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On the existence of a solution of a system $\dot{x} = f(t, x)$ which remains in a given set

by S. P. Hastings (Cleveland, Ohio)

Let $a(t, x, y)$ be continuous for all (t, x, y) and consider the equation

$$\ddot{x} = a(t, x, \dot{x}) . \tag{1}$$

In [6] Z. Opial considered the existence of a solution $x(\cdot)$ of (1) which is bounded on $(-\infty, \infty)$. Other authors (see for example [1], [2]) also discussed bounded solutions of (1), but the strongest results on existence seem to be those of Opial.

In this note we shall discuss the existence of at least one solution of a first order system

$$\dot{x} = f(t, x), \qquad x = (x^1, \ldots, x^n) \tag{2}$$

which remains in a given domain on its entire maximal interval of existence. Our discussion is based on a known result from topology (Lemma 1 below) and our results are in the spirit of Ważewski's topological approach to similar questions. By applying our technique to second order equations we can obtain some of Opial's results a little more easily than he did, but his strongest theorems require at least as much work with our methods as with his. On the other hand, we feel that our technique can more easily be extended to higher order equations and systems.

Our basic lemma, found in its dual form involving closed sets in [4], p. 137, is the following:

Lemma 1. *Let M_1 and M_2 be open subsets of m-dimensional Euclidean space E^m, with $m \geqslant 2$, and let N_1 and N_2 be (connected) components of M_1 and M_2 respectively, with $N_1 \cap N_2$ not connected. Then $M_1 \cup M_2 \neq E^m$.*

We shall consider the system (2), where x and $f(t, x)$ are real n-dimensional vectors, $n \geqslant 2$, and $f(t, x)$ is continuous in an open subset Ω of E^{n+1}. We assume that through each point (t_1, x_1) of Ω there passes a unique solution $x(\cdot, t_1, x_1)$ of (2), so that $x(t, t_1, x_1)$ is continuous in (t, t_1, x_1). Let Ω^0 be an open connected set with $\overline{\Omega^0} \subset \Omega$. ($\overline{\Omega^0} =$ closure of Ω^0.) Following [3] we shall call a point $(t_0, x_0) \in \partial\Omega^0 = \overline{\Omega^0} - \Omega^0$ an egress (ingress) point of Ω^0 if there exists an $\varepsilon > 0$ such that $(t, x(t, t_0, x_0)) \in \Omega^0$ for $t_0 - \varepsilon < t < t_0$ $(t_0 < t < t_0 + \varepsilon)$. If we can choose ε so that in addition $(t, x(t, t_0, x_0)) \notin \overline{\Omega^0}$ for $t_0 < t < t_0 + \varepsilon$ $(t_0 - \varepsilon < t < t_0)$, then (t_0, x_0) is called a strict egress (strict ingress) point of Ω^0. Denote by Ω_e^0 (Ω_i^0) the set of egress (ingress) points of Ω^0, and by Ω_{se}^0 (Ω_{si}^0) the set of strict egress (strict ingress) points of Ω^0. If, for some (t, t_1, x_1), $(t, x(t, t_1, x_1)) \in \Omega_e^0$ $(\Omega_{se}^0$, etc.), we say that $x(\cdot, t_1, x_1)$ meets Ω_e^0 $(\Omega_{se}^0$, etc.) at $(t, x(t, t_1, x_1))$.

Let S be a subset of Ω^0 which is homeomorphic to E^m for some m, $2 \leqslant m \leqslant n$. Let

$$F = \{(t_1, x_1) \in S \mid (t, x(t, t_1, x_1)) \in \partial\Omega^0 \text{ for some } t > t_1\} ,$$

$$G = \{(t_1, x_1) \in S \mid (t, x(t, t_1, x_1)) \in \partial\Omega^0 \text{ for some } t < t_1\} .$$

Since $\partial\Omega^0$ is closed, we can define maps $\pi_+ \colon F \to \partial\Omega^0$ and $\pi_- \colon G \to \partial\Omega^0$ by setting $\pi_+(t_1, x_1)$ equal to the first point $(t, x(t, t_1, x_1))$, $t > t_1$, after t_1 at which $x(\cdot, t_1, x_1)$ meets $\partial\Omega^0$ and $\pi_-(t_1, x_1)$ equal to the last point $(t, x(t, t_1, x_1))$, $t < t_1$, before t_1 at which $x(\cdot, t_1, x_1)$ meets $\partial\Omega^0$. Then $\pi_+ \colon F \to \Omega_e^0$ and $\pi_- \colon G \to \Omega_i^0$.

THEOREM 1. *Suppose that the image sets $\pi_+(F)$ and $\pi_-(G)$ are each disconnected:*

$$\pi_+(F) = Q_1 \cup Q_2, \quad \bar{Q}_1 \cap Q_2 = Q_1 \cap \bar{Q}_2 = \emptyset, \quad Q_1 \neq \emptyset, \quad Q_2 \neq \emptyset ,$$
$$\pi_-(G) = R_1 \cup R_2, \quad \bar{R}_1 \cap R_2 = R_1 \cap \bar{R}_2 = \emptyset, \quad R_1 \neq \emptyset, \quad R_2 \neq \emptyset .$$

Suppose also that the inverse images $\pi_+^{-1}(Q_1)$, $\pi_+^{-1}(Q_2)$, $\pi_-^{-1}(R_1)$, and $\pi_-^{-1}(R_2)$ have components S_1, S_2, T_1, and T_2, respectively, such that $T_i \cap S_j \neq \emptyset$, $i, j = 1, 2$.

Under these conditions, if $\pi_+(F) \subset \Omega_{se}^0$ and $\pi_-(G) \subset \Omega_{si}^0$, then there is a point $(t_1, x_1) \in S$ such that $(t, x(t, t_1, x_1)) \in \Omega^0$ on the whole maximal interval of existence of the solution $x(\cdot, t_1, x_1)$.

Proof. The theorem follows from Lemma 1 once we show that the sets $\pi_+^{-1}(Q_i)$ and $\pi_-^{-1}(R_i)$, $i = 1, 2$, are open relative to S. Suppose, for example, that $(\tau, \zeta) \in \pi_+^{-1}(Q_1)$. Let $\pi_+(\tau, \zeta) = (t_0, x(t_0, x(t_0, \tau, \zeta)) \in \Omega_{se}^0$ and pick $t_1 > t_0$ such that $(t, x(t, \tau, \zeta)) \notin \overline{\Omega^0}$ for $t_0 < t \leqslant t_1$. Let $\delta = \min_{\tau \leqslant s \leqslant t_1} |(s, x(s, \tau, \zeta)) - Q_2|$, where we use any of the usual metrics on E^{n+1} and the corresponding definition of the distance between a point and a set. Then $\delta > 0$ and we can find a neighborhood N of (τ, ζ) such that if $(\tau_1, \zeta_1) \in N$, then

$$|(s, x(s, \tau_1, \zeta_1)) - (s, x(s, \tau, \zeta))| < \min\{\delta/2, |(t_1, x(t_1, \tau, \zeta)) - \Omega^0|\}$$

for $\tau_1 \leqslant s \leqslant t_1$. This implies that $N \cap S \subset \pi_+^{-1}(Q_1)$, so $\pi_+^{-1}(Q_1)$ is open relative to S.

To apply Lemma 1, let $M_i = \pi_+^{-1}(Q_i) \cup \pi_-^{-1}(R_i)$ and let N_i be the component M_i which contains $S_i \cup T_i$, $i = 1, 2$. Then $S_1 \cap T_2$ and $S_2 \cap T_1$ will be distinct components of $N_1 \cap N_2$ because $\pi_+^{-1}(Q_1) \cap \pi_+^{-1}(Q_2)$ and $\pi_-^{-1}(R_1) \cap \pi_-^{-1}(R_2)$ are empty. By Lemma 1, $S-(M_1 \cup M_2) = S-(F \cup G)$ is not empty, and this proves the theorem.

In order to apply this theorem to a particular problem it is necessary to choose an appropriate Ω^0. Of course, one of the difficulties in this choice is the determination of the set of (strict) egress points and (strict) ingress points. We refer the reader to a paper [5] by Onuchic for a useful set of criteria for this purpose.

To illustrate an application of Theorem 1 to second order systems we consider the equations

$$\begin{aligned} \dot{x} &= f(t, x, y)\,, \\ \dot{y} &= g(t, x, y)\,, \end{aligned} \tag{2}$$

where we assume that f and g are C^1 in t, x, and y for all (t, x, y). In addition, suppose that the following conditions hold:

(i) There exist a and b, with $a < b$, such that

$$\frac{\partial f}{\partial t}(t, a, y) + g(t, a, y)\frac{\partial f}{\partial y}(t, a, y) < 0 \quad \text{whenever} \quad f(t, a, y) = 0$$

and

$$\frac{\partial f}{\partial t}(t, b, y) + g(t, b, y)\frac{\partial f}{\partial y}(t, b, y) > 0 \quad \text{whenever} \quad f(t, b, y) = 0\,.$$

(ii) There exist y_a and y_b such that $f(0, a, y)(y-y_a) > 0$ when $y \neq y_a$ and $f(0, b, y)(y-y_b) > 0$ when $y \neq y_b$.

(iii) For some $T > 0$, $\lim\limits_{y\to\infty} f(t, x, y) = -\lim\limits_{y\to-\infty} f(t, x, y) = \infty$ uniformly for $a \leqslant x \leqslant b$, $|t| \leqslant T$, and $g(t, x, y)$ is bounded for $|t| \leqslant T$, $a \leqslant x \leqslant b$, $-\infty < y < \infty$.

THEOREM 2. *Under the above hypotheses the system* (2) *has at least one solution* $(x(\cdot), y(\cdot))$ *such that on its maximal interval of existence,* $a < x(t) < b$.

Proof. We set $\Omega^0 = \{(t, x, y) | a < x < b\}$ so that $\partial\Omega^0 = P_a \cup P_b$, where P_a is the plane $x = a$ and P_b is the plane $x = b$. Then by (i), $\Omega^0_e = \Omega^0_{se}$ and $\Omega^0_i = \Omega^0_{si}$, and by (iii), P_a and P_b each contain points of both Ω^0_e and Ω^0_i. Let $S = \Omega^0 \cap \{(t, x, y) | t = 0\}$. Also, in the terminology of Theorem 1, let $Q_1 = \pi_+(F) \cap P_b$, $Q_2 = \pi_+(F) \cap P_a$, $R_1 = \pi_-(G) \cap P_b$, and $R_2 = \pi_-(G) \cap P_a$.

From (iii) we see that for sufficiently large y and $a < x < b$, $\pi_+(0, x, y) \in P_b$ and $\pi_-(0, x, y) \in P_a$, while for sufficiently large values of $(-y)$, and $a < x < b$, $\pi_+(0, x, y) \in P_a$ and $\pi_-(0, x, y) \in P_b$. This shows that there exists $M > 0$ such that $y > M$ and $a < x < b \Rightarrow (0, x, y) \in \pi_+^{-1}(Q_1) \cap \pi_-^{-1}(R_2)$, while $y < -M$ and $a < x < b \Rightarrow (0, x, y) \in \pi_+^{-1}(Q_2) \cap \pi_-^{-1}(R_1)$.

On the other hand, it follows from (ii) and (iii) that for any $y > y_b$ there is an $\varepsilon(y)$ such that $b - \varepsilon(y) < x < b \Rightarrow \pi_+(0, x, y) \in P_b$. Furthermore, (i) and (iii) imply that for $x < b$ and $b - x$ sufficiently small, $\pi_+(0, x, y_b) \in P_b$. From this it can be shown that there is a semi-infinite "strip" ω in S which is bounded above by the line $x = b$ and which intersects the line $y = y_b$ such that $(0, x, y) \in \omega \Rightarrow \pi_+(0, x, y) \in P_b$. We let S_1 be the component of $\pi_+^{-1}(Q_1)$ which contains this strip. Similarly we define S_2, T_1, and T_2 and the above results show that the hypotheses of Theorem 1 are satisfied, thus proving Theorem 2. (See Fig. 1.)

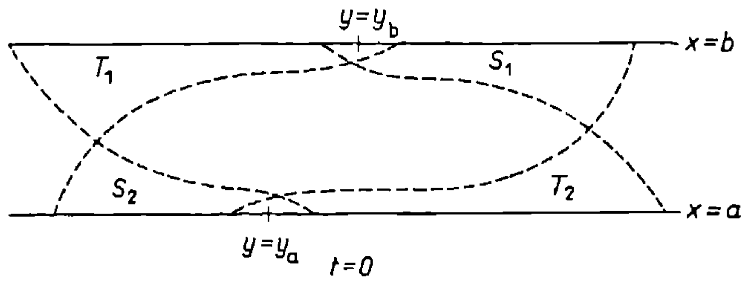

Fig. 1

We remark that, as pointed out by Opial, the strict inequalities in hypothesis (i) of Theorem 2 can be replaced by weak inequalities and the regularity conditions of f and g can be weakened. In fact, if the systems

$$(3-n) \qquad \begin{aligned} \dot{x} &= f_n(t, x, y), \\ \dot{y} &= g_n(t, x, y) \end{aligned}$$

satisfy the hypotheses of Theorem 2 and $f_n \to f$, $g_n \to g$ uniformly in every bounded subset of E^3, then there will be a sequence of solutions of $(3-n)$ which converges to a solution of (2) having the desired boundedness property.

The same result can also be obtained by modifying Theorem 1 slightly. To do this, we first define an egression (ingression) point of Ω^0 to be an egress (ingress) point $(t, x) \in \partial\Omega^0$ such that there is a $\bar{t} > t$ $(\bar{t} < t)$ with $(\bar{t}, x(\bar{t}, t, x)) \notin \bar{\Omega}^0$ and $(s, x(s, t, x)) \notin \Omega^0$ for $t \leqslant s \leqslant \bar{t}$ $(\bar{t} \leqslant s \leqslant t)$. Denote the set of egression (ingression) points by Ω^0_{E} (Ω^0_{I}). In Theorem 1, then, the condition that $\pi_+(F) \subset \Omega^0_{se}$ and $\pi_-(G) \subset \Omega^0_{si}$ can be replaced by the following two conditions:

(a) $\pi_+(F) \subset \Omega^0_{\mathrm{E}}$, $\pi_-(G) \subset \Omega^0_{\mathrm{I}}$,

(b) $\partial\Omega^0$ is disconnected, with Q_1 in a different component of $\partial\Omega^0$ from Q_2 and R_1 in a different component of $\partial\Omega^0$ from R_2.

To see this, we note that if $\pi_+(\tau, \zeta) \in Q_1$, then (a) and (b) allow us to show that for some $\bar{t} > \tau$, $(\bar{t}, x(\bar{t}, \tau, \zeta)) \notin \bar{\Omega}^0$ and $|Q_2 - (s, x(s, \tau, \zeta))| > 0$ for $\tau \leqslant s \leqslant \bar{t}$. Using this we can show as above that $\pi_+^{-1}(Q_1)$ is open relative to S. Similarly, $\pi_+^{-1}(Q_2)$, $\pi_-^{-1}(R_1)$, and $\pi_-^{-1}(R_2)$ are open relative to S, and the proof proceeds as before.

References

[1] M. M. Belova, *On bounded solutions of non-linear second-order differential equations*, Mat. Sb. (N. S.) 56 (98) (1962), pp. 469-503.

[2] C. Corduneanu, *Quelques Problèmes Globaux Concernant les Equations Différentielles Non-linéaires du Second Ordre*, Acad. R. P. Romîne. Fil. Iași. Stud Cerc. Şti. Mat. 7 (1956), pp. 1-7.

[3] P. Hartman, *Ordinary Differential Equations*, New York 1964.

[4] M. H. A. Newman, *Elements of the Topology of Plane Sets of Points*, Second Edition, Cambridge 1961.

[5] N. Onuchic, *Applications of the topological method of Ważewski to certain problems of asymptotic behaviour in ordinary differential equations*, Pacific J. Math. 11 (1961), pp. 1511-1527.

[6] Z. Opial, *Sur les intégrales bornées de l'équation $u'' = f(t, u, u')$*, Ann. Polon. Math. 4 (1958), pp. 314-324.

CASE INSTITUTE OF TECHNOLOGY
CLEVELAND, OHIO

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