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On the least principal fundamental solution of a parabolic differential equation

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Abstract. This article is a continuation of paper [3], where the concept of a principal fundamental solution of a second order parabolic equation was introduced. In this paper, using a method due to Besala [2], we prove the existence of the least principal fundamental solution.

The notion of the principal fundamental solution was introduced in [3], where the existence and uniqueness of such a solution was proved for a parabolic equation with bounded and smooth coefficients. The purpose of this paper is to construct the principal fundamental solution of a second order parabolic equation under assumptions which do not ensure the uniqueness of this solution. The principal fundamental solution which we construct is the least possible. The construction used here is based on the method applied in papers [1], [2] and [3]. This result includes an earlier result given in [3].

Denote by $x = (x_1, \ldots, x_n)$ points of the Euclidean n-space E_n $(n \geqslant 1)$ and by t points of the Euclidean space E $(E = E_1)$.

We consider the differential equation:

$$(1) \qquad Lu = \sum_{i,j=1}^{n} a_{ij}(t, x) u_{x_i x_j} + \sum_{j=1}^{n} b_j(t, x) u_{x_j} + c(t, x) u - u_t = 0$$

for $(t, x) \in E_{n+1}$, where the coefficients a_{ij}, b_j, c are defined in E_{n+1} and satisfy the following conditions (see [2]):

I. a_{ij}, b_j, c are Hölder continuous with respect to (t, x) on every compact subset of E_{n+1}, $a_{ij} = a_{ji}$, and for each $t \in E$ there exist weak derivatives $(a_{ij})_{x_i}, (a_{ij})_{x_i x_j}, (b_j)_{x_j}$ in E_n for $i, j = 1, \ldots, n$.

Moreover, there is a constant $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t, x) \xi_i \xi_j \geqslant \lambda |\xi|^2 \quad \text{for } (t, x) \in E_{n+1} \text{ and } \xi(\xi_1, \ldots, \xi_n) \in E_n.$$

II. There exists a function $h(t, x) \in C^2(E_{n+1})$ with Hölder continuous second order x-derivatives on every compact subset of E_{n+1} and such that $h(t, x) > 0$ on E_{n+1} and

$$Lh \leqslant -Kh, \tag{2}$$

$$Lh + h\Big[\sum_{i,j=1}^{n} (a_{ij})_{x_i x_j} - \sum_{j=1}^{n} (\tilde{b}_j)_{x_j}\Big] \leqslant 0, \tag{3}$$

where

$$\tilde{b}_j = 2h^{-1}\sum_{i=1}^{n} a_{ij} h_{x_i} + b_j \tag{4}$$

for each $t \in E$ and almost all $x \in E_n$; K is a positive constant.

Observe that then

$$Lh + \eta h\Big[\sum_{i,j=1}^{n} (a_{ij})_{x_i x_j} - \sum_{j=1}^{n} (\tilde{b}_j)_{x_j}\Big] \leqslant 0$$

for $\eta \in [0, 1]$, as shown in [2].

DEFINITION. A function $\Gamma(t, x, \tau, y)$ defined in D: $-\infty < \tau < t < +\infty$, $x, y \in E_n$ is said to be a *principal fundamental solution* of (1) if it satisfies the following conditions:

1° For any fixed $(\tau, y) \in E_{n+1}$, $\Gamma(t, x, \tau, y)$ as a function of (t, x) has continuous derivatives $\Gamma_t, \Gamma_{x_i}, \Gamma_{x_i x_j}$ $(i, j = 1, \ldots, n)$ and satisfies equation (1) in $(\tau, +\infty) \times E_n$.

2° For every bounded function f which is locally Hölder continuous (that means, Hölder continuous on every compact subset of E_{n+1}) we have

$$L\Big[\int_{-\infty}^{t} d\tau \int_{E_n} \Gamma(t, x, \tau, y) f(\tau, y)\, dy\Big] = -f(t, x) \quad \text{for } (t, x) \in E_{n+1}.$$

We begin by proving the existence of the principal fundamental solution.

THEOREM 1. *If Assumptions* I, II *are satisfied, then there exists a principal fundamental solution* $\Gamma(t, x, \tau, y)$ *of equation* (1) *which satisfies the inequalities*:

$$0 \leqslant \Gamma(t, x, \tau, y) \leqslant C(t-\tau)^{-n/2} h(t, x)/h(\tau, y) \quad \text{in } D, \tag{5}$$

$$\int_{E_n} \Gamma(t, x, \tau, y) h(\tau, y)\, dy \leqslant h(t, x) \quad \text{for } -\infty < \tau < t < +\infty,\ x \in E_n, \tag{6}$$

$$\int_{E_n} \Gamma(t, x, \tau, y)/h(t, x)\, dx \leqslant 1/h(\tau, y) \quad \text{for } -\infty < \tau < t < +\infty,\ y \in E_n, \tag{7}$$

C being a positive constant depending only on n and λ.

Set $u(t,x) = v(t,x)h(t,x)$ into (1). We obtain for v the equation:

$$\tilde{L}v = \sum_{i,j=1}^{n} a_{ij}v_{x_ix_j} + \sum_{j=1}^{n} \tilde{b}_j v_{x_j} + \tilde{c}v - v_t = 0, \tag{8}$$

where $\tilde{b}_j$ is given by (4) and $\tilde{c} = Lh/h$. Evidently, if $\gamma(t,x,\tau,y)$ is a principal fundamental solution of equation (8), then

$$\Gamma(t,x,\tau,y) = \gamma(t,x,\tau,y)h(t,x)/h(\tau,y)$$

is a principal fundamental solution of (1).

Thus Theorem 1 is an immediate consequence of the following

THEOREM 2. *If Assumptions* I, II *are satisfied, then there exists a principal fundamental solution* $\gamma(t,x,\tau,y)$ *of equation* (8) *which satisfies the inequalities*

$$0 \leqslant \gamma(t,x,\tau,y) \leqslant C(t-\tau)^{-n/2} \quad \textit{in } D, \tag{9}$$

$$\int_{E_n} \gamma(t,x,\tau,y)\,dy \leqslant 1 \quad \textit{for } -\infty < \tau < t < +\infty,\ x \in E_n, \tag{10}$$

$$\int_{E_n} \gamma(t,x,\tau,y)\,dx \leqslant 1 \quad \textit{for } -\infty < \tau < t < +\infty,\ y \in E_n, \tag{11}$$

C being the same as in (5).

Proof. The proof is similar to that of Theorem 2 in Besala's paper [2], if we replace the sets $S_m = (0,T\rangle \times (|x| < m)$ by $D_m = (-m, m\rangle \times \times(|x| < m)$.

Considering the sequence of Green's functions $\gamma_m(t,x,\tau,y)$ of (8) in D_m, we see that $\{\gamma_m\}$ is a non-decreasing sequence and satisfies the inequalities:

$$\int_{|y|<m} \gamma_m(t,x,\tau,y)\,dy \leqslant 1 \quad \text{for } |x| < m,\ -m < \tau < t \leqslant m,$$

$$\int_{|x|<m} \gamma_m(t,x,\tau,y)\,dx \leqslant 1 \quad \text{for } |y| < m,\ -m < \tau < t \leqslant m,$$

$$0 \leqslant \gamma_m(t,x,\tau,y) \leqslant C(t-\tau)^{-n/2} \quad \text{for } |x| < m,\ |y| < m,\ -m < \tau < t \leqslant m.$$

Next, we show that $\gamma(t,x,\tau,y) = \lim_{m\to\infty} \gamma_m(t,x,\tau,y)$ is a principal fundamental solution of equation (8). Applying the Schauder interior estimates (see [5], Chapter 3, Section 2), we show that $\gamma(t,x,\tau,y)$ satisfies (8) as a function of $(t,x) \in (\tau, +\infty) \times E_n$ and has continuous derivatives $\gamma_t, \gamma_{x_i}, \gamma_{x_ix_j}$. To prove that condition 2° of the definition of the principal fundamental solution is satisfied, consider the following Cauchy problem:

$$\tilde{L}v_m = f(t,x) \quad \text{for } (t,x) \in D_m, \tag{12}$$

$$v_m(t,x) = 0 \quad \text{for } (t,x) \in \partial_p D_m, \tag{13}$$

where

$$\partial_p D_m = \{(t = -m)\times(|x| < m)\}\cup\{(-m < t \leqslant m)\times(|x| = m)\}$$

and f is a bounded and locally Hölder continuous function.

It is well known that the unique solution of problem (12)–(13) is given by

$$v_m(t, x) = -\int_{-m}^{t} d\tau \int_{|y|<m} \gamma_m(t, x, \tau, y) f(\tau, y)\, dy.$$

Extending the domain of definition of $\gamma_m(t, x, \tau, y)$ by setting

$$\gamma_m = 0 \quad \text{for } |x| \geqslant m \quad \text{or} \quad |y| \geqslant m \quad \text{or} \quad t > m \quad \text{or} \quad \tau \leqslant -m,$$

we have $\gamma_m \leqslant \gamma_{m+1}$ in D. Hence, if $f \geqslant 0$, then

$$-\int_{-m}^{t} d\tau \int_{|y|<m} \gamma_m(t, x, \tau, y) f(\tau, y)\, dy \geqslant -\int_{-(m+1)}^{t} d\tau \int_{|y|<m+1} \gamma_{m+1}(t, x, \tau, y) f(\tau, y)\, dy,$$

i.e., $\{v_m\}$ is non-increasing. By the inequalities

$$\tilde{L}\left(v_m + \frac{\sup f}{K}\right) = f + \tilde{c}\frac{\sup f}{K} \leqslant f - \sup f \leqslant 0 \quad \text{in } D_m,$$

$$v_m + \frac{\sup f}{K} \geqslant 0 \quad \text{for } (t, x) \in \partial_p D_m,$$

and by the maximum principle, we obtain

$$-\frac{\sup f}{K} \leqslant v_m \leqslant 0.$$

Thus the sequence $\{v_m\}$ is convergent and bounded.

If f is any function, then $f = f^+ - f^-$, where

$$f^+(t, x) = \begin{cases} 0 & \text{for } \{(t, x) \in E_{n+1}\colon f(t, x) < 0\}, \\ f(t, x) & \text{for } \{(t, x) \in E_{n+1}\colon f(t, x) \geqslant 0\}, \end{cases}$$

$$f^-(t, x) = \begin{cases} -f(t, x) & \text{for } \{(t, x) \in E_{n+1}\colon f(t, x) < 0\}, \\ 0 & \text{for } \{(t, x) \in E_{n+1}\colon f(t, x) \geqslant 0\}, \end{cases}$$

and $f^+ \geqslant 0$, $f^- \geqslant 0$. Hence the functions

$$v_m^+(t, x) = -\int_{-m}^{t} d\tau \int_{|y|<m} \gamma_m(t, x, \tau, y) f^+(\tau, y)\, dy$$

and

$$v_m^-(t, x) = -\int_{-m}^{t} d\tau \int_{|y|<m} \gamma_m(t, x, \tau, y) f^-(\tau, y)\, dy$$

are convergent and bounded.

According to the Friedman–Schauder interior estimates there exists a subsequence of $\{v_m\}$ which converges to the function

$$v(t, x) = -\int_{-\infty}^{t} d\tau \int_{E_n} \gamma(t, x, \tau, y) f(\tau, y)\, dy$$

satisfying the equation $\tilde{L}v = f$.

The proof of the fact that γ satisfies inequalities (9), (10), (11) is technically similar to the proof of Theorem 2 in [2], therefore we omit the details.

Examples of equations satisfying Assumption II can be found in [3] and [4]. In these examples the existence of h is ensured by suitable growth conditions imposed on the coefficients.

Now, if we define the principal fundamental solution as a function satisfying 1°, 2° and

3° $\Gamma(t, x, \tau, y) \geqslant 0$ for $x, y \in E_n$, $-\infty < \tau < t < +\infty$,

then we have the following

THEOREM 3. *If Assumptions* I *and* II *are fulfilled, then*

$$\Gamma_h(t, x, \tau, y) = \gamma(t, x, \tau, y) h(t, x)/h(\tau, y)$$

is the least principal fundamental solution of (1). Γ_h *is independent of* $h \in H$, *where* H *denotes the set of all functions satisfying Assumption* II.

Proof. Let $G(t, x, \tau, y)$ be any non-negative principal fundamental solution of (1). Let f be a locally Hölder continuous non-negative function with compact support. Take m so large that $\operatorname{supp} f \subset D_m$. By the properties of Green's function (see [5], Chapter 3, Section 7) we have

$$\lim_{\substack{(t,x)\to(\bar{t},\bar{x})\\ (\bar{t},\bar{x})\in\partial_p D_m}} \int_{-m}^{t} d\tau \int_{|y|<m} \gamma_m(t, x, \tau, y) \frac{h(t, x)}{h(\tau, y)} f(\tau, y)\, dy$$

$$\leqslant \lim_{\substack{(t,x)\to(\bar{t},\bar{x})\\ (\bar{t},\bar{x})\in\partial_p D_m}} \int_{-\infty}^{t} d\tau \int_{E_n} G(t, x, \tau, y) f(\tau, y)\, dy.$$

Hence, by the maximum principle we obtain

$$\int_{-m}^{t} d\tau \int_{|y|<m} \gamma_m(t, x, \tau, y) \frac{h(t, x)}{h(\tau, y)} f(\tau, y)\, dy \leqslant \int_{\infty -}^{t} d\tau \int_{E_n} G(t, x, \tau, y) f(\tau, y)\, dy$$

for $(t, x) \in D_m$. Thus, $\{\gamma_m\}$ being a non-decreasing sequence of non-negative functions, we obtain by letting $m \to \infty$

$$\int_{-\infty}^{t} d\tau \int_{E_n} \gamma(t, x, \tau, y) \frac{h(t, x)}{h(\tau, y)} f(\tau, y) dy \leqslant \int_{-\infty}^{t} d\tau \int_{E_n} G(t, x, \tau, y) f(\tau, y) dy.$$

It follows from the last inequality that

$$\Gamma_h(t, x, \tau, y) \leqslant G(t, x, \tau, y).$$

Putting consecutively Γ_{h_1} and Γ_{h_2} for G we get

$$\Gamma_{h_1} = \Gamma_{h_2} \quad \text{for } h_1, h_2 \in H.$$

Thus Γ_h is independent of $h \in H$ and is the least principal fundamental solution as asserted. From now on the least principal fundamental solution will be denoted simply by Γ.

Similarly we can prove the following

THEOREM 4. *Let Assumptions* I, II *be satisfied and let $f(t, x)$ be a bounded non-negative and locally Hölder continuous function defined for $(t, x) \in E_{n+1}$. Then*

$$u(t, x) = \int_{-\infty}^{t} d\tau \int_{E_n} \Gamma(t, x, \tau, y) f(\tau, y) dy$$

is the least non-negative solution of the equation

$$Lu = -f(t, x) \quad \textit{for } (t, x) \in E_{n+1}.$$

Proof. It follows from the earlier considerations that

$$u(t, x) = \int_{-\infty}^{t} d\tau \int_{E_n} \Gamma(t, x, \tau, y) f(\tau, y) dy$$

is the solution of $Lu = -f$. Let us introduce the sequence:

$$u_m(t, x) = \int_{-m}^{t} d\tau \int_{|y|<m} \gamma_m(t, x, \tau, y) \frac{h(t, x)}{h(\tau, y)} f(\tau, y) dy.$$

By the maximum principle we obtain

$$u_m(t, x) \leqslant z(t, x) \quad \text{on } D_m,$$

where $z(t, x)$ is an arbitrary non-negative solution of $Lu = -f$. Letting $m \to \infty$ we get $u(t, x) \leqslant z(t, x)$ for $(t, x) \in E_{n+1}$.

Let $\Gamma_{T_1 T_2}(t, x, \tau, y)$ denote the fundamental solution of the Cauchy problem in $[T_1; T_2] \times E_n$ for equation (1) constructed by Besala in [2]. Then we have:

THEOREM 5.

$$\Gamma_{T_1T_2}(t,x,\tau,y) = \Gamma(t,x,\tau,y) \quad \textit{for } T_1 \leqslant \tau < t \leqslant T_2,\ x, y \in E_n.$$

Proof. Let $\gamma_m^{T_1T_2}$ denote Green's function of (8) in $H_m^{T_1T_2} = [T_1, T_2] \times \times (|x| < m)$ and let γ_m denote Green's function of (8) in D_m. Take m so large that $[T_1, T_2] \subset (-m, m)$ and a locally Hölder continuous function f such that $\operatorname{supp} f \subset (T_1, T_2) \times E_n$. Then, according to the well-known properties of Green's function we have

$$L\Big[-\int_{T_1}^{t} d\tau \int_{|y|<m} \gamma_m^{T_1T_2}(t,x,\tau,y) f(\tau,y)\,dy + \\ + \int_{-m}^{t} d\tau \int_{|y|<m} \gamma_m(t,x,\tau,y) f(\tau,y)\,dy\Big] = 0$$

for $(t,x) \in H_m^{T_1T_2}$,

$$\lim_{\substack{(t,x)\to(\bar t,\bar x)\\ (\bar t,\bar x)\in\partial_p H_m^{T_1T_2}}} \Big[-\int_{T_1}^{t} d\tau \int_{|y|<m} \gamma_m^{T_1T_2}(t,x,\tau,y) f(\tau,y)\,dy + \\ + \int_{-m}^{t} d\tau \int_{|y|<m} \gamma_m(t,x,\tau,y) f(\tau,y)\,dy\Big] = 0,$$

where

$$\partial_p H_m^{T_1T_2} = \{(t = T_1) \times (|x| < m)\} \cup \{(T_1 \leqslant t \leqslant T_2) \times (|x| = m)\}.$$

By the maximum principle we obtain

$$\gamma_m^{T_1T_2}(t,x,\tau,y) = \gamma_m(t,x,\tau,y) \quad \text{for } T_1 \leqslant \tau < t \leqslant T_2,\ |x| < m,\ |y| < m.$$

Taking $m \to \infty$ we get

$$\gamma^{T_1T_2}(t,x,\tau,y) = \gamma(t,x,\tau,y) \quad \text{for } T_1 \leqslant \tau < t \leqslant T_2,\ x, y \in E_n.$$

Consequently

$$\Gamma_{T_1T_2}(t,x,\tau,y) = \Gamma(t,x,\tau,y) \quad \text{for } T_1 \leqslant \tau < t \leqslant T_2,\ x, y \in E_n.$$

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