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Markov operators defined by Volterra type integrals with advanced argument

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Abstract. Sufficient conditions for asymptotical stability of some Markov operators defined by stochastic kernels are given. These operators appear in mathematical models of the cell cycle. They are also related with linear Volterra equations with advanced argument.

0. Introduction. In the mathematical theory of the cell cycle an important role plays a class of integral Markov operators of the form

$$Pf(x) = \int_0^{\lambda(x)} K(x, y) f(y)\, dy, \tag{0.1}$$

where

$$K(x, y) = -\frac{\partial}{\partial x} \exp\left\{-\int_y^{\lambda(x)} q(z)\, dz\right\} \tag{0.2}$$

and q, λ are nonnegative given functions.

In a recent paper [7], J. Tyrcha proved the asymptotical stability of the iterates $\{p^n\}$, assuming that the conditions

$$\liminf_{x\to\infty} [\lambda(x) - x] = \lambda_0, \quad 0 < \lambda_0 < \infty, \quad \lambda(0) = 0$$

and

$$\lim_{x\to\infty} q(x) = q_0, \quad \lambda_0 q_0 > 1, \quad 0 < q_0 < \infty$$

are satisfied.

Setting $\lambda(x) = x/\sigma$ $(0 < \sigma < 1)$ and

$$q(x) = \begin{cases} 0 & \text{for } 0 \leqslant x < 1, \\ c/x & \text{for } \quad x \geqslant 1, \end{cases}$$

we obtain a special form of (0.1), (0.2), namely,

$$Pf(x) = \int_0^{x/\sigma} K(x, y)f(y)dy, \tag{0.3}$$

$$K(x, y) = \begin{cases} 0 & \text{for } 0 \leqslant x < \sigma, \\ (c/\sigma)(x/\sigma)^{-1-c} & \text{for } 0 \leqslant y < 1,\ x \geqslant \sigma, \\ (c/\sigma)(x/\sigma)^{-1-c}y^c & \text{for } 1 \leqslant y \leqslant x/\sigma,\ x \geqslant \sigma. \end{cases} \tag{0.4}$$

Operators (0.3), (0.4) were considered by Tyson and Hannsgen [8] who established asymptotic stability of $\{P^n\}$ for $1/c < 1-\sigma$. This result was extended by Tyrcha [7]. She proved that the inequality

$$1/c < -\ln\sigma \tag{0.5}$$

is a necessary and sufficient condition for the asymptotic stability of $\{P^n\}$. Tyrcha's proof of necessity is quite sophisticated and based on the Central Limit Theorem.

Our paper unifies and generalizes all these results. From Theorems 2.1 and 2.2 it follows that Tyrcha's condition $\lambda_0 q_0 > 1$ is the best possible for asymptotic stability. Further in the case $\lambda_0 q_0 < 1$ the operator (0.1) with kernel (0.2) is not asymptotically stable but has another property which we call *sweeping*. This part of our paper was stimulated by recent results of Komorowski and Tyrcha [2].

The organization of the paper is as follows. In Section 1 we give the basic definitions and theorems concerning the stability of Markov operators. Sections 2 and 3 are devoted to a study of integral operators of the form (0.1). Section 4 contains applications.

1. Markov operators. Let $L^1 = L^1(\boldsymbol{R}_+)$ be the space of all integrable functions on the half line $\boldsymbol{R}_+ = [0, +\infty)$. A linear mapping $P\colon L^1 \to L^1$ is called a *Markov operator* if it satisfies the following conditions:

$$Pf \geqslant 0 \quad \text{for } f \geqslant 0,\ f \in L^1, \tag{1.1}$$

$$\|Pf\| = \|f\| \quad \text{for } f \geqslant 0,\ f \in L^1, \tag{1.2}$$

where $\|\cdot\|$ stands for the norm in L^1.

By D we denote the set of all (normalized) densities on $\boldsymbol{R}_+$, i.e.

$$D = \{f \in L^1\colon f \geqslant 0 \text{ and } \|f\| = 1\}.$$

Let a Markov operator P be given. A density f is called *stationary* if $Pf = f$.

A Markov operator $P\colon L^1 \to L^1$ is called *asymptotically stable* if there is a unique stationary $f_* \in D$ and if

$$\lim_{n\to\infty} \|P^n f - f_*\| = 0 \quad \text{for every } f \in D. \tag{1.3}$$

A Markov operator P: $L^1 \to L^1$ is called *sweeping* if

$$(1.4) \qquad \lim_{n\to\infty} \int_0^c P^n f\, dx = 0 \quad \text{for every } f \in D \text{ and } c > 0.$$

In what follows, we will exclusively consider Markov operators of the form

$$(1.5) \qquad Pf(x) = \int_0^\infty K(x, y) f(y)\, dy,$$

where K is a stochastic kernel, i.e. K: $\boldsymbol{R}_+^2 \to \boldsymbol{R}_+$ is measurable, and

$$(1.6) \qquad \int_0^\infty K(x, y)\, dx = 1 \quad \text{for } y \geqslant 0.$$

For operators (1.6), it is easy to find simple sufficient conditions for asymptotic stability and sweeping using Lyapunov and Bielecki functions.

A function V: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ is called a *Lyapunov function* if it is measurable, locally bounded (bounded on compact subsets of $\boldsymbol{R}_+$) and satisfies

$$(1.7) \qquad \lim_{x\to\infty} V(x) = \infty.$$

A function V: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ will be called a *Bielecki function* if it is measurable, locally bounded, and

$$(1.8) \qquad \inf_{0 \leqslant x \leqslant c} V(x) > 0 \quad \text{for every } c > 0.$$

Proposition 1.1. *Let* K: $\boldsymbol{R}_+^2 \to \boldsymbol{R}_+$ *be a stochastic kernel and* P *the corresponding Markov operator* (1.5). *Assume that there exists a Lyapunov function* V *and nonnegative constants* δ, $\gamma < 1$ *such that*

$$(1.9) \qquad \int_0^\infty V(x) K(x, y)\, dx \leqslant \delta + \gamma V(y) \quad \textit{for } y \geqslant 0.$$

Moreover, assume that

$$(1.10) \qquad \int_0^\infty \inf_{0 \leqslant y \leqslant a} K(x, y)\, dx > 0 \quad \textit{for } a > 0.$$

Then the operator P *is asymptotically stable.*

The proof of Proposition 1.1 can be found in [4].

Proposition 1.2. *Let* P *be a Markov operator. If there exists a Bielecki function* V *and a nonnegative constant* $\gamma < 1$ *such that*

$$(1.11) \qquad \int_0^\infty V(x) Pf(x)\, dx \leqslant \gamma \int_0^\infty V(x) f(x)\, dx \quad \textit{for } f \in D,$$

then the operator P is sweeping.

The proof is straightforward.

Remark 1.1. The definitions of asymptotic stability and sweeping for a Markov operator $P\colon L^1(X) \to L^1(X)$ can be formulated for more general situations. In particular, asymptotic stability is defined when X is a measure space [4] and sweeping when X is simultaneously a measure space and a topological space [2]. It is worthwhile to compare these definitions with the property of strong and weak mixing [5].

2. Asymptotic stability and sweeping. Setting

$$K(x, y) = -1_{[y,\infty]}(\lambda(x))\frac{d}{dx}\exp\left\{-\int_y^{\lambda(x)} q(z)dz\right\}, \tag{2.1}$$

where $1_{[y,\infty]}$ is the characteristic function of the interval $[y, \infty)$, we may rewrite the integral operator (0.1), (0.2) in the form (1.5). The kernel (2.1) is stochastic under quite general assumptions concerning q and λ. To fix ideas, we shall assume the following conditions:

(i) The function $\lambda\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is continuously differentiable. Moreover,

$$\lambda'(x) > 0 \quad \text{for } x \geqslant 0, \quad \lambda(0) = 0, \quad \text{and} \quad \lim_{x\to\infty} \lambda(x) = \infty.$$

(ii) The function $q\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is locally integrable and

$$\int_0^\infty q(x)dx = \infty.$$

An important role in our further consideration plays the function

$$H(x) = Q(\lambda(x)) - Q(x), \quad \text{where } Q(x) = \int_0^x q(y)dy. \tag{2.2}$$

Using (2.2), it is easy to formulate a sufficient condition for the asymptotic stability.

THEOREM 2.1. *If q and λ satisfy* (i), (ii) *and if*

$$\liminf_{x\to\infty} H(x) > 1, \tag{2.3}$$

then the operator P given by (0.1), (0.2) *is asymptotically stable.*

Proof. Let s be an arbitrary number satisfying $1 < s < \liminf_{x\to\infty} H(x)$ and let x_0 be such that

$$H(x) \geqslant s \quad \text{for } x \geqslant x_0. \tag{2.4}$$

Consider a Lyapunov function $V_h(x) = e^{hQ(x)}$ where the positive constant h will be chosen later. From (2.1) it follows immediately that

$$\int_0^\infty V_h(x)K(x, y)dx = I_1 + I_2, \tag{2.5}$$

where

$$I_1 = e^{Q(y)} \int_0^{x_0} \lambda'(x)q(\lambda(x))e^{-Q(\lambda(x))+hQ(x)} 1_{[y,\infty)}(\lambda(x))dx$$

and

$$I_2 = e^{Q(y)} \int_{x_0}^\infty \lambda'(x)q(\lambda(x))e^{-Q(\lambda(x))+hQ(x)} 1_{[y,\infty)}(\lambda(x))dx.$$

The integral I_1 is easy to evaluate. Namely,

$$I_1 \leqslant \int_0^{x_0} \lambda'(x)q(\lambda(x))e^{hQ(x)}dx \stackrel{\mathrm{df}}{=} \delta(h). \tag{2.6}$$

To evaluate I_2 observe that, according to (2.2) and (2.4),

$$-Q(\lambda(x)) + hQ(x) \leqslant -(1-h)Q(\lambda(x)) - hs \quad \text{for } x \geqslant x_0.$$

Consequently,

$$\begin{aligned} I_2 &\leqslant e^{Q(y)-hs} \int_{x_0}^\infty \lambda'(x)q(\lambda(x))e^{-(1-h)Q(\lambda(x))} 1_{[y,\infty)}(\lambda(x))dx \\ &= e^{Q(y)-hs} \int_{\lambda(x_0)}^\infty q(z)e^{-(1-h)Q(z)} 1_{[y,\infty)}(z)dz \\ &\leqslant e^{Q(y)-hs} \int_y^\infty q(z)e^{-(1-z)Q(z)}dz = \frac{e^{-hs}}{1-h} V_h(y). \end{aligned} \tag{2.7}$$

Now define

$$\gamma(h) = e^{-hs}/(1-h)$$

and observe that $\gamma(0) = 1$, $\gamma'(0) = 1-s < 0$. Thus there exists an $h \in (0, 1)$ such that $\gamma(h) < 1$. From (2.5), (2.6) and (2.7) it follows that condition (1.9) of Proposition 1.1 is satisfied. To verify (1.10) observe that

$$\inf_{0 \leqslant y \leqslant a} K(x, y) \geqslant \lambda'(x)q(\lambda(x)) \exp\{-\int_0^{\lambda(x)} q(z)dz\} \quad \text{for } \lambda(x) \geqslant a,$$

and consequently,

$$\int_0^\infty \inf_{0 \leqslant y \leqslant a} K(x, y)dx \geqslant \exp\{-\int_0^a q(z)dz\} > 0.$$

The proof is completed.

The proof of Theorem 2.1 was relatively short due the special choice of the Lyapunov function V_h. Such functions are considered in the theory of differential equations with advanced argument [1]. Using similar but more sophisticated technique, we may prove a sufficient condition for sweeping.

THEOREM 2.2. *If q and λ satisfy* (i), (ii) *and*

$$\limsup_{x\to\infty} H(x) < 1, \tag{2.8}$$

then the operator P given by (0.1), (0.2) *is sweeping.*

Proof. Let w be an arbitrary nonnegative number satisfying $\limsup H < w < 1$. Further, let x_0 be the smallest nonnegative number such that

$$H(x) \leqslant w \quad \text{for } x \geqslant x_0. \tag{2.9}$$

Due to the continuity of H we have either $H(x_0) = w$ or $x_0 = 0$. In both the cases we have $H(x_0) \geqslant 0$ and consequently, $\lambda(x_0) \geqslant x_0$.

Now define a Bielecki function by the formula

$$V_h(x) = \begin{cases} e^{hQ(x_0)} & \text{for } 0 \leqslant x < x_0, \\ e^{hQ(x)} & \text{for } x_0 \leqslant x, \end{cases}$$

where a negative constant h will be chosen later. An immediate calculation shows that

$$\int_0^\infty V_h(x)K(x, y)dx = I_1 + I_2, \tag{2.10}$$

where

$$I_1 = \begin{cases} 0 & \text{for } y \geqslant \lambda(x_0), \\ e^{hQ(x_0)} - e^{Q(y)+hQ(x_0)-Q(\lambda(x_0))} & \text{for } y < \lambda(x_0), \end{cases} \tag{2.11}$$

and I_2 is the same as in formula (2.5). From (2.9) it follows that $Q(x) \geqslant Q(\lambda(x)) - w$ and consequently

$$I_2 \leqslant e^{Q(y)} \int_{x_0}^\infty \lambda'(x)q(\lambda(x))e^{-(1-h)Q(\lambda(x))-hw} 1_{[y,\infty)}(\lambda(x))dx$$

$$= e^{Q(y)-hw} \int_{\lambda(x_0)}^\infty q(z)e^{-(1-h)Q(z)} 1_{[y,\infty)}(z)dz.$$

Therefore,

$$I_2 \leqslant \begin{cases} \dfrac{e^{-hw}}{1-h} e^{hQ(y)} & \text{for } y \geqslant \lambda(x_0), \\ \dfrac{e^{-hw}}{1-h} e^{Q(y)-(1-h)Q(\lambda(x_0))} & \text{for } y < \lambda(x_0). \end{cases} \tag{2.12}$$

Now we are going to show that the integral (2.10) satisfies the inequality

$$\int_0^\infty V_h(x)K(x, y)dx \leqslant \gamma(h)V_h(y) \tag{2.13}$$

with a coefficient $\gamma(h)$ such that $\gamma(h) < 1$ for sufficiently small absolute value of h. We will consider three cases.

Case 1. $y \geqslant \lambda(x_0)$. In this case, $V_h(y) = e^{hQ(y)}$. From (2.10), (2.11) and (2.12) it follows that (2.13) is satisfied with

$$\gamma_1(h) = e^{-hw}/(1-h). \tag{2.14}$$

Since $\gamma_1(0) = 1$ and $\gamma_1'(0) = 1-w > 0$, there exists an $h_1 < 0$ such that $\gamma_1(h) < 1$ for $h_1 < h < 0$.

Case 2. $x_0 \leqslant y < \lambda(x_0)$. Again, in this case, $V_h(y) = e^{hQ(y)}$ and from (2.11), (2.12) it follows that

$$I_1 + I_2 \leqslant V_h(y)\big[e^{h(Q)(x_0) - Q(y))} - e^{(1-h)Q(y) + hQ(x_0) - Q(\lambda(x_0))} + \frac{e^{-hw}}{1-h} e^{(1-h)(Q(y) - Q(\lambda(x_0)))}\big]. \tag{2.15}$$

Since $\lambda(x_0) > 0$, we have $x_0 > 0$. Consequently, $H(x_0) = w$ and $Q(x_0) = Q(\lambda(x_0)) - w$. Using this, we may rewrite (2.15) in the form

$$I_1 + I_2 \leqslant V_h(y)e^{-hw}\left[e^{h(Q(\lambda(x_0)) - Q(y))} + \frac{h}{1-h} e^{(1-h)(Q(y) - Q(\lambda(x_0)))}\right].$$

The last inequality may be written shorter. Namely,

$$I_1 + I_2 \leqslant V_h(y)e^{-hw}\left[e^{hz} + \frac{h}{1-h} e^{(1-h)z}\right],$$

where $z = Q(\lambda(x_0)) - Q(y) \geqslant 0$. For $z \geqslant 0$, the term in brackets is a decreasing function of z and admits its maximal value for $z = 0$. Thus

$$I_1 + I_2 \leqslant V_h(y)e^{-hw}(1 + h/(1-h))$$

which again implies (2.13) with γ_1 given by (2.14).

Case 3. $0 \leqslant y < x_0$. In this case, $V_h(y) = e^{hQ(x_0)}$ and from (2.11), (2.12) we obtain

$$I_1 + I_2 \leqslant V_h(y)\left[1 - e^{Q(y) - Q(\lambda(x_0))} + \frac{e^{-hw}}{1-h} e^{Q(y) - hQ(x_0) - (1-h)Q(\lambda(x_0))}\right].$$

Since $x_0 > 0$, we have $Q(x_0) = Q(\lambda(x_0)) - w$ and consequently

$$I_1 + I_2 \leqslant V_h(y)\left[1 + \frac{h}{1-h} e^{Q(y) - Q(\lambda(x_0))}\right].$$

The term in brackets admits its maximal value for $r = 0$. Thus (2.13) is satisfied with

$$\gamma_3(h) = 1 + \frac{h}{1-h} e^{-Q(\lambda(x_0))}.$$

Evidently, $\gamma_3(h) < 1$ for all $h < 1$.

Thus, condition (2.13) is satisfied in all cases if $\gamma = \min(\gamma_1, \gamma_3)$. Moreover, $\gamma(h) < 1$ if $h_1 < h < 0$ and h_1 is defined as in Case 1. According to Proposition 1.2 the proof is completed.

3. A nonexistence theorems. Theorems 2.1 and 2.2 do not describe the asymptotic behaviour of $\{P^n f\}$ when $\lim_{x\to\infty} H(x) = 1$. A partial answer to this problem will be given in this section. We shall assume that the functions λ and q satisfy two additional conditions:

(iii) There is an $a \geqslant 0$ such that

$$q(x) > 0 \quad \text{for } x > a \quad \text{and} \quad q(x) = 0 \quad \text{for } x < a.$$

(If $a = 0$ only the first part of condition (3.1) is required.)

(iv) $\lambda(x) > x$ for $x > 0$.

THEOREM 3.1. *If q and λ satisfy conditions* (i)–(iv) *and if*

$$H(x) \leqslant 1 \quad \textit{for } x \geqslant 0, \tag{3.1}$$

then the operator P defined by (0.1), (0.2) *has no stationary density.*

Proof. Suppose not. Let f be a stationary density.

We have

$$f(x) = \lambda'(x) q(\lambda(x)) e^{-Q(\lambda(x))} \int_0^{\lambda(x)} e^{Q(y)} f(y)\, dy \quad \text{for } x \geqslant 0. \tag{3.2}$$

Using inequality $\lambda(x) > x$, it is easy to verify that $f(x) > 0$ for $x > \lambda^{-1}(a)$. Now consider a Bielecki function

$$V_h(x) = e^{-hQ(x)} \quad (h \geqslant 0) \tag{3.3}$$

and define

$$E(h) = \int_0^{\infty} V_h(x) f(x)\, dx. \tag{3.4}$$

Our goal is to evaluate $E(h)$ using (3.3) and to show that this evaluation leads to a contradiction. Substituting (3.2) and (3.3) into (3.4) and changing the order of integration, we obtain

$$E(h) = \int_0^\infty f(y) F_h(y) dy, \tag{3.5}$$

where

$$F_h(y) = e^{Q(y)} \int_{\lambda^{-1}(y)}^{\infty} \lambda'(x) q(\lambda(x)) e^{-hQ(x) - Q(\lambda(x))} dx.$$

Setting $\bar{H}(x) = H(\lambda^{-1}(x))$ or $Q(x) = Q(\lambda(x)) - \bar{H}(\lambda(x))$, we may rewrite $F_h(y)$ in the form

$$F_h(y) = e^{Q(y)} \int_y^\infty q(z) e^{-(1+h)Q(z) + h\bar{H}(z)} dz. \tag{3.6}$$

Since $\bar{H} \leqslant 1$, this gives immediately

$$F_h(y) \leqslant e^{h + Q(y)} \int_y^\infty q(z) e^{-(1+h)Q(z)} dz = \frac{e^h}{1+h} V_h(y) \quad \text{for } y \geqslant 0. \tag{3.7}$$

Observe that inequality (3.7) is valid for all possible values of y. Now we are going to evaluate $F_h(y)$ for y in a neighbourhood of the point a. Evidently,

$$\bar{H}(a) = Q(a) - Q(\lambda^{-1}(a)) = \int_{\lambda^{-1}(a)}^{a} q(z) dz = 0$$

and due to the continuity of $\bar{H}$ there is an $r > 0$ such that

$$\bar{H}(y) \leqslant 1/2 \quad \text{for } y \leqslant a + 2r.$$

Using (3.6) for $y \leqslant 2r$, we obtain

$$\begin{aligned} F_h(y) &\leqslant e^{Q(y)} \left[\int_y^{a+2r} q(x) e^{h - (1+h)Q(x)} dx + \int_{a+2r}^{\infty} q(x) e^{h-(1+h)Q(x)} dx \right] \\ &= \frac{V_h(y)}{1+h} e^{1/2h} (1 - e^z) + e^{h+z}, \end{aligned}$$

where $z = (1+h)(Q(y) - Q(a+2r))$. The term in brackets is an increasing function of z and

$$z \leqslant - \int_{a+r}^{a+2r} q(x) dx < 0 \quad \text{for } y \leqslant a + r.$$

Therefore,

$$F_h(y) \leqslant \frac{V_h(y)}{1+h} [(1-k) e^{1/2h} + k e^h] \quad \text{for } y \geqslant r, \tag{3.8}$$

where $k = \max_{y \leqslant r} e^z < 1$. Now write $\varrho = \frac{1}{4}(1-k)$. Using the inequalities

$$\frac{e^h}{1+h} \leqslant 1 + h^2, \quad \frac{1}{1+h} [(1-k) e^{1/2h} + k e^h] \leqslant 1 - \varrho h$$

which are valid for sufficiently small h (say $0 \leqslant h \leqslant h_0$), we may rewrite (3.7) and (3.8) in the final form

$$F_h(y) \leqslant (1+h^2)V_h(y) \quad \text{for } y \geqslant 0, \qquad F_h(y) \leqslant (1-\varrho h)V_h(y) \quad \text{for } a \leqslant y \leqslant a+r.$$

Write $\Delta = [a,\ a+r]$. From this and (3.5) we obtain

$$\begin{aligned} E(h) &\leqslant \int_{R_+\setminus\Delta} f(y)F_h(y)dy + \int_\Delta f(y)F_h(y)dy \\ &\leqslant (1+h^2)\int_{R_+\setminus\Delta} f(y)V_h(y)dy + (1-\varrho h)\int_\Delta f(y)V_h(y)dy \\ &= (1+h^2)\int_0^\infty f(y)V_h(y)dy - (\varrho h + h^2)\int_\Delta f(y)V_h(y)dy \end{aligned}$$

or

$$0 \leqslant h^2 E(h) - (\varrho h + h^2)C(h), \tag{3.9}$$

where

$$C(h) = \int_\Delta f(y)V_h(y)dy.$$

Observe that $E(h)$ and $C(h)$ are continuous functions of h and that

$$E(0) = \int_0^\infty f(y)dy = 1, \qquad C(0) = \int_a^{a+r} f(y)dy.$$

Thus, dividing (3.9) by $hE(h)$ and passing to the limit as $h \to 0$, we obtain

$$0 \leqslant -\varrho \int_a^{a+r} f(y)dy.$$

The last inequality is impossible since $\varrho > 0$ and $f(y) > 0$ for $y > a$ (even for $y > \lambda^{-1}(a)$). This contradiction completes the proof.

Remark 3.1. A crucial role in our proofs is played by the possibility of changing variables via substitution $\lambda(x) = y$ or $q(y) = z$. Thus it is necessary to assume that (a) λ, Q and $Q \circ \lambda$ are absolutely continuous, and (b) $(d/dx)(Q(\lambda(x))) = q(\lambda(x))\lambda'(x)$ almost everywhere. Thus in assumptions (i), (ii) the conditions $\lambda'(x) > 0$, the continuity of λ' and the local integrability of q may be replaced, for example, by (a), (b) and $\lambda' \geqslant 0$. In this case, however, it is necessary to add the inequality $\lambda'(x) > 0$ in assumption (iv).

4. Applications. Consider the integral operator P given by formulas (0.1), (0.2) and assume that q and λ satisfy conditions (i) and (ii). We shall apply Theorems 2.1 and 2.2 in two special cases related with the behaviour of $q(x)$ as $x \to \infty$.

First denote

$$
(4.1)\qquad \begin{aligned} q_0 &= \liminf_{x\to\infty} q(x), & q_1 &= \limsup_{x\to\infty} q(x),\\ \lambda_0 &= \liminf_{x\to\infty}(\lambda(x)-x), & \lambda_1 &= \limsup_{x\to\infty}(\lambda(x)-x). \end{aligned}
$$

Using the equality

$$
H(x) = \int_x^{\lambda(x)} q(z)dz,
$$

we obtain immediately that

$$
(4.2)\qquad \liminf_{x\to\infty} H(x) \geqslant q_0\lambda_0, \quad \limsup_{x\to\infty} H(x) \leqslant \lambda_1 q_1.
$$

From Theorems 2.1 and 2.2 it follows

COROLLARY 4.1. *If* $\lambda_0 q_0 > 1$ *then the operator* P *defined by* (0.1), (0.2) *is asymptotically stable, if* $\lambda_1 q_1 < 1$ *then* P *is sweeping.*

This corollary extends a result of Tyrcha [7] who proved the asymptotic stability of P if $\lambda_0 q_0 > 1$. It also contains as a special case a stability theorem of Lasota–Mackey [3], where the conditions $\lambda(x) = 2x$ and $q_0 > 0$ are assumed.

Now admit $\lambda(x) = x/\sigma$ $(0 < \sigma < 1)$ and consider the integral operator

$$
(4.3)\qquad Pf(x) = \int_0^{x/\sigma} K(x, y)f(y)dy,
$$

where

$$
(4.4)\qquad K(x, y) = -\frac{d}{dx}\exp\{-\int_y^{x/\sigma} q(z)dz\}.
$$

Define

$$
(4.5)\qquad c_0 = \liminf_{x\to\infty}(xq(x)) \quad \text{and} \quad c_1 = \limsup_{x\to\infty}(xq(x)).
$$

Using the formula

$$
H(x) = \int_x^{x/\sigma} q(z)dz,
$$

it is easy to verify that

$$
(4.6)\qquad \liminf_{x\to\infty} H(x) \geqslant -c_0\ln\sigma \quad \text{and} \quad \limsup_{x\to\infty} H(x) \leqslant -c_1\ln\sigma.
$$

Consequently, from Theorems 2.1 and 2.2 the following follows:

COROLLARY 4.2. *If* $1/c_0 < -\ln\sigma$ *then the operator* P *defined by* (4.3), (4.4) *is asymptotically stable, if* $1/c_1 > -\ln\sigma$ *then* P *is sweeping.*

For the Tyson–Hannsgen operator (0.3), (0.4) considered in the introduction, we have $q(x) = cx$ if $x \geqslant 1$ and $q(x) = 0$ if $x < 1$. In this case, $c_0 = c_1 = c$ and the regularity conditions (iii) and (iv) are satisfied. Thus from Corollary 4.2 and Theorem 3.1 it follows immediately that inequality (0.5) is a necessary and sufficient condition for the asymptotic stability.

In the case considered by Lasota–Mackey [3] ($\lambda(x) = 2x$, q arbitrary) Corollary 4.2 shows that the best possible evaluation for the asymptotic stability is $1/c_0 > \ln 2$.

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