ANNALES
POLONICI MATHEMATICI
XIV (1964)

On the compositions of integral means with Borel methods of summability

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Introduction. In this paper the compositions of Borel methods of summability of exponential and integral types with Cesàro methods for functions have been investigated. Only series with real terms have been considered. However, the results obtained may easily be extended to the case of complex numbers.

I wish to thank here Doc. Dr L. Włodarski for his valuable suggestions and remarks concerning this paper.

Let us begin with the definitions of the methods in question.

DEFINITION 1. We say that the series $\sum_{n=0}^{\infty} a_n$ is *summable to the number s by the exponential method* $B_{a,\gamma}$ $(a > 0, \gamma \geqslant 0)$, if the function

$$s(t) = ae^{-t}\sum_{n=0}^{\infty}\frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)}s_n\,, \qquad s_n = a_0+a_1+\ldots+a_n \tag{1}$$

is defined for any $t \geqslant 0$ and if the limit $\lim_{t\to\infty} s(t) = s$ exists. Then we write briefly

$$\sum_{n=0}^{\infty} a_n = s(B_{a,\gamma})\,. \tag{2}$$

DEFINITION 2. A series $\sum_{n=0}^{\infty} a_n$ is said to be *summable to s by the integral method* $\widetilde{B}_{a,\gamma}$ $(a > 0, \gamma \geqslant 0)$, if the function

$$a(t) = e^{-t}\sum_{n=0}^{\infty}\frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)}a_n \tag{3}$$

is defined for any $t \geqslant 0$ and if the limit $\lim_{t\to\infty}\int_0^t a(\tau)d\tau = s$ exists. Then we write briefly

$$\sum_{n=0}^{\infty} a_n = s(\widetilde{B}_{a,\gamma})\,. \tag{4}$$

Putting in the above definitions $a = 1$ and $\gamma = 0$ we obtain the classical Borel methods (the exponential and the integral methods). The exponential methods have been systematically investigated by L. Włodarski in his papers [11], [12], [14]. These methods, as L. Włodarski has proved, are permanent, consistent with one another and also with the Abel method. The consistency of the methods $B_{a,0}$ has been established independently of L. Włodarski by D. Borwein in paper [2]. D. Borwein has also proved in [1] that the summability $B_{a,0}$ of a series implies its $\widetilde{B}_{a,0}$ summability to the same number. It can be proved, by an easy modification of Borwein's argument, that the summability of a series by an exponential method $B_{a,\gamma}$ with $\gamma > 0$ implies its summability to the same number by an integral method $\widetilde{B}_{a,\gamma}$. The integral methods have been investigated by many authors, for instance, by Borel, Hardy, Mittag-Leffler, Good, Borwein. The methods $\widetilde{B}_{a,\gamma}$ are permanent and, as I. J. Good has proved in [5], consistent with one another.

DEFINITION 3. A function $f(t)$ is said to be *limitable in infinity to l by the Cesàro method C_r* $(r > 0)$, if the limit

$$\lim_{x\to\infty} \frac{r}{x^r} \int_0^x (x-t)^{r-1} f(t)\,dt = l \tag{5}$$

exists, which may be briefly written as

$$C_r\text{--}\lim_{t\to\infty} f(t) = l\,. \tag{6}$$

For symmetry we shall write also $C_0\text{--}\lim_{t\to\infty} f(t) = l$ instead of $\lim_{t\to\infty} f(t) = l$. The methods C_r are permanent; moreover, if $C_r\text{--}\lim_{t\to\infty} f(t) = l$ and $q > r$, then also $C_q\text{--}\lim_{t\to\infty} f(t) = l$.

In what follows we shall generalize the methods $B_{a,\gamma}$ and $\widetilde{B}_{a,\gamma}$ composing them with the methods C_r.

DEFINITION 4. A series $\sum_{n=0}^{\infty} a_n$ is said to be *summable by the method $B_{a,\gamma}^r$ to s*, if function (1) is defined for $t \geqslant 0$ and limitable to s by the method C_r. We shall write briefly

$$\sum_{n=0}^{\infty} a_n = s(B_{a,\gamma}^r)\,. \tag{7}$$

DEFINITION 5. A series $\sum_{n=0}^{\infty} a_n$ is said to be *summable to s by the method $\widetilde{B}_{a,\gamma}^r$*, if function (3) is defined for $t \geqslant 0$ and if $C_r\text{--}\lim_{t\to\infty} \int_0^t a(\tau)\,d\tau = s$, where the function $a(t)$ is given by formula (3). We shall write briefly

$$\sum_{n=0}^{\infty} a_n = s(\widetilde{B}^r_{a,\gamma}) . \tag{8}$$

The methods $B^r_{a,\gamma}$ and $\widetilde{B}^r_{a,\gamma}$ have thus been defined for $a > 0$, $\gamma \geqslant 0$, $r \geqslant 0$.

G. Doetsch ([3]) has investigated the methods $B^r_{1,0}$, i.e. the compositions of the classical Borel method of exponential type with Cesàro methods. By putting in definitions 4 and 5 $r = 0$ we obtain the methods $B_{a,\gamma}$ and $\widetilde{B}_{a,\gamma}$ described above. It immediately follows from the definition of the methods $B^r_{a,\gamma}$ and $\widetilde{B}^r_{a,\gamma}$ that they are permanent and also that the summability of a series by the method $B^r_{a,\gamma}$ (or by the method $\widetilde{B}^r_{a,\gamma}$) implies its summability to the same number by the method $B^q_{a,\gamma}$ (or $\widetilde{B}^q_{a,\gamma}$) provided $q > r$.

§ 1. Consistency theorems.

LEMMA 1. *If the functions $f(t)$ and $g(t)$ are continuous and if $g(t) \geqslant 0$, $\int_0^\infty g(\tau)d\tau = K < \infty$, $f(t) \sim At^a$, $a > 0$, then $f(t) * g(t) \sim KAt^a$* (1).

Proof. Let us put $f(t) = At^a + \varphi(t)$. Then

$$\varphi(t) = o(t^a) \qquad \text{and} \qquad f(t)*g(t) = At^a*g(t) + \varphi(t)*g(t) .$$

By the permanence of the Cesàro method C_a we have

$$\lim_{t\to\infty} \frac{1}{t^a}\big(At^a*g(t)\big) = \lim_{t\to\infty} \frac{A}{t^a} \int_0^t (t-\tau)^a g(\tau)d\tau$$
$$= \lim_{t\to\infty} \frac{Aa}{t^a} \int_0^t (t-\tau)^{a-1}d\tau \int_0^\tau g(\sigma)d\sigma = AK .$$

Now let an $\varepsilon > 0$ be given and let $|\varphi(t)| < \varepsilon t^a$ for $t > T$, $M = \sup_{0<\tau<T} |\varphi(\tau)|$. Then we have again by the regularity of the C_a method

$$|\varphi(t)*g(t)| \leqslant \int_0^T |\varphi(\tau) g(t-\tau)d\tau + \int_T^t |\varphi(\tau)| g(t-\tau)d\tau$$
$$\leqslant MK + \varepsilon \int_0^t (t-\tau)^a g(\tau)d\tau \sim \varepsilon K t^a ,$$

which, since ε has been chosen arbitrarily, proves our theorem.

Lemma 1 implies the following theorem:

(1) By $f(t)*g(t)$ we shall denote the convolution of the two functions $f(t)$ and $g(t)$, i.e. the function defined by the formula $h(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

THEOREM 1. *If a series* $\sum_{n=0}^{\infty} a_n$ *is summable to* s *by the method* $B^r_{a,\gamma}$, *then it is also summable to* s *by the method* $B^r_{a,\gamma+\delta}$ $(\delta > 0)$.

Proof. The summability of a series $\sum_{n=0}^{\infty} a_n$ by a method $B^r_{a,\gamma}$ to s implies

$$f(t) = \frac{t^{r-1}}{\Gamma(r)} * ae^{-t} \sum_{n=0}^{\infty} \frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)} s_n \sim \frac{t^r}{\Gamma(r+1)} s .$$

Putting $g(t) = e^{-t}t^{\delta-1}/\Gamma(\delta)$, and making use of the formula

$$e^{-t}t^{a}/\Gamma(a+1) * e^{-t}t^{\beta}/\Gamma(\beta+1) = e^{-t}t^{a+\beta+1}/\Gamma(a+\beta+2) \quad \text{for} \quad a, \beta > -1$$

and also of the commutativity and associativity of the convolution we have by lemma 1

$$f(t) * g(t) = \frac{t^{r-1}}{\Gamma(r)} * ae^{-t} \sum_{n=0}^{\infty} \frac{t^{na+\gamma+\delta}}{\Gamma(na+\gamma+\delta+1)} s_n \sim \frac{t^r}{\Gamma(r+1)} s ,$$

which means that the series $\sum_{n=0}^{\infty} a_n$ is summable to s by the method $B^r_{a,\gamma+\delta}$.

In the sequel we shall use the following lemmas:

LEMMA 2. *If* $0 < \theta < 1$, $\beta > 0$, $\int_0^{\infty} e^{-\sigma\tau}|f(\tau)|d\tau < \infty$ *for a certain* $\sigma > 0$, $F^*(s) = \int_0^{\infty} e^{-st} f(t)dt$, *then*

$$\int_0^{\infty} f(\tau) L^{-1}\left\{\frac{1}{s^{\beta}} e^{-\tau s^{\theta}}\right\} d\tau = L^{-1}\left\{\frac{1}{s^{\beta}} F^*(s^{\theta})\right\}, \tag{9}$$

where L^{-1} *denote a transform converse to the Laplace transform.*

Formula (9) is a particular case of the Efros formula ([10]).

LEMMA 3. *The function* $\Phi_{\beta,\theta}(t,\tau) = L^{-1}\left\{\frac{1}{s^{\beta}} e^{-\tau s^{\theta}}\right\}$ *is a real, continuous and non-negative function for any* $t > 0$, $\tau > 0$, $\beta > 0$ *and* $0 < \theta < 1$. (Mikusiński [9], Włodarski [13].)

LEMMA 4. *If* $f(t) \sim At^{k_1}/\Gamma(k_1+1)$, $g(t) \sim Bt^{k_2}/\Gamma(k_2+1)$, *then* $f(t)*g(t) \sim ABt^{k_1+k_2+1}/\Gamma(k_1+k_2+2)$. (Hardy [6], p. 99.)

Consider now the transform

$$W^{rk}_{\theta,\beta}\{f;\, t\} = \frac{\theta^{k+1}(2k)!}{t^{2k}} \left[\frac{t^{k-1}}{(k-1)!} * \left(e^{-t} \int_0^{\infty} e^{\tau} \frac{\tau^k}{k!} L^{-1}\left\{\frac{1}{s^{\beta}} e^{-\tau s^{\theta}}\right\} f(\tau)d\tau\right)\right] \tag{10}$$

for $0 < \theta < 1$, $\beta > 0$ and $k = 1, 2, \ldots$

It is easily seen that the operation $W^k_{\theta,\beta}$ is linear, i.e. that

$$W^k_{\theta,\beta}\{af+bg;\, t\} = aW^k_{\theta,\beta}\{f;\, t\}+bW^k_{\theta,\beta}\{g;\, t\} \tag{11}$$

provided the right-hand side exists.

THEOREM 2. *If $f(\tau)$ is a function defined and continuous for $\tau \geqslant 0$ and if $\lim\limits_{\tau\to\infty} f(\tau) = m$ (m may be finite or equal to $+\infty$), then*

$$\lim_{t\to\infty} W^k_{\theta,\beta}\{f;\, t\} = m\,.$$

Proof. Case 1°: $f(\tau) \equiv m$. Then we have

$$W^k_{\theta,\beta}\{f;\, t\} = m\frac{\theta^{k+1}(2k)!}{t^{2k}}\left[\frac{t^{k-1}}{(k-1)!} * \left(e^{-t}\int\limits_0^\infty e^{\tau}\frac{\tau^k}{k!}L^{-1}\left\{\frac{1}{s^\beta}e^{-\tau s^\theta}\right\}d\tau\right)\right].$$

By lemma 2 and the well-known properties of the Laplace transform we have

$$\int\limits_0^\infty e^{\tau}\frac{\tau^k}{k!}L^{-1}\left\{\frac{1}{s^\beta}e^{-\tau s^\theta}\right\}d\tau = L^{-1}\left\{\frac{1}{s^{\beta+(k+1)\theta}}\cdot\frac{1}{(1-s^{-\theta})^{k+1}}\right\} \tag{12}$$

$$= L^{-1}\left\{\frac{1}{s^{\beta+(k+1)\theta}}\sum_{n=0}^{\infty}\binom{n+k}{k}\frac{1}{s^{n\theta}}\right\} = \sum_{n=0}^{\infty}\binom{n+k}{k}\frac{t^{n\theta+\beta+(k+1)\theta-1}}{\Gamma\big(n\theta+\beta+(k+1)\theta\big)}.$$

Let us notice now that

$$e^{-t}\sum_{n=0}^{\infty}\frac{t^{n\theta+\beta+(k+1)\theta-1}}{\Gamma\big(n\theta+\beta+(k+1)\theta\big)}\binom{n+k}{k}\sim\frac{t^k}{k!\,\theta^{k+1}}. \tag{13}$$

Indeed, it is enough to make use of the properties of the Euler Γ-function and of the fact that for any real w we have

$$\lim_{t\to\infty}\theta e^{-t}\sum_{n=0}^{\infty}\frac{t^{n\theta+w}}{\Gamma(n\theta+w+1)}s_n = s$$

provided $\lim\limits_{n\to\infty} s_n = s$ (Włodarski [14]). Since obviously $t^{k-1}/(k-1)!\sim \sim t^{k-1}/(k-1)!$, we have by lemma 4 and (11), (12), (13) $\lim\limits_{t\to\infty} W^k_{\theta,\beta}\{f;\, t\} = m$.

Case 2°: $\lim\limits_{\tau\to\infty} f(\tau) = 0$. In this case we shall make use of an estimation given by L. Włodarski ([14]) for the function $L^{-1}\left\{\frac{1}{s^\beta}e^{-\tau s^\theta}\right\}$

$$L^{-1}\left\{\frac{1}{s^\beta}e^{-\tau s^\theta}\right\} \leqslant Ae^{t/2}+B\,, \tag{14}$$

where A and B are some constants greater than zero. Estimation (14) holds for $t > 1$, $\tau > 0$, $\beta > 0$, $0 < \theta < 1$. Now let

$$(15) \qquad |f(\tau)| < \varepsilon \quad \text{for} \quad \tau > T, \quad M = \sup_{0<\tau<\infty} |f(\tau)|.$$

By (14), (15) and lemma 3, we have for $t > 1$

$$(16) \quad \left| \int_0^\infty e^\tau \frac{\tau^k}{k!} f(\tau) L^{-1}\left\{\frac{1}{s^\beta} e^{-\tau s^\theta}\right\} d\tau \right|$$
$$\leqslant \frac{T^{k+1}}{k!} e^T M(Ae^{t/2}+B) + \varepsilon \int_T^\infty e^\tau \frac{\tau^k}{k!} L^{-1}\left\{\frac{1}{s^\beta} e^{-\tau s^\theta}\right\} d\tau$$
$$\leqslant \frac{T^{k+1}}{k!} e^T M(Ae^{t/2}+B) + \varepsilon \int_0^\infty e^\tau \frac{\tau^k}{k!} L^{-1}\left\{\frac{1}{s^\beta} e^{-\tau s^\theta}\right\} d\tau .$$

Consequently

$$(17) \quad |W^k_{\theta,\beta}\{f;\, t\}|$$
$$\leqslant \frac{\theta^{k+1}(2k)!}{t^{2k}} \left[\frac{t^{k-1}}{(k-1)!} * \frac{T^{k+1}}{k!} e^T M(Ae^{-t/2}+Be^{-t}) + \varepsilon W^k_{\theta,\beta}\{e;\, t\} \right],$$

where $e(\tau) \equiv 1$.

The first term on the right side of inequality (17) tends to zero because of the regularity of the Cesàro method C_k, we have even

$$\frac{1}{t^k}\left[t^{k-1} * \frac{T^{k+1}}{k!} e^T M(Ae^{-t/2}+Be^{-t}) \right] \to 0 \qquad \text{as } t \to \infty .$$

The second term tends to ε, because of case 1°.

Hence it follows that $\lim\limits_{t\to\infty} W^k_{\theta,\beta}\{f;\, t\} = 0$.

Case 3°: $\lim\limits_{\tau\to\infty} f(\tau) = m$ (m—finite). This case may immediately be reduced to the preceding cases by considering the function $g(\tau) = f(\tau) - m$.

Case 4°: $\lim f(\tau) = +\infty$. Let $K > 0$ be given. Let $f(\tau) > K$ for $\tau > T$, $M = \sup\limits_{0<\tau<T} |f(\tau)|$. By lemma 3 and estimation (14) we have for $t > 1$

$$(18) \quad \int_0^\infty e^\tau \frac{\tau^k}{k!} f(\tau) L^{-1}\left\{\frac{1}{s^\beta} e^{-\tau s^\theta}\right\} d\tau$$
$$\geqslant K \int_T^\infty e^\tau \frac{\tau^k}{k!} L^{-1}\left\{\frac{1}{s^\beta} e^{-\tau s^\theta}\right\} d\tau - M \frac{T^{k+1}}{k!} e^T (Ae^{t/2}+B).$$

Thus we have

$$(19)\quad W^k_{\theta,\beta}\{f;t\} \geqslant KW^k_{\theta,\beta}\{e;t\} - \frac{\theta^{k+1}(2k)!}{t^{2k}}\left[\frac{t^{k-1}}{(k-1)!} * \frac{T^{k+1}}{k!}Me^{T}(Ae^{-t/2}+Be^{-t})\right].$$

The first term on the right-hand side of inequality (19) tends to K as $t\to\infty$, and the second tends to zero. Thus $W^k_{\theta,\beta}\{f(\tau)\} > K/2$ for t sufficiently large, which, since K is arbitrary, ends the proof.

Now we proceed to the proof of the following consistency theorem.

THEOREM 3. *If the series* $\sum_{n=0}^{\infty} a_n$ *is summable to* s *by the method* $B^k_{a,\gamma}$ *and it is summable to* $\hat{s}$ *by the method* $B^k_{\theta a,\delta}$, *where* $0<\theta<1$, $\delta > \theta(\gamma+k+1)-1$, *then* $s=\hat{s}$.

Proof. Let us put

$$(20)\quad \psi_n(\tau) = \frac{k!}{\tau^k}\left[\frac{\tau^{k-1}}{(k-1)!} * e^{-\tau}\frac{\tau^{na+\gamma}}{\Gamma(na+\gamma+1)}\right].$$

Let us take $\beta = \delta-\theta(\gamma+k+1)+1$. Obviously we have $\beta>0$.

$$W^k_{\theta,\beta}\{\psi_n;t\} = \frac{\theta^{k+1}(2k)!}{t^{2k}}\left[\frac{t^{k-1}}{(k-1)!} * \left(e^{-t}\int_0^{\infty} e^{\tau}\left(\frac{\tau^{k-1}}{(k-1)!} * e^{-\tau}\frac{\tau^{na+\gamma}}{\Gamma(na+\gamma+1)}\right)L^{-1}\left\{\frac{1}{s^{\beta}}e^{-\tau s^{\theta}}\right\}d\tau\right)\right].$$

Since

$$e^{\tau}\left(\frac{\tau^{k-1}}{(k-1)!} * e^{-\tau}\frac{\tau^{na+\gamma}}{\Gamma(na+\gamma+1)}\right) = \frac{e^{\tau}\tau^{k-1}}{(k-1)!} * \frac{\tau^{na+\gamma}}{\Gamma(na+\gamma+1)},$$

we receive by lemma 2 and the well-known properties of the Laplace-transformation

$$(21)\quad W^k_{\theta,\beta}\{\psi_n;t\} = \frac{\theta^{k+1}(2k)!}{t^{2k}}\left[\frac{t^{k-1}}{(k-1)!} * \left(e^{-t}L^{-1}\left\{\frac{1}{s^{\beta}}\cdot\frac{1}{(s^{\theta}-1)^k}\cdot\frac{1}{s^{n\theta a+\theta(\gamma+1)}}\right\}\right)\right]$$
$$= \frac{\theta^{k+1}(2k)!}{t^{2k}}\left[\left(\frac{t^{k-1}}{(k-1)!} * e^{-t}\frac{t^{n\theta a+\delta}}{\Gamma(n\theta a+\delta+1)}\right) * e^{-t}\sum_{\nu=0}^{\infty}\binom{\nu+k-1}{k-1}\frac{t^{\nu\theta-1}}{\Gamma(\nu\theta)}\right].$$

Let us notice now that from the summability of series $\sum_{n=0}^{\infty} a_n$ by a method $B^k_{a,\gamma}$ to s it immediately follows that the function

$$(22)\quad \Phi(t) = a\sum_{n=0}^{\infty} s_n\psi_n(t), \qquad s_n = a_0+a_1+\ldots+a_n$$

has in infinity a limit equal to s.

Let us now put

$$(23)\qquad \varphi_n(\tau) = \left(e^{\tau}\frac{\tau^{k-1}}{(k-1)!} * \frac{\tau^{n\alpha+\gamma}}{\Gamma(n\alpha+\gamma+1)} s_n\right) L^{-1}\left\{\frac{1}{s^{\beta}} e^{-\tau s^{\theta}}\right\}.$$

We shall show that the integral

$$(24)\qquad \int_0^{\infty}\Big[\sum_{n=0}^{\infty}\varphi_n(\tau)\Big]d\tau$$

exists and the equality

$$(25)\qquad \int_0^{\infty}\Big[\sum_{n=0}^{\infty}\varphi_n(\tau)\Big]d\tau = \sum_{n=0}^{\infty}\int_0^{\infty}\varphi_n(\tau)\,d\tau$$

holds.

To this end it suffices to prove that

1° the integrals $\int_0^{\infty}|\varphi_n(\tau)|\,d\tau$ $(n = 0, 1, \ldots)$ converge,

2° the series $\sum_{n=0}^{\infty}\varphi_n(\tau)$ converges uniformly in any finite interval $[0, T]$,

3° $\sum_{n=0}^{\infty}\int_0^{\infty}|\varphi_n(\tau)|\,d\tau < \infty$,

Since by lemmas 2 and 3 we have for any $t > 0$

$$\int_0^{\infty}|\varphi_n(\tau)|\,d\tau = |s_n| L^{-1}\left\{\frac{1}{(1-s^{-\theta})^k}\right\} * L^{-1}\left\{\frac{1}{s^{n\theta\alpha+\delta+1}}\right\}$$

condition 1° is satisfied.

Conditions 2° and 3° follow from the summability of the series $\sum_{n=0}^{\infty} a_n$ by the methods $B^k_{\alpha,\gamma}$ and $B^k_{\theta\alpha,\delta}$ and from some elementary properties of power series.

From (22), (23) and (25) the formula

$$(26)\qquad W^k_{\theta,\beta}\{\Phi;\, t\} = a\sum_{k=0}^{\infty} s_n W^k_{\theta,\beta}\{\psi_n;\, t\}$$

can easily be deduced.

Making use of our previous results we have

$$(27)\qquad W^k_{\theta,\beta}\{\Phi;\, t\}$$
$$= \frac{(2k)!}{t^{2k}}\left[\left(\frac{t^{k-1}}{(k-1)!} * \theta e^{-t}\sum_{n=0}^{\infty}\frac{t^{n\theta\alpha+\delta}}{\Gamma(n\theta\alpha+\delta+1)}\right) * \theta^k e^{-t}\sum_{\nu=0}^{\infty}\binom{\nu+k-1}{k-1}\frac{t^{\nu\theta-1}}{\Gamma(\nu\theta)}\right].$$

Since by hypothesis

$$\frac{t^{k-1}}{(k-1)!} * \theta a e^{-t} \sum_{n=0}^{\infty} \frac{t^{n\theta a+\delta}}{\Gamma(n\theta a+\delta+1)} s_n \sim \frac{t^k}{k!}\hat{s} \tag{28}$$

and

$$\theta^k e^{-t} \sum_{\nu=0}^{\infty} \binom{\nu+k-1}{k-1} \frac{t^{\nu\theta-1}}{\Gamma(\nu\theta)} \sim \frac{t^{k-1}}{(k-1)!} \tag{29}$$

(compare the proof of the theorem 2, case 1°), thus by lemma 4

$$\lim_{t\to\infty} W^k_{\theta,\beta}\{\Phi;\, t\} = \hat{s}\,. \tag{30}$$

In view of the regularity of the transformation $W^k_{\theta,\beta}$ we also have

$$\lim_{t\to\infty} W^k_{\theta,\beta}\{\Phi;\, t\} = s \tag{31}$$

and thus $\hat{s} = s$, which ends the proof.

From theorems 1 and 3 and the fundamental properties of the Cesàro methods the following general consistency theorem follows.

THEOREM 4. *If a series is summable to s by the method $B^{r_1}_{a_1,\gamma_1}$ and if it is summable to $\hat{s}$ by the method $B^{r_2}_{a_2,\gamma_2}$, where $a_i > 0$, $\gamma_i \geqslant 0$, $r_i \geqslant 0$ $(i = 1, 2)$, then $s = \hat{s}$.*

REMARK. Theorem 4 enables us to consider a three-parameter family of methods $B^r_{a,\gamma}$ as one method of summability of type (B); namely a series is said to be summable by the method (B) if it is summable by one of the methods $B^r_{a,\gamma}$. Theorem 4 guarantees the unambiguity of such a definition.

In the sequel we shall use the following lemmas.

LEMMA 5. *If a continuous function $f(t)$ is limitable by a method C_r, $0 \leqslant r \leqslant 1$, then its Laplace transform*

$$F^*(s) = \int_0^{\infty} e^{-st} f(t)\,dt \tag{32}$$

is defined for $R(s) > 0$, the limit $\lim\limits_{s\to 0+} sF^*(s)$ exists and the equality

$$\lim_{s\to 0+} sF^*(s) = C_r\text{-}\lim_{t\to\infty} f(t) \tag{33}$$

holds.

LEMMA 6. *If a continuous function $f(t)$ is limitable by a method C_r, $r > 1$, and if Laplace transform* (32) *exists for $R(s) > 0$, then the limit $\lim\limits_{s\to 0+} sF^*(s)$ exists and equality* (33) *holds.* (G. Doetsch [4], p. 462).

THEOREM 5. *If a series* $\sum_{n=0}^{\infty} a_n$ *is summable to* σ *by a method* $B_{a,\gamma}^{r}$ $(0 \leqslant r \leqslant 1)$ *and its transform by the Abel method exists,*

$$A(t) = \sum_{n=0}^{\infty} a_n t^n, \quad |t| < 1 \tag{34}$$

then the series $\sum_{n=0}^{\infty} a_n$ *is summable by the Abel method to* σ, *i.e.*

$$\lim_{t \to 1-} A(t) = \sigma . \tag{35}$$

Proof. By our hypothesis the function

$$f(t) = ae^{-t} \sum_{n=0}^{\infty} \frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)} s_n, \quad s_n = a_0 + a_1 + \ldots + a_n$$

is defined for any $t \geqslant 0$ and the generalized limit $C_r - \lim_{t \to \infty} f(t) = \sigma$ exists. By lemma 5 transform (32) exists for $R(s) > 0$. Further we have

$$sF^*(s) = as \sum_{n=0}^{\infty} \frac{s_n}{(s+1)^{na+\gamma+1}} = \frac{as}{(s+1)^{\gamma+1}} \sum_{n=0}^{\infty} \frac{s_n}{(s+1)^{na}} . \tag{36}$$

On substitution $t = 1/(s+1)^a$ we receive

$$sF^*(s) = b(t) \sum_{n=0}^{\infty} a_n t^n, \quad \text{where} \quad b(t) \to 1 \quad \text{as} \quad t \to 1- . \tag{37}$$

By lemma 5 the left-hand side of formula (37) tends to σ as $s \to 0+$, and thus the right tends to σ as $t \to 1-$, which implies (35).

Making use of lemma 6 in an analogous way the following theorem may be proved.

THEOREM 6. *If a series* $\sum_{n=0}^{\infty} a_n$ *is summable to* σ *by a method* $B_{a,\gamma}^{r}$ $(r > 1)$, *the integral*

$$\int_0^{\infty} e^{-(s+1)t} \sum_{n=0}^{\infty} \frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)} s_n \, dt$$

converges for $R(s) > 0$ *and transform* (34) *exists, then the series is summable to* σ *by the Abel method.*

Since for bound series Abel transform (34) always exists, and since the integral

$$\int_0^\infty e^{-(s+1)t} \sum_{n=0}^\infty \frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)} |s_n| \, dt$$

is finite for $R(s) > 0$, the following corollary follows from theorem 6.

COROLLARY. *If a bounded series is summable by the method $B^r_{a,\gamma}$ for some $r \geqslant 0$, then it is also summable by the Abel method and consequently by first means method.*

DEFINITIONS 6. A summability method is said to be *translative to the right* if the summability of the series $a_0 + a_1 + a_2 + \dots$ implies the summability of the series $0 + a_0 + a_1 + \dots$

THEOREM 7. *The methods $B^r_{a,\gamma}$ are translative to the right.*

Proof. Let

$$C_r - \lim_{t \to \infty} a e^{-t} \sum_{n=0}^\infty \frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)} s_n = s \,, \qquad s_n = a_0 + a_1 + \dots + a_n \,.$$

We shall put

$$\hat{s}_n = \begin{cases} 0 & \text{for} \quad n = 0 \,, \\ s_{n-1} & \text{for} \quad n \geqslant 1 \,. \end{cases} \tag{38}$$

By hypothesis we have

$$f(t) = \frac{t^{r-1}}{\Gamma(r)} * a e^{-t} \sum_{n=0}^\infty \frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)} s_n \sim \frac{t^r}{\Gamma(r+1)} s \,. \tag{39}$$

Let $\varphi(t) = e^{-t} t^{a-1} / \Gamma(a)$. Thus we have $\varphi(t) \geqslant 0$, $\int_0^\infty \varphi(\tau) d\tau = 1$. By lemma 1 we have

$$\varphi(t) * f(t) \sim \frac{t^r}{\Gamma(r+1)} s \,. \tag{40}$$

Moreover, it is easily seen that

$$\begin{aligned} \varphi(t) * f(t) &= a e^{-t} \sum_{n=0}^\infty \frac{t^{na+a+\gamma}}{\Gamma(na+\gamma+a+1)} s_n * \frac{t^{r-1}}{\Gamma(r)} \\ &= \left(a e^{-t} \sum_{n=0}^\infty \frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)} \hat{s}_n + a a_0 e^{-t} \frac{t^{a+\gamma}}{\Gamma(a+\gamma+1)} \right) * \frac{t^{r-1}}{\Gamma(r)} \,. \end{aligned} \tag{41}$$

Since evidently

$$\frac{r}{t^r} \left[t^{r-1} * a a_0 e^{-t} \frac{t^{a+\gamma}}{\Gamma(a+\gamma+1)} \right] \to 0 \quad \text{as} \quad t \to \infty \,,$$

by (40) and (41) we have

$$C_r - \lim_{t\to\infty} ae^{-t} \sum_{n=0}^{\infty} \frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)} \hat{s}_n = s\,.$$

§ 2. Theorems on the multiplication of series. In the sequel we shall use the following result of G. Doetsch ([3], p. 33).

LEMMA 7. *Let $f(t)$ and $g(t)$ and their first derivatives be continuous functions, $f(0)=0$, $r_1, r_2 \geqslant 0$. If $C_{r_1} - \lim_{t\to\infty} f(t) = A$ and $C_{r_2} - \lim_{x\to\infty} \int_0^x g(t)\,dt = B$, then $C_{r_1+r_2+1} - \lim \big(f(t) * g(t)\big)$ exists and is equal to AB.*

THEOREM 8. *If $\sum_{n=0}^{\infty} a_n = A(B_{a,\gamma_1}^{r_1})$, $\sum_{n=0}^{\infty} b_n = B(\widetilde{B}_{a,\gamma_2}^{r_2})$, then*

$$\sum_{n=0}^{\infty} c_n = AB(B_{a,\gamma_1+\gamma_2+1}^{r_1+r_2+1})\ (^2)\,.$$

Proof. In view of the additivity and regularity of the methods $B_{a,\gamma}^{r}$ we may assume, without restricting the generality of our considerations that $a_0 = 0$. Then the functions

$$f(t) = ae^{-t} \sum_{n=0}^{\infty} \frac{t^{na+\gamma_1}}{\Gamma(na+\gamma_1+1)} s_n\,, \qquad s_n = a_0 + a_1 + \ldots + a_n$$

and

$$g(t) = ae^{-t} \sum_{n=0}^{\infty} \frac{t^{na+\gamma_2}}{\Gamma(na+\gamma_2+1)} b_n$$

satisfy the conditions of lemma 7.

Moreover, by the properties of the convolution and the elementary properties of power series we have

$$f(t) * g(t) = ae^{-t} \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} s_{n-\nu} b_\nu \left(\frac{t^{(n-\nu)a+\gamma_1}}{\Gamma\big((n-\nu)a+\gamma_1+1\big)} * \frac{t^{\nu a+\gamma_2}}{\Gamma(\nu a+\gamma_2+1)}\right)$$

$$= ae^{-t} \sum_{n=0}^{\infty} \frac{t^{na+\gamma_1+\gamma_2+1}}{\Gamma(na+\gamma_1+\gamma_2+2)} (c_0 + c_1 + \ldots + c_n)\,.$$

Whence by lemma 7 it immediately follows that $\sum_{n=0}^{\infty} c_n = AB(B_{a,\gamma_1+\gamma_2+1}^{r_1+r_2+1})$.

(2) By $\sum_{n=0}^{\infty} c_n$ we shall denote the Cauchy product of series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, i.e. $c_n = \sum_{\nu=0}^{n} a_{n-\nu} b_\nu$.

THEOREM 9. *If a series* $\sum_{n=0}^{\infty} a_n$ *is summable to* A *by the method* $\tilde{B}^r_{a,\gamma}$ *then it is also summable to* A *by the method* $B^{r+1}_{a,\gamma+1}$.

Proof. This theorem is an immediate corollary from theorem 8. In fact, consider the series $\sum_{n=0}^{\infty} b_n$, in which $b_0 = 1$, $b_n = 0$ for $n \geqslant 1$. It follows from the regularity of the method $B^0_{a,0}$ that $\sum_{n=0}^{\infty} b_n = 1\,(B^0_{a,0})$.

Thus by theorem 8 we have

$$\sum_{n=0}^{\infty} c_n = A\,(B^{r+1}_{a,\gamma+1})\,.$$

But in our case $c_n = a_n$, which ends the proof.

The following general consistency theorem is an immediate corollary from theorems 4 and 9.

THEOREM 10. *If* $\sum_{n=0}^{\infty} a_n = s(\tilde{B}^{r_1}_{a_1,\gamma_1})$, $\sum_{n=0}^{\infty} a_n = \hat{s}(\tilde{B}^{r_2}_{a_2,\gamma_2})$, *then* $s = \hat{s}$.

THEOREM 11. *If a series* $\sum_{n=0}^{\infty} a_n$ *is summable to* s *by the method* $B^r_{a,\gamma}$, *then it is also summable to* s *by the method* $\tilde{B}^r_{a,\gamma}$.

Proof. D. Borwein has proved this theorem for $r = 0$. Let us assume then that $r > 0$. It may be easily verified that, if $\sum_{n=0}^{\infty} a_n t^{na+\gamma}/\Gamma(na+\gamma+1)$ is an integral function, then $\sum_{n=0}^{\infty} s_n t^{na+\gamma}/\Gamma(na+\gamma+1)$ is an integral function. The converse assertion is also true.

We have the following relation [3]

$$(42)\qquad \int_0^t e^{-\tau} \sum_{n=0}^{\infty} \frac{\tau^{na+\gamma}}{\Gamma(na+\gamma+1)} a_n d\tau$$
$$= \left(ae^{-t} \sum_{n=0}^{\infty} \frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)} s_n\right) * \left(\frac{1}{\Gamma(a+1)} \int_t^{\infty} \tau^{a-1} e^{-\tau} d\tau\right).$$

Let us notice that

$$\lim_{x\to\infty} \int_0^x dt \int_t^{\infty} \tau^{a-1} e^{-\tau} d\tau = \Gamma(a+1)\,.$$

Let us multiply in the sense of the convolution both sides of (42) by $t^{r-1}/\Gamma(r)$. Then, making use of lemma 1 and also of the properties of the

[3] The proof of a relation analogous to (42) is to be found in D. Bornwein's paper [1].

convolution, we have

$$\frac{t^{r-1}}{\Gamma(r)} * \int_0^t e^{-\tau} \sum_{n=0}^{\infty} \frac{\tau^{na+\gamma}}{\Gamma(na+\gamma+1)} a_n d\tau$$

$$= \left(\frac{t^{r-1}}{\Gamma(r)} * ae^{-t} \sum_{n=0}^{\infty} \frac{t^{na+\gamma}}{\Gamma(na+\gamma+1)} s_n\right) * \left(\frac{1}{\Gamma(a+1)} \int_t^{\infty} \tau^{a-1} e^{-\tau} d\tau\right) \sim \frac{st^r}{\Gamma(r+1)},$$

which means that the series is summable to s by the method $\widetilde{B}^r_{a,\gamma}$.

It follows from theorems 9 and 11 that the summability of a series by one of the methods $B^r_{a,\gamma}$ implies its summability by some method $\widetilde{B}^r_{a,\gamma}$ and conversely. In this sense the range of all integral methods $\widetilde{B}^r_{a,\gamma}$ and exponential methods $B^r_{a,\gamma}$ is the same.

In theorems 8 and 9 the indices γ and r behave similarly. It is quite natural to ask whether the index γ may also be regarded as an "exponent" of the superposition with the integral mean. We shall show that in fact this is the case.

DEFINITION 7. The function $h(t)$ given by formula

$$h(t) = \int_1^t f\left(\frac{t}{\tau}\right) g(\tau) \frac{d\tau}{\tau} \tag{43}$$

is called the *convolution of the two functions $f(t)$ and $g(t)$ in the sense of Dirichlet* (4), which shall briefly be written

$$h(t) = f(t) \wedge g(t). \tag{44}$$

The Dirichlet convolution, similarly as the Cauchy convolution, is commutative, associative, distributive with respect to addition and homogeneous with respect to either factors. There is a close relation between these two convolutions. Namely we have

$$f(t) \wedge g(t) = F(s) * G(s), \tag{45}$$

where $F(\ln x) = f(x)$, $G(\ln x) = g(x)$, $s = \ln t$.

The following formula which will be used in the sequel, holds for $a > -1$, $b > -1$

$$t^c \frac{(\ln t)^a}{\Gamma(a+1)} \wedge t^c \frac{(\ln t)^b}{\Gamma(b+1)} = t^c \frac{(\ln t)^{a+b+1}}{\Gamma(a+b+2)} \text{ (5)}. \tag{46}$$

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(4) By analogy to the multiplication of series, the convolution of the two functions $f(t)$ and $g(t)$ given by $\int_0^t f(t-\tau) g(\tau) d\tau$, will be called consequently the *Cauchy convolution*.

(5) Formula (46) can easily be obtained from a well-known relation between the Euler functions β and Γ.

Let us notice now that

$$\frac{1}{t}\wedge g(t) = \frac{1}{t}\int_1^t g(\tau)\,d\tau\,. \tag{47}$$

Thus multiplying a function $g(t)$ by $1/t$ in the sense of the Dirichlet convolution we receive the first means method transform of the function $g(t)$ (our argument is obviously not influenced by considering the integral $\int_1^t$ instead of $\int_0^t$). Iterating k-times operation (47) we receive the transform of the k-th Hölder mean.

In view of the associativity of the Dirichlet convolution, this transform may briefly be written in the form

$$(1/t)^k_\wedge \wedge g(t) \tag{48}$$

where $(1/t)^k_\wedge$ denotes the k-th power in the sense of the convolution of the function $1/t$. It may easily be proved by the method of induction that

$$(1/t)^k_\wedge = (\ln t)^{k-1}/(k-1)!t\,,\qquad k = 1, 2, \ldots \tag{49}$$

Writting

$$(1/t)^{\beta+1}_\wedge \stackrel{\text{def}}{=} (\ln t)^\beta/\Gamma(\beta+1)t\,,\qquad \beta > -1 \tag{50}$$

we generalize formula (49) for real exponents, similarly as in Mikusiński's operational calculus (compare [8], p. 59). Formula (50) enables us to extend in a natural manner the Hölder methods for functions to non-integral indices. The Hölder transform of order $\beta > 0$ for a function $g(t)$ is defined by the formula

$$(1/t)^\beta_\wedge \wedge g(t)\,. \tag{51}$$

If $\lim_{t\to\infty}[(1/t)^\beta_\wedge \wedge g(t)] = l$, then the function $g(t)$ is said to be limitable to l by the method H_β.

The composition of a method H_a with a method H_β gives the method $H_{a+\beta}$. This follows immediately from the properties of the convolution and from formula (46). The H_a methods are equivalent to the Cesàro methods C_a for all $a > 0$ (K. Knopp [7]) (6).

(6) As a marginal note let us mention that the transform of a function $g(t)$: $C_a(t) = \frac{a}{t^a}\int_1^t (t-\tau)^{a-1} g(\tau)\,d\tau$, equivalent to the Cesàro transform of order a, may be expressed in the form of the Dirichlet convolution. Namely we have $C_a(t) = \left[\frac{a}{t}\left(1-\frac{1}{t}\right)^{a-1}\right]\wedge g(t)$.

Now returning to the Borel methods we shall notice that the transform

$$\hat{s}_{a,\gamma}(t) = \frac{a}{t}\sum_{n=0}^{\infty}\frac{(\ln t)^{na+\gamma}}{\Gamma(na+\gamma+1)}s_n$$

is equivalent to transform (1) and that

$$(1/t)^{\beta}_{\wedge} \wedge \hat{s}_{a,\gamma}(t) = \hat{s}_{a,\gamma+\beta}(t)\,. \tag{52}$$

We may thus say that the method $B_{a,\gamma+\beta}$ is the result of composition of the method $B_{a,\gamma}$ with the Hölder mean of order β "on the level" $\ln t$. Consequently, the index γ in the Borel methods, similarly to the index r in $B^{r}_{a,\gamma}$ may be regarded as the "exponent" of superposition with the integral mean.

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Reçu par la Rédaction le 5. 7. 1961