $\boldsymbol{a} \neq \boldsymbol{b}$, the difference $\boldsymbol{b} - \boldsymbol{a}$ does not belong to the null space of any n-term convex sum of the Jacobian $T'$ taken along the segment $[\boldsymbol{a}, \boldsymbol{b}]$. That is, $\forall \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^n$, with $\boldsymbol{a} \neq \boldsymbol{b}$,*

$$(5) \quad \Big[\sum_{k=1}^{n} \alpha_k T'(\boldsymbol{x}_k)\Big] \cdot (\boldsymbol{b} - \boldsymbol{a}) \neq 0 \quad \text{when } \alpha_k \geqslant 0,\ \sum \alpha_k = 1, \text{ and } \boldsymbol{x}_k \in [\boldsymbol{a}, \boldsymbol{b}].$$

The proof is immediate from McLeod's Mean Value Theorem [3] which allows us to express $T(\boldsymbol{b}) - T(\boldsymbol{a})$ in the form of the left-hand side of (5).

Note that condition (iii) of Theorem 3 is stronger than the condition

$$\text{(iv)} \quad \forall \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^n \quad \text{and} \quad \forall \boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}], \quad T'(\boldsymbol{x})(\boldsymbol{b} - \boldsymbol{a}) \neq 0 \quad \text{unless } \boldsymbol{a} = \boldsymbol{b},$$

which is equivalent to the classical condition

$$\text{(v)} \qquad \forall \boldsymbol{x} \in \boldsymbol{R}^n, \quad \det T'(\boldsymbol{x}) \neq 0.$$

Recall that (v) implies only that $T$ is *locally* one-to-one, but is too weak to imply global injectivity.

In [7] we have established that *global injectivity* for *polynomial* maps $T\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$, satisfying $\det T' \neq 0$, is equivalent to the *completeness* of the differential equation

$$\dot{\boldsymbol{x}} = [T'(\boldsymbol{x})^{-1}]\boldsymbol{a},$$

for each vector $\boldsymbol{a}$ in $\boldsymbol{R}^n$. This is related to the celebrated "Jacobian Conjecture" of Keller.

Finally, we remark that the converse of Theorem 1 is not true, even for polynomial mappings. The polynomial mapping

$$T(x, y) = \big(x+(y+x^2)^2,\ y+x^2\big)$$

is bijective with a polynomial inverse, but does not satisfy condition (ii) of Theorem 1.

**Appendix. McLeod's Mean Value Theorem.** We state here, for the reader's convenience, a version of McLeod's Mean Value Theorem as it can be simplified for $C^1$ maps of $\boldsymbol{R}^m$ into $\boldsymbol{R}^n$. See McLeod [3] for details and more general statements.

THEOREM (McLeod [3]). *For each class $C^1$ function $F\colon \boldsymbol{R}^m \to \boldsymbol{R}^n$, and for each pair of vectors $a$, $b$ in $\boldsymbol{R}^m$ there exist vectors $x_1, \ldots, x_n$ in the interval $[a, b] \equiv \{tb+(1-t)a\colon 0 \leqslant t \leqslant 1\}$ and nonnegative real numbers $c_1, \ldots, c_n$ such that $\sum c_k = 1$ and*

$$F(b)-F(a) = \sum_{k=1}^{n} c_k F'(x_k)(b-a).$$

Here $F'$ denotes the Jacobian matrix of $F$, and it would suffice for the domain $F$ to be open convex.

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