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Quasi-starlike functions

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Introduction. The class S^* of starlike functions plays a very significant role in the investigations conducted in the class S of normalized and univalent functions in the unit disc. Thus, it seems natural and useful to introduce and analyze functions analogous to the starlike ones in the class S_M of normalized and univalent functions bounded by M in the unit disc. The class $\tilde{S}^*_M$ of quasi-starlike normalized functions seems to realize such an analogy.

We now undertake to define precisely this class of functions.

Suppose that

$$F(\zeta) = \zeta + \ldots \quad \text{for } |\zeta| < 1 \tag{0}$$

denotes an arbitrary starlike function of S^*. Further, let $G(\zeta)$ be a starlike function of the form

$$G(\zeta) = \frac{\zeta}{\prod_{k=1}^{m} (1-\sigma_k z)^{\beta_k}} \quad \text{for } |\zeta| < 1, \tag{1}$$

$$\sigma_k = e^{i\varphi_k} \quad \text{and} \quad \sigma_i \neq \sigma_j \quad \text{when } i \neq j \ (i, j = 1, \ldots, m), \tag{2}$$

$$\sum_{k=1}^{m} \beta_k = 2, \tag{2'}$$

where φ_k $(k = 1, \ldots, m)$ run over all real numbers, and β_k run over all positive numbers.

Finally, let M denote an arbitrary fixed number belonging to the interval $(1, \infty)$.

Under the above notation the class of functions $g(z)$ satisfying the equation

$$F\big(g(z)\big) = \frac{1}{M} F(z) \quad \text{for } |z| < 1 \ (M > 1) \tag{3}$$

will be denoted by $\mathfrak{G}^M$; the subclass $\mathfrak{G}^M$ of the functions $g(z)$ satisfying the equation

$$G(g(z)) = \frac{1}{M}G(z) \quad \text{for } |z| < 1, \tag{4}$$

where $G(\zeta)$ is a function determined by formula (1) will be denoted by $\mathfrak{G}_m^M$.

Then the class $\tilde{S}_M^*$, mentioned at the beginning, consists of the normalized functions $g(z)$, i.e., of the functions $Mg(z) = z + \ldots$, where $g(z) \epsilon \mathfrak{G}^M$. The function of the classes $\mathfrak{G}^M$ and $\tilde{S}_M^*$ will be called *quasi-starlike* and *normalized quasi-starlike*, respectively. It is clear that the investigations carried out in $\mathfrak{G}^M$ and $\tilde{S}_M^*$ are in fact equivalent.

Moreover, one can easily prove that every function $g(z)$, determined by equation (3) or (4) is unambiguously determined by this equation, being at the same time holomorphic and univalent in the unit disc. Taking into consideration the compactnes of the class of starlike functions (0) we can also easily show that the class of quasi-starlike functions and consequently $\tilde{S}_M^*$ is compact; moreover, making use of the theorem on approximation of a starlike function by starlike functions of form (1), we can prove that the quasi-starlike functions can be approximated by functions of the class $\mathfrak{G}_m^M$ and that the sum $\bigcup_{m=1}^{k} \mathfrak{G}_m^M$ $(k = 1, 2, \ldots)$ is compact.

The present paper deals mainly with the examination of extremal properties of the class $\mathfrak{G}^M$ and consequently of $\tilde{S}_M^*$. The results obtained in the first part are analogous to those obtained by Hummel [4] and Zamorski [5] in the class S^*; it seems attractive, however, to undertake other, equally interesting investigations concerning the structure of this class.

In this place the author would like to express his deepest gratitude and cordinal thanks to Professor Zygmunt Charzyński for many valuable suggestions and advice given during the author's investigations concerning the above problem.

1. Extremal functions. We introduce the following notation:

$$G(\zeta) = \zeta + b_2\zeta^2 + \ldots \quad \text{for } |\zeta| < 1, \tag{5}$$

where $G(\zeta)$ is any function which appears is (1) and (4),

$$g(z) = a_1 z + a_2 z^2 + \ldots \quad \text{for } |z| < 1, \tag{6}$$

$$(g(z))^p = a_p^{(p)} z^p + a_{p+1}^{(p)} z^{p+1} + \ldots \quad (p = 1, 2, \ldots), \tag{6'}$$

$$a_n = x_n + iy_n \quad (n = 2, 3, \ldots), \tag{7}$$

where $g(z)$ is any function of the class $\mathfrak{G}^M$.

Let V represent in the Euclidean space R_{2N-2} a set of points whose coordinates $(x_2, \ldots, x_N, y_2, \ldots, y_N)$ $(N \geqslant 2)$ are determined by means of formula (7) by the first $N-1$ coefficients of (6).

Finally, let us be given in $\mathfrak{G}^M$ a functional defined for every function (6) of (3) by the formula

$$(8) \qquad H_g = H(x_2, \ldots, x_N,\ y_2, \ldots, y_N) = H\left(\frac{a_2+\bar{a}_2}{2}, \ldots, \frac{a_N-\bar{a}_N}{2i}\right),$$

where H denote a real function of $2N-2$ real variables, having continuous partial derivatives of the first order which do not disappear simultaneously in a sufficiently large domain containing the set V.

We shall sufficiently prove the following result:

FUNDAMENTAL THEOREM 1. *If the functional H_g attains in $\mathfrak{G}_m^M$ its extremal value for an extremal function* (6) *of* (4), *this function satisfies under the notation* (1), (5), (6), (6′), (7), (8) *the following differential equations*:

$$(9) \qquad \frac{g'(z)}{g(z)}\frac{\mathscr{L}\big(g(z)\big)}{\mathscr{R}\big(g(z)\big)} = \frac{1}{z}\frac{\mathscr{L}(z)}{\mathscr{R}(z)},$$

where

$$(10) \qquad \mathscr{L}(\zeta) = \sum_{p=1}^{N-1}\left(\frac{\mathscr{E}_p}{\zeta^p} + \bar{\mathscr{E}}_p\zeta^p\right) + \mathscr{E}_0,$$

$$(11) \qquad \mathscr{R}(\zeta) = \sum_{p=1}^{N-1}\left(\frac{\mathscr{D}_p}{\zeta^p} - \bar{\mathscr{D}}_p\zeta^p\right),$$

$$(10') \qquad \mathscr{E}_p = \sum_{n=1}^{N-p}(a_{n+p}^{(p+1)} - na_n)H_{n+p} \qquad (p = 0, 1, \ldots, N-1),$$

$$(11') \qquad \mathscr{D}_p = \sum_{l=1}^{N-p} c_l \sum_{k=l}^{N-p}(a_{k+p}^{(l+p)} - a_k^{(l)})H_{k+p} \qquad (p = 1, \ldots, N-1),$$

$$(12) \qquad H_n = H'_{x_n}(x_2, \ldots, y_N) - iH'_{y_n}(x_2, \ldots, y_N) \qquad (n = 2, \ldots, N),$$

$$(13) \qquad c_l = \begin{vmatrix} d_0 & 0 & \ldots & 0 & 1 \\ d_1 & d_0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ d_{l-1} & d_{l-2} & \ldots & d_1 & 0 \end{vmatrix} \qquad (l = 1, 2, \ldots),$$

$$(14) \qquad d_l = \sum_{k=1}^{m}\beta_k\sigma_k^l \qquad (l = 1, 2, \ldots),$$

$$(14') \qquad d_0 = 1,$$

and

$$(15)\qquad \frac{g'(z)}{g(z)}\frac{\tilde{\mathscr{L}}(g(z))}{\tilde{\mathscr{R}}(g(z))} = \frac{1}{z}\frac{\tilde{\mathscr{L}}(z)}{\tilde{\mathscr{R}}(z)},$$

where

$$(16)\qquad \tilde{\mathscr{L}}(\zeta) = \sum_{p=0}^{N-1}\left(\frac{\tilde{\mathscr{E}}_p}{\zeta^p} - \bar{\tilde{\mathscr{E}}}_p\zeta^p\right),$$

$$(17)\qquad \tilde{\mathscr{R}}(\zeta) = \sum_{p=1}^{N-1}\frac{1}{p}\left(\frac{\mathscr{D}_p}{\zeta^p} + \bar{\mathscr{D}}_p\zeta^p\right) + \lambda,$$

$$(16')\qquad \tilde{\mathscr{E}}_p = \sum_{l=p}^{N-1}\frac{\mathscr{D}_l}{l}d_{l-p}, \qquad p = (1, \ldots, N-1),$$

$$(16'')\qquad \tilde{\mathscr{E}}_0 = \frac{1}{2}\sum_{l=1}^{N-1}\frac{\mathscr{D}_l}{l}d_l,$$

$$(17')\qquad \lambda = -\frac{1}{2}\sum_{l=1}^{N-1}\frac{1}{l}(\mathscr{D}_l d_l + \bar{\mathscr{D}}_l\bar{d}_l);$$

at the same time the numbers $\bar{\sigma}_k$ ($k = 1, \ldots, m$) *of* (1) *are roots of the function* $\mathscr{R}(\zeta)$ *and double roots of the function* $\tilde{\mathscr{R}}(\zeta)$.

Proof. Since the family $\mathfrak{G}^M$ is compact, the existence of the extremal functions is obvious because of the continuity of the function (8).

We shall now proceed to derive equation (9) taking as a basis Lagrange's method of multipliers, applied by Charzyński [1] to extremal problems in the theory of complex functions.

We conjecture that function (6) of (4) belonging to $\mathfrak{G}_m^M$ is extremal.

From relationships (1) and (5) it follows that the coefficients $b_1, b_2, \ldots$ are functions of the parameters $\varphi_1, \ldots, \varphi_m, \beta_1, \ldots, \beta_m$; simultaneously from (4) it follows that between the coefficients of the functions (5) and (6) the relationships

$$(18)\qquad b_1 a_k^{(1)} + b_2 a_k^{(2)} + \ldots + b_k a_k^{(k)} - \frac{1}{M}b_k = 0 \qquad (k = 1, 2, \ldots;\ b_1 = 1)$$

hold. Consequently, as is easy to see, the value of the functional H_g for an arbitrary function (6) of (4) coincides with the value of the function appearing on the right-hand side of formula (8) at the point $P = (a_2, \ldots, a_N, \varphi_1, \ldots, \varphi_m, \beta_1, \ldots, \beta_m)$ of a $2N-2+2m$ dimensional Euclidean space, provided the additional conditions (18) are satisfied for $k = 2, \ldots, N$.

Applying the above-mentioned multipliers, we conclude that for the point $P = (a_2, \ldots, a_N, \varphi_1, \ldots, \varphi_m, \beta_1, \ldots, \beta_m)$ connected by (1),

(4) and (18) with function (6) realizing the extremum of H_g in $\mathfrak{G}_m^M$ the equations

$$\sum_{k=2}^{N}\hat{\lambda}_k\sum_{j=1}^{k}\frac{\partial a_k^{(j)}}{\partial a_s}b_j = \tau\left(\frac{\partial H}{\partial a_s}+\overline{\frac{\partial H}{\partial \bar{a}_s}}\right)\quad (s = 2,\ldots,N),\tag{19}$$

$$\sum_{k=2}^{N}\left(\hat{\lambda}_k\sum_{j=2}^{k}\frac{\partial b_j}{\partial \varphi_t}a_k^{(j)}-\frac{1}{M}\hat{\lambda}_k\frac{\partial b_k}{\partial \varphi_t}+\bar{\hat{\lambda}}_k\sum_{j=2}^{k}\overline{\frac{\partial b_j}{\partial \varphi_t}}\bar{a}_k^{(j)}-\frac{1}{M}\bar{\hat{\lambda}}_k\overline{\frac{\partial b_k}{\partial \varphi_t}}\right) = 0\tag{20}$$
$$(t = 1,\ldots,m),$$

$$\sum_{k=2}^{N}\left(\hat{\lambda}_k\sum_{j=2}^{k}\frac{\partial b_j}{\partial \beta_t}a_k^{(j)}-\frac{1}{M}\hat{\lambda}_k\frac{\partial b_k}{\partial \beta_t}+\bar{\hat{\lambda}}_k\sum_{j=2}^{k}\overline{\frac{\partial b_j}{\partial \beta_t}}\bar{a}_k^{(j)}-\frac{1}{M}\bar{\hat{\lambda}}_k\overline{\frac{\partial b_k}{\partial \beta_t}}\right)+\hat{\lambda}_1 = 0\tag{21}$$
$$(t = 1,\ldots,m)$$

hold, where $\hat{\lambda}_1,\ldots,\hat{\lambda}_N,\tau$ are a non-trivial set of Lagrange's multipliers.

Moreover, we shall show that $\tau\neq 0$. In fact, we assume that it is not so and notice that it follows from relations (6′) that the coefficient $a_p^{(j)}$ for $2\leqslant j\leqslant p$ depends on the coefficients $a_1,\ldots,a_{p-1}$, only. It follows directly from our assumption and the above remark that the last equation of the set (19), corresponding to $s = N$, is of the form $\hat{\lambda}_N b_1 = 0$, and so $\hat{\lambda}_N = 0$.

It follows from the above formula and the preceding remark that the last but one equation of the set (19), corresponding to $s = N-1$, is of the form $\hat{\lambda}_{N-1}b_1 = 0$, and so $\hat{\lambda}_{N-1} = 0$.

Repeating the above argumentation one shall easily find that all the coefficients $\hat{\lambda}_k$ $(k = 2,\ldots,N)$ are zeros, i.e., the set of numbers $\hat{\lambda}_1,\ldots,\hat{\lambda}_N,\tau$ is trivial.

Moreover, from the definition of numbers (2) we have the relations

$$\frac{\partial b_j}{\partial \varphi_t} = i\sigma_t\frac{\partial b_j}{\partial \sigma_t},\qquad \overline{\frac{\partial b_j}{\partial \varphi_t}} = -i\bar{\sigma}_t\overline{\frac{\partial b_j}{\partial \sigma_t}}\qquad (t = 1,\ldots,m,\ j = 1,2,\ldots).$$

Next, if we use the above relations in equations (20) and divide the set of equations (19)-(21) by τ, we shall obtain the set of equations

$$\sum_{k=2}^{N}\lambda_k\sum_{j=1}^{k}\frac{\partial a_k^{(j)}}{\partial a_s}b_j = \left(\frac{\partial H}{\partial a_s}\right)+\left(\overline{\frac{\partial H}{\partial \bar{a}_s}}\right)\quad (s = 2,\ldots,N),\tag{22}$$

$$\sum_{k=2}^{N}\left(\lambda_k\sum_{j=2}^{k}\sigma_t\frac{\partial b_j}{\partial \sigma_t}a_k^{(j)}-\frac{1}{M}\lambda_k\sigma_t\frac{\partial b_k}{\partial \sigma_t}-\bar{\lambda}_k\sum_{j=2}^{k}\bar{\sigma}_t\overline{\frac{\partial b_j}{\partial \sigma_t}}\bar{a}_k^{(j)}+\frac{1}{M}\bar{\lambda}_k\bar{\sigma}_t\overline{\frac{\partial b_k}{\partial \sigma_t}}\right) = 0\tag{23}$$
$$(t = 1,\ldots,m),$$

$$(24)\quad \sum_{k=2}^{N}\left(\lambda_k\sum_{j=2}^{k}\frac{\partial b_j}{\partial\beta_t}a_k^{(j)}-\frac{1}{M}\lambda_k\frac{\partial b_k}{\partial\beta_t}+\bar{\lambda}_k\sum_{j=2}^{k}\overline{\frac{\partial b_j}{\partial\beta_t}}\bar{a}_k^{(j)}-\frac{1}{M}\bar{\lambda}_k\overline{\frac{\partial b_k}{\partial\beta_t}}\right)+\lambda_1=0$$

$$(t=1,\ldots,m),$$

where

$$(25)\quad \lambda_k=\frac{1}{\tau}\hat{\lambda}_k\quad (k=1,\ldots,N).$$

Then, considering the expression

$$(26)\quad G\big(g(z)\big)-\frac{1}{M}G(z)$$

as a function of the independent variables $a_2,\ldots,a_N,\ \varphi_1,\ldots,\varphi_m,\ \beta_1,\ldots,\beta_m$ from (6), (4) and (1) and of a variable z from neighbourhood zero, and keeping the remaining coefficients $a_{N+1}, a_{N+2},\ldots$ of $g(z)$ constant, for example equal to those of the extremal function, we easily get

$$(27)\quad \sum_{j=1}^{k}\frac{\partial a_k^{(j)}}{\partial a_s}b_j=\frac{1}{k!}\left\{\frac{\partial^k}{\partial z_k}\Big(G'\big(g(z)\big)z^s\Big)\right\}_{z=0}\qquad (s=2,\ldots,N),$$

$$(28)\quad \sigma_t\left(\sum_{j=2}^{k}\frac{\partial b_j}{\partial\sigma_t}a_k^{(j)}-\frac{1}{M}\frac{\partial b_k}{\partial\sigma_t}\right)=\sigma_t\frac{\partial}{\partial\sigma_t}\left\{\frac{1}{k!}\frac{\partial^k}{\partial z^k}\Big(G\big(g(z)\big)-\frac{1}{M}G(z)\Big)\right\}_{z=0}$$

$$=\sigma_t\frac{1}{k!}\left\{\frac{\partial^k}{\partial z^k}\frac{\partial}{\partial\sigma_t}\Big(G\big(g(z)\big)-\frac{1}{M}G(z)\Big)\right\}_{z=0}$$

$$=\sigma_t\frac{1}{k!}\left\{\frac{\partial^k}{\partial z^k}\left(\frac{\beta_t g(z)}{1-\sigma_t g(z)}G\big(g(z)\big)-\frac{1}{M}\frac{\beta_t z}{1-\sigma_t z}G(z)\right)\right\}_{z=0}$$

$$=\beta_t\frac{1}{k!}\left\{\frac{\partial^k}{\partial z^k}\left(\left(\frac{\sigma_t g(z)}{1-\sigma_t g(z)}-\frac{\sigma_t z}{1-\sigma_t z}\right)G\big(g(z)\big)\right)\right\}_{z=0}\qquad (t=1,2,\ldots,m)$$

and, analogously,

$$(29)\quad \sum_{j=2}^{k}\frac{\partial b_j}{\partial\beta_t}a_k^{(j)}-\frac{1}{M}\frac{\partial b_k}{\partial\beta_k}$$

$$=\frac{1}{k!}\left\{\frac{\partial^k}{\partial z^k}\left(-\Big(\log\big(1-\sigma_t g(z)\big)-\log(1-\sigma_t z)\Big)G\big(g(z)\big)\right)\right\}_{z=0}(t=1,2,\ldots,m).$$

Taking formulas (27)-(29) we can write the set (22)-(24) in the form

$$(30)\quad \sum_{k=2}^{N}\frac{1}{k!}\lambda_k\left(\frac{\partial^k}{\partial z^k}\Big(G'\big(g(z)\big)z^s\Big)\right)_{z=0}=H_s\qquad (s=2,\ldots,N),$$

$$\sum_{k=2}^{N}\left\{\frac{1}{k!}\lambda_k\left(\frac{\partial^k}{\partial z^k}\left(\left(\frac{\sigma_t g(z)}{1-\sigma_t g(z)}-\frac{\sigma_t z}{1-\sigma_t z}\right)G\big(g(z)\big)\right)\right)_{z=0}-\right.$$

$$\left.-\frac{1}{k!}\bar{\lambda}_k\left(\frac{\partial^k}{\partial z^k}\left(\left(\frac{\bar{\sigma}_t\bar{g}(z)}{1-\bar{\sigma}_t\bar{g}(z)}-\frac{\bar{\sigma}_t z}{1-\bar{\sigma}_t z}\right)\bar{G}\big(\bar{g}(z)\big)\right)\right)_{z=0}\right\}=0 \quad (t=1,\dots,m), \tag{31}$$

$$\sum_{k=2}^{N}\left\{\frac{1}{k!}\bar{\lambda}_k\left(\frac{\partial^k}{\partial z^k}\Big(-\big(\log(1-\sigma_t g(z))-\log(1-\sigma_t z)\big)\,G\big(g(z)\big)\Big)\right)_{z=0}-\right.$$

$$\left.-\frac{1}{k!}\bar{\lambda}_k\left(\frac{\partial^k}{\partial z^k}\Big(-\big(\log(1-\bar{\sigma}_t\bar{g}(z))-\log(1-\bar{\sigma}_t z)\big)\,\bar{G}\big(\bar{g}(z)\big)\Big)\right)_{z=0}\right\}+\lambda_1=0$$

$$(t=1,\dots,m), \tag{32}$$

where the following notation has been adopted:

$$\bar{G}(\zeta)=\zeta+\bar{b}_2\zeta^2+\bar{b}_3\zeta^3+\dots \quad \text{for } |\zeta|<1,$$

$$\bar{g}(z)=\bar{a}_1 z+\bar{a}_2 z^2+\bar{a}_3 z^3+\dots \quad \text{for } |z|<1,$$

$$H_s=\frac{\partial H}{\partial a_s}+\overline{\frac{\partial H}{\partial \bar{a}_s}} \quad (s=2,\dots,N). \tag{33}$$

Now, we assume for an arbitrary holomorphic function

$$f(z)=a_0+a_1 z+\dots \quad \text{for } |z|<1$$

and for given numbers $\lambda_2,\dots,\lambda_N$ of (25) the operator

$$K\big(f(z)\big)=\lambda_2 a_2+\dots+\lambda_N a_N \tag{34}$$

and, analogously,

$$\bar{K}\big(f(z)\big)=\bar{\lambda}_2 a_2+\dots+\bar{\lambda}_N a_N. \tag{34'}$$

Finally we introduce the function

$$H^*\left(\frac{1}{z}\right)=\sum_{s=2}^{N}\frac{H_s}{z^s}. \tag{35}$$

Using the above notation, we can write the set (30) in the form

$$K\big(G'(g(z))\,z^s\big)=H_s \quad (s=2,\dots,N); \tag{36}$$

moreover, for every function $f_0(z)$ of the form

$$f_0(z)=e_2 z^2+e_3 z^3+\dots \quad \text{for } |z|<1$$

we have

$$(37)\qquad K\big(G'(g(z))f_0(z)\big) = \sum_{s=2}^{N} e_s H_s = \frac{1}{2\pi i}\int_c H^*\left(\frac{1}{z}\right) f_0(z)\frac{dz}{z},$$

$$(37')\qquad K\big(\bar{G}'(\bar{g}(z))\bar{f}_0(z)\big) = \sum_{s=2}^{N} \bar{e}_s \bar{H}_s = \frac{1}{2\pi i}\int_c \bar{H}^*\left(\frac{1}{z}\right) \bar{f}_0(z)\frac{dz}{z},$$

where c is a sufficiently small circumference with the centre zero.

In particular, if we put

$$f_t(z) = \left(\frac{\sigma_t g(z)}{1-\sigma_t g(z)} - \frac{\sigma_t z}{1-\sigma_t z}\right)\frac{G(g(z))}{G'(g(z))} \qquad (t = 1, \ldots, m),$$

$$\bar{f}_t(z) = \left(\frac{\bar{\sigma}_t \bar{g}(z)}{1-\bar{\sigma}_t \bar{g}(z)} - \frac{\bar{\sigma}_t z}{1-\bar{\sigma}_t z}\right)\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))} \qquad (t = 1, \ldots, m),$$

then (31) can be written, in view of (37), (37′), in the form

$$(38)\qquad K\big(G'(g(z))f_t(z)\big) + \bar{K}\big(\bar{G}'(\bar{g}(z))\bar{f}_t(z)\big)$$

$$= \frac{1}{2\pi i}\int_c H^*\left(\frac{1}{z}\right)\frac{\sigma_t g(z)}{1-\sigma_t g(z)}\frac{G(g(z))}{G'(g(z))}\frac{dz}{z} - \frac{1}{2\pi i}\int_c H^*\left(\frac{1}{z}\right)\frac{\sigma_t z}{1-\sigma_t z}\frac{G(g(z))}{G'(g(z))}\frac{dz}{z} -$$

$$-\frac{1}{2\pi i}\int_c \bar{H}^*\left(\frac{1}{z}\right)\frac{\bar{\sigma}_t \bar{g}(z)}{1-\bar{\sigma}_t \bar{g}(z)}\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))}\frac{dz}{z} + \frac{1}{2\pi i}\int_c \bar{H}^*\left(\frac{1}{z}\right)\frac{\bar{\sigma}_t z}{1-\bar{\sigma}_t z}\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))}\frac{dz}{z}$$

$$(t = 1, \ldots, m).$$

Now we consider the following function of the variable ζ

$$(39)\qquad \mathscr{R}(\zeta) = \frac{1}{2\pi i}\int_c H^*\left(\frac{1}{z}\right)\frac{g(z)}{\zeta - g(z)}\frac{G(g(z))}{G'(g(z))}\frac{dz}{z} -$$

$$-\frac{1}{2\pi i}\int_c H^*\left(\frac{1}{z}\right)\frac{z}{\zeta - z}\frac{G(g(z))}{G'(g(z))}\frac{dz}{z} - \frac{1}{2\pi i}\int_c \bar{H}^*\left(\frac{1}{z}\right)\frac{\zeta\bar{g}(z)}{1-\zeta\bar{g}(z)}\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))}\frac{dz}{z} +$$

$$+\frac{1}{2\pi i}\int_c \bar{H}^*\left(\frac{1}{z}\right)\frac{\zeta z}{1-\zeta z}\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))}\frac{dz}{z}.$$

The function $\mathscr{R}(\zeta)$ thus determined is holomorphic in the full plane except the point $0, \infty$. Hence, the Laurent expansion at the origin of thus function is of the form

$$(39')\qquad \mathscr{R}(\zeta) = \sum_{-\infty}^{\infty} \mathscr{D}_p \zeta^p, \qquad \text{where } \mathscr{D}_0 = 0.$$

Next, taking into consideration (38) we have

$$\mathscr{R}(\bar{\sigma}_t) = 0 \quad (t = 1, \ldots, m). \tag{40}$$

Moreover, we introduce another function determined by the formula

$$\mathscr{L}(\zeta) = \mathscr{R}(\zeta)\frac{G'(\zeta)\zeta}{G(\zeta)}. \tag{41}$$

One easily sees from (39) and the formula

$$\frac{G'(\zeta)\zeta}{G(\zeta)} = \sum_{k=1}^{m}\beta_k\frac{1+\sigma_k\zeta}{1-\sigma_k\zeta} = \sum_{n=0}^{\infty} d_n\zeta^n \tag{41'}$$

that the function $\mathscr{L}(\zeta)$ is also holomorphic in the full plane except the points 0 and ∞, because the poles of (41′) and the roots of cancel each other. Hence, the Laurent expansion at the origin of this function is of the form

$$\mathscr{L}(\zeta) = \sum_{-\infty}^{\infty}\mathscr{E}_p\zeta^p. \tag{41''}$$

Next, replacing ζ by $g(z)$ in (41) we have

$$\mathscr{L}\big(g(z)\big) = \mathscr{R}\big(g(z)\big)\frac{G'\big(g(z)\big)g(z)}{G\big(g(z)\big)}. \tag{42}$$

Now, if we take into consideration the relations

$$G\big(g(z)\big) = \frac{1}{M}G(z), \tag{43}$$

$$G'\big(g(z)\big)g'(z) = \frac{1}{M}G'(z), \tag{43'}$$

then equation (9) of an extremal function follows immediately from (41), (42), (43) and (43′).

Another equation of an extremal function will be derived in a similar way. We shall use the following notation:

$$h_t(z) = -\big(\log\big(1-\sigma_t g(z)\big)-\log(1-\sigma_t z)\big)\frac{G\big(g(z)\big)}{G'\big(g(z)\big)} \qquad (t = 1, \ldots, m),$$

$$\bar{h}_t(z) = -\big(\log\big(1-\bar{\sigma}_t \bar{g}(z)\big)-\log(1-\bar{\sigma}_t z)\big)\frac{\bar{G}\big(\bar{g}(z)\big)}{\bar{G}'\big(\bar{g}(z)\big)} \qquad (t = 1, \ldots, m),$$

where the branches of the logarithm are egual to 0 for z equals to 0.

Then, relation (32) can be written in the form

$$(44)\quad K\big(G'(g(z))h_t(z)\big)+\bar{K}\big(\bar{G}'(\bar{g}(z))\bar{h}_t(z)\big)$$
$$= -\frac{1}{2\pi i}\int_c H^*\left(\frac{1}{z}\right)\log\big(1-\sigma_t g(z)\big)\frac{G(g(z))}{G'(g(z))}\frac{dz}{z}+$$
$$+\frac{1}{2\pi i}\int_c H^*\left(\frac{1}{z}\right)\log(1-\sigma_t z)\frac{G(g(z))}{G'(g(z))}\frac{dz}{z}-$$
$$-\frac{1}{2\pi i}\int_c \bar{H}^*\left(\frac{1}{z}\right)\log\big(1-\bar{\sigma}_t\bar{g}(z)\big)\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))}\frac{dz}{z}+$$
$$+\frac{1}{2\pi i}\int_c \bar{H}^*\left(\frac{1}{z}\right)\log(1-\bar{\sigma}_t z)\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))}\frac{dz}{z}+\lambda_1 \quad (t=1,\ldots,m).$$

Now, we consider the function

$$(45)\quad \tilde{\mathscr{R}}(\zeta) = -\frac{1}{2\pi i}\int_c H^*\left(\frac{1}{z}\right)\log\left(1-\frac{1}{\zeta}g(z)\right)\frac{G(g(z))}{G'(g(z))}\frac{dz}{z}+$$
$$+\frac{1}{2\pi i}\int_c H^*\left(\frac{1}{z}\right)\log\left(1-\frac{1}{\zeta}z\right)\frac{G(g(z))}{G'(g(z))}\frac{dz}{z}-$$
$$-\frac{1}{2\pi i}\int_c \bar{H}^*\left(\frac{1}{z}\right)\log\big(1-\zeta\bar{g}(z)\big)\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))}\frac{dz}{z}+$$
$$+\frac{1}{2\pi i}\int_c \bar{H}^*\left(\frac{1}{z}\right)\log(1-\zeta z)\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))}\frac{dz}{z}+\lambda_1.$$

Hence and from (44) we have

$$(46)\quad \tilde{\mathscr{R}}(\bar{\sigma}_t) = 0 \quad (t=1,\ldots,m).$$

In addition, from (45) and (39) there follows the relation

$$(47)\quad \tilde{\mathscr{R}}'(\zeta) = -\frac{1}{\zeta}\mathscr{R}(\zeta).$$

It gives that $\bar{\sigma}_t$ are at least double roots of $\tilde{\mathscr{R}}(\zeta)$.

Moreover, we introduce the function determined by the formula

$$(48)\quad \tilde{\mathscr{L}}(\zeta) = \tilde{\mathscr{R}}(\zeta)\frac{G'(\zeta)\zeta}{G(\zeta)}$$

and using relations (43) and (43′), by means of analogous argumentation, we obtain equation (15).

We shall now proceed to derive formulas for the function $\mathscr{L}(\zeta)$, $\tilde{\mathscr{L}}(\zeta)$, $\tilde{\mathscr{R}}(\zeta)$ and $\mathscr{R}(\zeta)$. First we shall undertake to prove two lemmas.

LEMMA 1. *If $G(\zeta)$ is a starlike function of the form* (1), *and the function $g(z)$ is a quasi-starlike function determined by equation* (4), *the second integral of* (39)

$$\mathscr{I}(\zeta) = \frac{1}{2\pi i}\int_{c} H^*\left(\frac{1}{z}\right)\frac{z}{\zeta - z}\frac{G\big(g(z)\big)}{G'\big(g(z)\big)}\frac{dz}{z} \tag{49}$$

is equal to this part of the Laurent expansion at the origin of

$$-\operatorname{res}_{z=\zeta}\left(H^*\left(\frac{1}{z}\right)\frac{1}{\zeta - z}\frac{G(z)g'(z)}{G'(z)}\right), \tag{50}$$

which contains the negative powers of the variable ζ.

Proof. In fact, since c is a circumference with the centre zero and a sufficiently small radius r, we have

$$\left|\frac{z}{\zeta}\right| < 1 \quad \text{for } |z| = r \text{ and } \left|\frac{1}{\zeta}\right| < \frac{1}{r}, \tag{51}$$

thus, integral (49) with respect to (43) and (43′) can be written in form

$$\mathscr{I}(\zeta) = \frac{1}{2\pi i}\int_{c} H^*\left(\frac{1}{z}\right)\frac{G(z)g'(z)}{G'(z)}\sum_{n=1}^{\infty}\frac{z^n}{\zeta^n}\frac{dz}{z} \tag{52}$$

for all the values of $|\zeta| > r$. It can be seen, however, from formula (51) that function (49) depends only on the negative powers of ζ. On the other hand, let C denote a circumference with the centre zero and radius $R < 1$ sufficiently close to unity. Then we have

$$\left|\frac{\zeta}{z}\right| < 1 \quad \text{for } |z| = R \text{ and } |\zeta| < R.$$

Thus, the following integral analogous to (49) can be written in the form

$$\frac{1}{2\pi i}\int_{C} H^*\left(\frac{1}{z}\right)\frac{z}{\zeta - z}\frac{G\big(g(z)\big)}{G'\big(g(z)\big)}\frac{dz}{z} = -\frac{1}{2\pi i}\int_{C} H^*\left(\frac{1}{z}\right)\frac{G(z)g'(z)}{G'(z)}\sum_{n=0}^{\infty}\frac{\zeta^n}{z^n}\frac{dz}{z}. \tag{53}$$

It can be remarked that in the last integral there exist only non-negative powers of ζ.

Yet on the basis of the well-known theorem on residua an obvious conclusion can be drawn

$$\mathscr{I}(\zeta) = \frac{1}{2\pi i}\int_{C} H^*\left(\frac{1}{z}\right)\frac{z}{\zeta - z}\frac{G\big(g(z)\big)}{G'\big(g(z)\big)}\frac{dz}{z} - \operatorname{res}_{z=\zeta}\left(H^*\left(\frac{1}{z}\right)\frac{1}{\zeta - z}\frac{G'(z)g'(z)}{G'(z)}\right) \tag{54}$$

$$\text{for } r < |\zeta| < R,$$

which easily gives our lemma.

LEMMA 2. *If $G(\zeta)$ is a starlike function of form* (1), *the function $g(z)$ is a quasi-starlike function determined by equation* (4), *and the function $g^{-1}(w)$ is a function inverse to $w = g(z)$, then the first integral of* (39)

$$\mathscr{I}_1(\zeta) = \frac{1}{2\pi i}\int_c H^*\left(\frac{1}{z}\right)\frac{g(z)}{\zeta - g(z)}\frac{G(g(z))}{G'(g(z))}\frac{dz}{z} \tag{55}$$

is equal to that part of the Laurent expansion at the origin of

$$-\operatorname{res}_{w=\zeta}\left(H^*\left(\frac{1}{g^{-1}(w)}\right)\frac{w}{\zeta - w}\frac{G(w)}{G'(w)}\frac{g^{-1}(w)}{g^{-1}(w)}\right), \tag{56}$$

which contains the negative powers of the variable ζ.

Proof. First, let ϱ denote the distance of the point $w = 0$ from the boundary of the image of the unit disc under $g(z)$, r a sufficiently small positive number, and R a number sufficiently close to ϱ. Further, assume that γ is the image of the circumference c under $g(z)$. We then obtain

$$\mathscr{I}_1(\zeta) = \frac{1}{2\pi i}\int_\gamma H^*\left(\frac{1}{g^{-1}(w)}\right)\frac{w}{\zeta - w}\frac{G(w)g^{-1\prime}(w)}{G'(w)g^{-1}(w)}dw \tag{57}$$

and hence

$$\mathscr{I}_1(\zeta) = \frac{1}{2\pi i}\int_\gamma H^*\left(\frac{1}{g^{-1}(w)}\right)\frac{G(w)g^{-1\prime}(w)}{G'(w)g^{-1}(w)}\sum_{n=1}^{\infty}\frac{w^n}{\zeta^n}dw \quad \text{for } |\zeta| > r. \tag{58}$$

It follows directly from formula (58) that function (57) contains only the negative powers of the variable ζ.

On the other hand, the integral of the form

$$\frac{1}{2\pi i}\int_C H^*\left(\frac{1}{g^{-1}(w)}\right)\frac{w}{\zeta - w}\frac{G(w)g^{-1\prime}(w)}{G'(w)g^{-1}(w)}dw$$
$$= \frac{1}{2\pi i}\int_C H^*\left(\frac{1}{g^{-1}(w)}\right)\frac{G(w)g^{-1\prime}(w)}{G'(w)g^{-1}(w)}\sum_{n=0}^{\infty}\frac{\zeta^n}{w^n}dw \quad \text{for } |\zeta| < R,$$

where C is a circumference with the centre zero and the radius R, contains only the non-negative powers of the variable ζ. Similarly to the preceding considerations we obtain the obvious relation

$$\mathscr{I}_1(\zeta) = \frac{1}{2\pi i}\int_C H^*\left(\frac{1}{g^{-1}(w)}\right)\frac{w}{\zeta - w}\frac{G(w)g^{-1\prime}(w)}{G'(w)g^{-1}(w)}dw -$$
$$-\operatorname{res}_{w=\zeta}\left(H^*\left(\frac{1}{g^{-1}(w)}\right)\frac{w}{\zeta - w}\frac{G(w)g^{-1\prime}(w)}{G'(w)g^{-1}(w)}\right),$$

which leads directly to Lemma 2.

Moreover, we remark that the Laurent expansions at the origin of the next integrals of (39)

$$(59)\qquad \hat{\mathscr{I}}(\zeta) = \frac{1}{2\pi i}\int_{c} \bar{H}^*\left(\frac{1}{z}\right)\frac{\zeta z}{1-\zeta z}\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))}\frac{dz}{z}$$

and

$$(60)\qquad \hat{\mathscr{I}}_1(\zeta) = \frac{1}{2\pi i}\int_{c} \bar{H}^*\left(\frac{1}{z}\right)\frac{\zeta\bar{g}(z)}{1-\zeta\bar{g}(z)}\frac{\bar{G}(\bar{g}(z))}{\bar{G}'(\bar{g}(z))}\frac{dz}{z}$$

contain the positive powers of the variable ζ only.

Using the above lemmas we shall now undertake to calculate the coefficients of the function $\mathscr{L}(\zeta)$. From the definition of the function $\mathscr{L}(\zeta)$ by formula (41) and from (39), (49), (55), (59) and (60) it follows that

$$(61)\qquad \mathscr{L}(\zeta) = \left(-\mathscr{I}(\zeta)+\mathscr{I}_1(\zeta)\right)\frac{G'(\zeta)\zeta}{G(\zeta)} + \left(\hat{\mathscr{I}}(\zeta)-\hat{\mathscr{I}}_1(\zeta)\right)\frac{G'(\zeta)\zeta}{G(\zeta)}.$$

Then, from the results obtained from Lemmas 1, 2 and (41) it follows that the terms with non-positive powers of ζ in the Laurent expansion at the origin of the function $\mathscr{L}(\zeta)$ appear in the first part of the formula (61) only. Moreover, we easily observe that these terms in the function $\mathscr{L}(\zeta)$ and

$$-H^*\left(\frac{1}{\zeta}\right)g'(\zeta)\zeta + H^*\left(\frac{1}{g^{-1}(\zeta)}\right)\frac{\zeta^2 g^{-1\prime}(\zeta)}{g^{-1}(\zeta)}$$

are equal.

Using the above notation we now for have the coefficients of the cited terms the formulas

$$(62)\quad \mathscr{E}_p = \frac{1}{2\pi i}\int_{\gamma} H^*\left(\frac{1}{g^{-1}(w)}\right)\frac{w^{p+1}g^{-1\prime}(w)}{g^{-1}(w)}\,dw - \frac{1}{2\pi i}\int_{c} H^*\left(\frac{1}{z}\right)g'(z)z^p\,dz$$

$$(p = 0, 1, \ldots, N-1)$$

and

$$(62')\qquad \mathscr{E}_p = 0 \quad \text{for } p \geqslant N.$$

Relations (62), (62′), (6), (6′), (35) and the obvious relation

$$\frac{1}{2\pi i}\int_{\gamma} H^*\left(\frac{1}{g^{-1}(w)}\right)\frac{w^{p+1}g^{-1\prime}(w)}{g^{-1}(w)}\,dw = \frac{1}{2\pi i}\int_{c} H^*\left(\frac{1}{z}\right)g^{p+1}(z)\frac{dz}{z}$$

immediately imply formulas (10′).

It still remains to observe that the coefficients at the terms with positive powers of ζ in (41″) are conjugate to the corresponding coefficients at the negative powers of ζ. Indeed, let us notice that it follows from formula (39) that the function $\mathscr{R}(\zeta)$ on the circumference of the unit disc assumes imaginary values only and we can easily prove the same for function (41′) except a finite number of points at which it has the poles; from this statement and from the form of $\mathscr{L}(\zeta)$ it follows that on the unit circumference this function assumes real values only, which gives our observation.

We shall now proceed to calculate the coefficients of the function $\mathscr{R}(\zeta)$, which according to formula (39) and to the adopted notation (49), (55), (59), (60) is of the form

$$\mathscr{R}(\zeta) = \big(\mathscr{I}_1(\zeta)-\mathscr{I}(\zeta)\big)-\big(\hat{\mathscr{I}}(\zeta)-\hat{\mathscr{I}}_1(\zeta)\big). \tag{63}$$

First, we notice that the function

$$\chi(\zeta) = \frac{G(\zeta)}{G'(\zeta)} = \sum_{n=1}^{\infty} c_n\zeta^n \quad \text{for } |\zeta|<1 \tag{64}$$

has the coefficients c_n $(n = 1, 2, \ldots)$ determined by coefficients (14), (14′) of function (41′) in the formulas

$$\begin{aligned} d_0c_1 &= 1, \\ d_1c_1+d_0c_2 &= 0, \\ \ldots\ldots&\ldots, \\ d_{k-1}c_1+\ldots+d_0c_k &= 0, \end{aligned}$$

which lead directly to relations (13); moreover, from (64) we obtain

$$\chi\big(g(z)\big)g^p(z) = \sum_{n=p+1}^{k}\Big(\sum_{k=p+1}^{n} a_n^{(k)}c_{k-p}\Big)z^n \quad (p = 0, 1, \ldots). \tag{65}$$

If we now use Lemmas 1 and 2 and formula (64), then it will follow from (63) that this part of the function $\mathscr{R}(\zeta)$, which contains the negative powers of the variable ζ equals respectively that part of the expression

$$H^*\left(\frac{1}{g^{-1}(\zeta)}\right)\chi(\zeta)\frac{\zeta g^{-1'}(\zeta)}{g^{-1}(\zeta)} - H^*\left(\frac{1}{\zeta}\right)\chi\big(g(\zeta)\big),$$

which contains the negative powers of ζ. Hence we have

$$\mathscr{D}_p = \frac{1}{2\pi i}\int_{\gamma} H^*\left(\frac{1}{g^{-1}(w)}\right)\chi(w)\frac{w^p g^{-1'}(w)}{g^{-1}(w)}\,dw - \frac{1}{2\pi i}\int_{c} H^*\left(\frac{1}{z}\right)\chi\big(g(z)\big)z^{p-1}dz \tag{66}$$
$$(p = 1, \ldots, N-1)$$

and

$$\mathscr{D}_p = 0 \quad \text{for } p \geqslant N. \tag{66′}$$

Relations (66) and (66′) immediatly imply as before formulas (11′).

The second part of formula (11) for the function $\mathscr{R}(\zeta)$ is a consequence of (39′) and of the fact that this function on circumference of the unit disc assumes imaginary values only.

Next, we shall proceed to derive formula (17) for the function $\tilde{\mathscr{R}}(\zeta)$. Using relation (47), (11) and the second relation of (39′) we have

$$(67)\qquad \tilde{\mathscr{R}}(\zeta) = \sum_{p=1}^{N} \frac{1}{p}\left(\frac{\mathscr{D}_p}{\zeta^p} + \bar{\mathscr{D}}_p\zeta^p\right) + \lambda.$$

On the other hand, it follows from (46) and (72) that

$$(68)\qquad \sum_{p=1}^{N-1} \frac{1}{p}(\mathscr{D}_p\sigma_t^p + \bar{\mathscr{D}}_p\bar{\sigma}_t^p) + \lambda = 0.$$

Next, multiplying equalities (68) by the numbers $\beta_1, \ldots, \beta_m$ respectively, and summing then, we obtain

$$\sum_{p=1}^{N-1} \frac{1}{p}\left(\mathscr{D}_p \sum_{t=1}^{m} \beta_t\sigma_t^p + \bar{\mathscr{D}}_p \sum_{t=1}^{m} \beta_t\bar{\sigma}_t^p\right) + 2\lambda = 0;$$

hence and from (14) we obtain

$$(69)\qquad \lambda = -\frac{1}{2}\sum_{p=1}^{N-1} \frac{1}{p}(\mathscr{D}_p d_p + \bar{\mathscr{D}}_p d_p),$$

formulas (67) and (69) immediately give (17).

Finally, in order to prove that formula (16) is true for the function $\tilde{\mathscr{L}}(\zeta)$ we notice that if equalities (68) are multiplied by the numbers $\sigma_1^k\beta_1, \ldots, \sigma_m^k\beta_m$ $(k = 1, \ldots, m)$, respectively and summed, we obtain, as in the preceding considerations, the relations

$$\sum_{p=1}^{N-1} \frac{1}{p}\mathscr{D}_p d_{p+k} + \sum_{p=1}^{k-1} \frac{1}{p}\bar{\mathscr{D}}_p d_{k-p} + \frac{2}{k}\bar{\mathscr{D}}_k + \sum_{p=k+1}^{N-1} \frac{1}{p}\bar{\mathscr{D}}_p \bar{d}_{p-k} + \lambda_k d_k = 0$$
$$(k = 1, \ldots, N-1);$$

this and relations (69) and (48), by means of elementary calculations, directly imply formula (16) for the function $\tilde{\mathscr{L}}(\zeta)$, which ends the proof of the Fundamental Theorem 1.

Now we shall proceed to formulate and prove the following fact.

FUNDAMENTAL THEOREM 2. *There exists a quasi-starlike function $g(z)$ which is extremal in the class $\mathfrak{G}^M$ with respect to the functional H_g of* (8) *and which belongs to a class $\mathfrak{G}_m^M$, where $m \leqslant N-1$.*

Proof. Let H^* denote, for example, the maximum of the functional H_g in $\mathfrak{G}^M$ and let $H_k^* \leqslant H^*$ denote an analogous maximum in the compact subfamily $\bigcup_{m=1}^{k} \mathfrak{G}_m^M$ $(k = 1, 2, \ldots)$.

We observe first that $H_k^* = H_{N-1}^*$ for $k \geqslant N-1$ and consequently

$$\sup_k H_k^* = H_{N-1}^*. \tag{70}$$

Indeed, at last it is sufficient to prove that for $k \geqslant N-1$ the functional H_g considered in the set $\bigcup_{m=1}^{k} \mathfrak{G}_m^M$ attains its maximum for functions belonging to $\bigcup_{m=1}^{N-1} \mathfrak{G}_m^M$ only.

Suppose this is not the case. Then we can find a function $\hat{g}(z)$ of the class $\mathfrak{G}_m^M$ $(N-1 < m \leqslant k)$ realizing the above-mentioned maximum. Hence and from Theorem 1 it follows that there exist m different numbers σ_i of (2) which are double roots of the function $\tilde{\mathscr{R}}(\zeta)$ of (17), which is impossible, because the function $\tilde{\mathscr{R}}(\zeta)$ has at most $2N-2$ roots.

Secondly we observe that

$$H^* \leqslant H_{N-1}^*. \tag{70'}$$

Indeed, in the contrary case there would be a function $g^*(z) \in \mathfrak{G}^M$ such that

$$H_{g^*} > H_{N-1}^*.$$

Since every quasi-starlike function can be approximated by functions of the class $\mathfrak{G}_m^M$, there exists a function $g^{**}(z) \in \mathfrak{G}_{m_0}^M$ sufficiently close to $g^*(z)$ and such that $H_{g^{**}} > H_{N-1}^*$ and consequently $H_{m_0}^* > H_{N-1}^*$, which contradicts (70). Relations (70), (70′) give $H^* = H_{N-1}^*$, which ends the proof.

Starlike functions and quasi-starlike functions. In this part we shall give the necessary and sufficient conditions for a quasi-starlike function of a class $\mathfrak{G}_m^M$ to be a starlike function. The classes $\mathfrak{G}_m^M$ are of special interest here, as follows from the Fundamental Theorem 2 about the extremal functions.

We shall deduce the following result.

THEOREM 3. *The necessary and sufficient condition for a function $w = g(z)$ of the class $\mathfrak{G}_m^M$ generated by* (4) *to be starlike is that by the notation and relations* (2), (2′) *the following conditions be satisfied*

$$\beta_k = \frac{2}{m} \quad (k = 1, 2, \ldots, m), \tag{71}$$

$$\sigma_k = e^{i(2\pi k/m)}, \text{ when } m \text{ is an odd number } (k = 1, 2, \ldots, m), \tag{72}$$

(73) $\sigma_k = e^{i\left(\frac{4\pi}{m}\left[\frac{k}{2}\right]+(-1)^{k-1}\varphi\right)}$, *when m is an even number* $(k = 1, 2, \ldots, m)$, *where φ is an arbitrary real number belonging to the open interval* $(0, 2\pi/m)$.

Formulas (72), (73) *determine the numbers σ_k with an accuracy up to an arbitrary rotation angle.*

Proof. Suppose the function $g(z)$ from (4) is starlike and observe first that the corresponding starlike function $\omega = G(\zeta)$ from (1) maps the unit disc onto the full complex plane from which m semistraight lines l_k $(k = 1, \ldots, m)$ have been removed, the lines l_k being radial in relation to 0 with certain arguments $0 \leqslant \psi_1 < \ldots < \psi_m < 2\pi$. Then, for the sake of convenience, put

$$h(\omega) = \frac{1}{M}\omega$$

and notice that by (4)

$$g(z) = G^{-1}\Big(h\big(G(z)\big)\Big).$$

Hence it easily follows that the function $w = g(z)$ maps the unit disc onto the same disc from which m arcs s_k have been removed, the arcs being the images of the segments

$$p_k = h(l_k) - l_k \quad (k = 1, \ldots, m). \tag{74}$$

From the assumption made it also follows that the arcs s_k are radial segments with certain arguments Θ_k $(k = 1, \ldots, m)$.

Denote by

$$z = xe^{i\psi_k}, \quad x_1^{(k)} \leqslant x \leqslant x_2^{(k)}, \quad x_1^k > 0 \quad (k = 1, \ldots, m)$$

the parametrical equations of the segments p_k and by

$$w = ye^{i\Theta_k}, \quad y_1^{(k)} \leqslant y \leqslant y_2^{(k)}, \quad y_1^{(k)} > 0 \quad (k = 1, \ldots, m)$$

the parametrical equations of the segments s_k.

Then it follows from the above remarks that the function

$$y = h_k(x) = G^{-1}(xe^{i\psi_k})e^{-i\Theta_k} \quad (k = 1, \ldots, m) \tag{75}$$

maps real intervals $[x_1^{(k)}, x_2^{(k)}]$ onto real intervals $[y_1^{(k)}, y_2^{(k)}]$ and consequently is a real function of a real variable x. At the same time we can easily notice that, by using formula (75), the function $h_k(x)$ can be prolonged to the inteval $[0, x_2^k]$. If we consider the fact that

$$G^{-1}(0) = 0, \quad G^{-1\prime}(0) = 1,$$

we shall see that the expansion of this function in a power series is of the form

$$h_k(x) = \sum_{n=1}^{\infty} d_n^{(k)} x^n, \quad 0 \leqslant x \leqslant x_2^{(k)}, \tag{75'}$$

where

$$(75'') \qquad \operatorname{im} d_n^{(k)} = 0 \quad (n = 1, 2, \ldots, \; k = 1, \ldots, m).$$

Next, using the expansion (5) of the function $G(\zeta)$ and (75), (75′), we obtain

$$(76) \qquad \big(h_k(x)\big)e^{i\Theta_k} + b_2\big(h_k(x)\big)^2 e^{2i\Theta_k} + \ldots = xe^{i\psi_k}, \quad 0 \leqslant x \leqslant x_2^{(k)};$$

hence from (75″) it follows that $e^{i\Theta_k} = \pm e^{i\psi_k}$ $(k = 1, \ldots, m)$, and (76) can be written in the form

$$(76') \qquad h_k(x) + b_2\big(h_k(x)\big)^2 e^{i\Theta_k} + \ldots = \pm x, \quad 0 \leqslant x \leqslant x_2^{(k)};$$

the terms $\big(h_k(x)\big)^n$ being real it follows from relation (76′) that $b_n e^{i(n-1)\Theta_k}$ and so $b_n e^{i(n-1)\psi_k}$ $(n = 1, \ldots, \; k = 1, \ldots, m)$ are real. That means that the functions

$$(77) \qquad G(ze^{i\psi_k})e^{-i\psi_k} \quad (k = 1, \ldots, m)$$

have real coefficients.

We shall show now that the numbers $e^{i\psi_k}$ are the m-th degree roots of unity with an accuracy up a constant factor $e^{i\psi}$.

Put

$$(77') \qquad \psi = \psi_1, \quad \psi_k - \psi = \hat{\psi}_k \quad (k = 1, \ldots, m)$$

and

$$(78) \qquad \hat{G}(z) = G(ze^{i\psi})e^{-i\psi} = \frac{z}{\prod_{k=1}^{m}(1-\hat{\sigma}_k z)^{\beta_k}}, \qquad (78') \qquad \hat{\sigma}_k = \sigma_k e^{i\psi}.$$

Function (78) obviously maps the unit disc onto the full plane from which m semistraight lines $\hat{l}_k$ have been removed, the lines being radial in relation to 0 with the arguments

$$0 = \hat{\psi}_1 < \hat{\psi}_2 < \ldots < \hat{\psi}_m < 2\pi.$$

Then, according to the above information regarding (77) the functions

$$(79) \qquad \hat{G}(ze^{i\hat{\psi}_k})e^{-i\hat{\psi}_k} = G(ze^{i\psi_k})e^{-i\psi_k} \quad (k = 1, \ldots, m)$$

have real coefficients and map the unit disc onto the full plane from which m analogous semistraight lines have been removed, the lines lying symmetrically with respect to the real axis. These lines we obtain from $\hat{l}_1, \ldots, \hat{l}_m$ by rotation about an angle $-\hat{\psi}_k$.

To prove the above-mentioned property of numbers $e^{i\hat{\psi}_k}$ it suffices to prove that

$$(80) \qquad \hat{\psi}_{k+1} - \hat{\psi}_k = \text{const} \quad (k = 1, \ldots, m)$$

under the convention $\hat{\psi}_{m+1} = \psi_1 + 2\pi$.

Contrariwise suppose first that there exists such a number $1 < k < m$ that

$$\hat{\psi}_{k+1} - \hat{\psi}_k < \hat{\psi}_k - \hat{\psi}_{k-1} \quad \text{or} \quad 2\hat{\psi}_k > \hat{\psi}_{k-1} + \hat{\psi}_{k+1} \quad \text{or} \quad \hat{\psi}_{k-1} < 2\hat{\psi}_k - \hat{\psi}_{k+1} < \hat{\psi}_k.$$

Then, we conclude by the symmetry of functions (79) that for the argument $\hat{\psi}_{k+1} - \hat{\psi}_k$ there exists a symmetrical argument $\hat{\psi}_l - \hat{\psi}_k$, i.e. such that $2\pi s - (\hat{\psi}_{k+1} - \hat{\psi}_k) = \hat{\psi}_l - \hat{\psi}_k$, where $1 \leqslant l < m$ and s are integers. But on account of $\hat{\psi}_l - 2\pi s = 2\hat{\psi}_k - \hat{\psi}_{k+1}$, according to what has been said before, we would have

$$\hat{\psi}_{k-1} < \hat{\psi}_l - 2\pi s < \hat{\psi}_k,$$

which contradicts the obvious fact that numbers (77′) lie in the interval $[0, 2\pi)$. The argument is similar if $k = 1$ or $k = m$, and analogously, when $\hat{\psi}_{k+1} - \hat{\psi}_k > \hat{\psi}_k - \hat{\psi}_{k+1}$. This yields (80) and clearly $c = 2\pi/m$. At last we have obtained

$$(80') \qquad \hat{\psi}_1 = 0, \ \hat{\psi}_2 = \frac{2\pi}{m}, \ \ldots, \ \hat{\psi}_m = \frac{2(m-1)\pi}{m}.$$

Basing our considerations on the result obtained, we shall now proceed to find the general form of the numbers σ_k and β_k.

First let us notice that according to what has been said above function (79) have real coefficients and thus their singular points must lie symmetrically with respect to the real axis, or which is the same, the singular points

$$(80'') \qquad \hat{\bar{\sigma}}_k = e^{-i\psi}\,\bar{\sigma}_k = e^{-i(\psi+\varphi_k)} \qquad (k = 1, \ldots, m)$$

of (78) must lie symmetrically with respect to the radial semistraight lines with the directions

$$(81) \qquad e^{i\hat{\psi}_k}, \ldots, e^{i\hat{\psi}_m}.$$

In addition, comparing the order of magnitude of function (78) in the neighbourhood of their arbitrary two different singular points from (80″) which lie symmetrically with respect to the real axis, we observe that the exponents assigned to them must be equal. To determine the position of the singular points $\hat{\bar{\sigma}}_k$ on the unit circumference we first prove that no singular point of (80″) has the direction of $\hat{l}_1$, i.e.

$$(81') \qquad \hat{\bar{\sigma}}_k \neq e^{i\hat{\psi}_1} = 1.$$

In fact, let us assume that for certain k_0 we have $\hat{\sigma}_{k_0} = 1$. As easily follows from the form of function (78), the anti-image of the semistraight line $\hat{l}_1$ is an arc which does not contain any point (80″). Moreover, from the symmetry of the function $\hat{G}(z)$ it follows that the points $\hat{\bar{\sigma}}_k$ lie symmetrically with respect to the real axis. Hence, if m is even there exists a k_1 such that $\hat{\sigma}_{k_1} = -1$ too and the anti-image of $\hat{l}_1$, by the symmetry of function (78) is a symmetrical arc with respect to the real axis and this arc must contain the point -1 or $+1$. On the other hand, the anti-image of $\hat{l}_1$ does not contain any point of (80″) particularly $\bar{\hat{\sigma}}_{k_0}$ and $\bar{\hat{\sigma}}_{k_1}$. So we have obtained a contradiction. Next, we suppose that m is odd; then the anti-image of $\hat{l}_1$ is symmetrical arc with respect to the real axis, as previously, and it must contain -1 or $+1$. On the other hand, this arc does not contain $1 = \hat{\sigma}_{k_0}$; consequently it contains -1, and therefore -1 does not occur among the numbers $\bar{\hat{\sigma}}_k$. Then, by joining the singular points in pairs, we can write function (78) in the form

$$(81'') \qquad \hat{G}(z) = \frac{z}{(1-z)^{\beta_{k_0}} \prod\limits_{\substack{k=0 \\ k\neq k_0}}^{(m-1)/2} (1-2z\cos(\psi+\varphi_k)+z^2)^{\beta_k}}.$$

From (81″) we see that the function $\hat{G}(z)$ has negative values for negative values of z, particulary for $z = -1$, which is impossible, because the image of the point -1 lies on $\hat{l}_1$, and so it is a positive number. Thus we have obtained a contradiction and consequently relations (81′). If we make use of (79) instead of (78) and we rotate the semistraight line $\hat{l}_k$ about the angle $-\hat{\psi}_k$, we shall show that generally

$$(81''') \qquad \bar{\hat{\sigma}}_k \neq e^{i\hat{\varphi}_1}, e^{i\hat{\varphi}_2}, \ldots, e^{i\hat{\varphi}_m} \qquad (k = 1, \ldots, m).$$

Making use of the above results (81′), (81) we see that the singular points (80″) of the function $\hat{G}(z)$ lie between the points (81), exactly one point $\bar{\hat{\sigma}}_{kl}$ of (80″) being situated between two successive points $e^{i\cdot 2\pi(l-1)/m}$ and $e^{i\cdot 2\pi l/m}$ of (81) on the unit circumference, precisely. In the contrary case the numbers of the points (80″) and directions (81) would be different. Next putting

$$\bar{\hat{\sigma}}_{k_1} = e^{i\varphi} \qquad (0 < \varphi < 2\pi/m),$$

we obtain by easy induction from the above-mentioned symmetry of the singular points (80″) $\bar{\hat{\sigma}}_{k_l}, \bar{\hat{\sigma}}_{k_{l+1}}$ in relation to the direction $e^{i\cdot 2\pi l/m}$ and from the equality of the exponents β_{k_l} and $\beta_{k_{l+1}}$, the formulas

$$(82) \qquad \sigma_{k_l} = e^{i\left(\frac{4\pi}{m}\left[\frac{l}{2}\right]+(-1)^{l-1}\varphi\right)} \qquad (l = 1, \ldots, m) \text{ [1]},$$

$$(82') \qquad \beta_{k_l} = 2/m \qquad (l = 1, \ldots, m)$$

[1] $[x]$ denotes here an integer not larger than x.

and from the symmetry with respect to the direction $e^{i\cdot 2\pi\cdot 0/m}$ of the real axis the equality

$$\hat{\sigma}_{k_m} = \overline{\hat{\sigma}}_{k_1}.$$

If m is even, the last equality is satisfied for every $\varphi \in (0, 2\pi/m)$ and if m is odd, it is satisfied for $\varphi = \pi/m$ only. Hence and from (82), (82′), by suitable notation, we can easily obtain on view of (78′) formulas (71), (72) and (73).

Now we shall show that the inverse theorem is also true. We assume that conditions (71), (72) or (73) for $G(z)$ of (1) are satisfied and we consider the functions

$$G_1(z) = \frac{z}{(1-z)^2} \tag{83}$$

and

$$G_2(z) = \frac{z}{1-2z\cos\varphi+z^2} \quad (0 < \varphi < 2\pi/2). \tag{84}$$

We may easily find that these functions have real coefficients and that the first of them maps the unit circumference onto a semistraight line lying on the positive part of the real axis, while the second maps it onto two semistraight lines lying on the real axis symmetricaly in relation to the origin. Thus the corresponding functions $g_j(z)$ $(j = 1, 2)$ generated by equation (4) with functions (83) or (84) instead of (1) have real coefficients and map the unit disc onto the unit disc without a segment of the real axis in the first case and without two symmetrical segments of the real axis in the second. Hence it immediately follows that $g_j(z)$ $(j = 1, 2)$ are starlike functions.

Then we observe that every function $G(z)$ of (1) for which conditions (71), (72), or (73) are satisfied can be defined by the formula

$$G(z) = \sqrt[\mu]{G_j(z^\mu)},$$

where for odd m we take: $j = 1$ and $\mu = m$, while for even m we take: $j = 2$ and $\mu = m/2$. Thus $G(z)$ is a μ-symmetrical function. Moreover, the corresponding quasi-starlike function $g(z)$ is then defined by the formula

$$g(z) = \sqrt[\mu]{g_j(z^\mu)}$$

and thus is also μ-symmetrical. Hence and from the remark made earlier about the starlikeness of the function $g_j(z)$ the starlikeness of $g(z)$ immediately follows, which ends the proof.

Remark. Finding the conditions for an arbitrary function $g(z)$ of the class $\mathfrak{G}^M$ to be starlike seems to be difficult, since we shall show that having an arbitrary starlike function we can, by means of it, easily find a quasi-starlike function which will at the same time be starlike and which does not belong to any class $\mathfrak{G}_m^M$. In order to show this we take an arbitrary holomorphic, univalent and starlike function $F(\zeta)$ of (0) and we let $g(z)$ denote a quasi-starlike function determined by equation (3); then

$$g(z) = F^{-1}\left(\frac{1}{M}F(z)\right), \quad |z| < 1, \tag{85}$$

where $z = F^{-1}(w)$ is a function inverse to the function $w = F(z)$. Next, if r denotes the starlikeness radius of the function $g(z)$, then the function

$$g(rz) = F^{-1}\left(\frac{1}{M}F(rz)\right) \quad \text{for } |z| < 1, \tag{86}$$

is starlike in the unit disc. Now we shall show that the starlike function

$$g_1(z) = \frac{1}{r}g(rz), \quad |z| < 1, \tag{86'}$$

is also quasi-starlike. From (86) and (86′) we have

$$rg_1(z) = F^{-1}\left(\frac{1}{M}F(rz)\right), \tag{87}$$

$$F\big(rg_1(z)\big) = \frac{1}{M}F(rz). \tag{87'}$$

On the other hand, the function

$$F_1(z) = \frac{1}{r}F(rz)$$

is of course a starlike function, and taking into consideration (87′), we have

$$F_1\big(g_1(z)\big) = \frac{1}{M}F_1(z), \quad |z| < 1;$$

hence it immediately follows that the function $g_1(z)$ is quasi-starlike and we can easily see that the function $g_1(z)$ is holomorphic in the closed unit disc, and so it does not belong to any class $\mathfrak{G}_m^M$.

The above argumentation also shows that the class of quasi-starlike functions contains quite a big subclass of starlike functions. It raises the natural question whether every starlike function bounded in the unit disc and properly normalized is a quasi-starlike function. The answer has not been found yet.

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