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On a case of mixed inequalities between solutions of first order partial differential equations

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In part IV of paper [1] we considered the mutual position of two solutions $u(x, y_1, \ldots, y_n)$ and $v(x, y_1, \ldots, y_n)$ of the partial differential equation

$$\frac{\partial z}{\partial x} = f\left(x, y_1, \ldots, y_n, z, \frac{\partial z}{\partial y_1}, \ldots, \frac{\partial z}{\partial y_n}\right). \tag{1}$$

The solutions were supposed to be defined in the set D determined by the inequalities

$$|x-x_0| < a\,, \quad |y_i - y_i^0| \leqslant a_i - M|x-x_0|\,, \quad i = 1, \ldots, n\,, \tag{2}$$

where

$$a > 0\,, \quad M > 0\,, \quad a_i > 0\,, \quad a < a_i/M\,.$$

Let $\bar{G}$ be the intersection of D with the plane $x = x_0$. In Theorem 5 of [1] it is assumed that $u \underset{(<)}{>} v$ on a part of $\bar{G}$ while $u = v$ on the remaining part. Thus, in Theorem 5 of [1], the inequality $u \underset{(\leqslant)}{\geqslant} v$ is assumed to hold on the whole domain $\bar{G}$.

In the present paper we extend the Theorem 5 of [1] to the case of mixed inequalities assuming that $\bar{G}$ is divided into three parts such that $u \underset{(<)}{>} v$ on one part, $u \underset{(>)}{<} v$ on the second part and $u = v$ on the remaining part of $\bar{G}$. The first two parts are assumed to be partitioned off from each other by the part where $u = v$.

We shall only deal with the case where together with the relation $u = v$ also the relations $u_{y_i} = v_{y_i}$ $(i = 1, \ldots, n)$ are satisfied in the same part of $\bar{G}$.

Here as in Theorem 5 of [1] we suppose that the solutions u and v are generated by characteristics according to the definition given by J. Szarski ([5], p. 2-3).

In contrast to the assumption (δ) in [1] (p. 322) we do not assume in the present paper that the curves C which are the projections of the characteristic on the plane $x, y_1, \ldots, y_n$ reach the boundary of a domain D_1

containing a closed domain $\bar{\Delta}$ (specified in assumptions (β) of Theorem 1 of the present paper as well as in Theorem 5 of [1]).

This property follows from Theorem 2 of paper [2].

Theorem 2 of [2] and a lemma from which the theorem follows immediately will be stated without proof in the first part of the proof of the following main

THEOREM 1. *We make the following assumptions:*

(α) *The function $f(x, y_1, \ldots, y_n, z, q_1, \ldots, q_n)$ and its first derivatives with respect to $y_1, \ldots, y_n, z, q_1, \ldots, q_n$ are continuous in a domain $\mathcal{M}_1$ (of the $(2n+2)$-dimensional space) whose projection on the $(n+1)$-space $x, y_1, \ldots, y_n$ covers the set D defined by inequalities (2). The derivatives f_{y_i}, f_z, f_{q_i} satisfy in $\mathcal{M}_1$ a Lipschitz condition with respect to $y_1, \ldots, y_n, z, q_1, \ldots, q_n$ and the inequalities*

$$|f_{q_i}| < M, \quad i = 1, \ldots, n,$$

are fulfilled.

The solutions $u(x, y_1, \ldots, y_n)$ and $v(x, y_1, \ldots, y_n)$ are of class C^1 (it would be sufficient to assume that these solutions possess Stotz's differentials on the side surface of D and the continuous derivatives u_{y_k}, v_{y_k} $(k = 1, \ldots, n)$ in the interior of D) and their elements of contact belong to $\mathcal{M}_1$. Moreover, the solutions u and v are generated by characteristics.

(β) *$\bar{\Delta}$ will stand for the closed subdomain of D for which $|x - x_0| \leqslant c$, where $0 < c < a$, and $\bar{G}_0$ for the intersection of $\bar{\Delta}$ with the plane $x = x_0$.*

(γ) *Let G_1 and G_2 denote n-dimensional domains such that $\bar{G}_1 \subset \bar{G}_0$, $G_2 \subset G_1$ and let B_0, B_1, B_2 be the boundaries of G_0, G_1 and G_2 respectively.*

(δ) *Denote by P a point lying on a characteristic generating the solution u or v. Let C denote the curve in $\bar{\Delta}$ which is the projection on the plane $x, y_1, \ldots, y_n$ of the characteristic issuing from such a point $P \in \mathcal{M}_1$ that the projection Q of that point belongs to $\bar{\Delta}$.*

Under these assumptions we have:

1° *If*

$$u(x_0, y_1, \ldots, y_n) = v(x_0, y_1, \ldots, y_n) \quad in \quad \bar{G}_1 - G_2 \tag{3}$$

and

$$u(x_0, y_1, \ldots, y_n) \underset{(<)}{>} v(x_0, y_1, \ldots, y_n) \quad in \quad \bar{G}_0 - (\bar{G}_1 - G_2), \tag{4}$$

then

$$u(x, y_1, \ldots, y_n) = v(x, y_1, \ldots, y_n) \quad in \quad \bar{\Delta}^*$$

and

$$u(x, y_1, \ldots, y_n) \underset{(<)}{>} v(x, y_1, \ldots, y_n) \quad in \quad \bar{\Delta} - \bar{\Delta}^*, \tag{5}$$

where $\bar{\Delta}^*$ is a closed set generated by curves C which correspond, according to assumption (δ), to characteristics issuing from all the points $(Px_0, y_1, \ldots, y_n,$

$u(x_0, y_1, \ldots, y_n)$, $u_{y_1}(x_0, y_1, \ldots, y_n), \ldots, u_{y_n}(x_0, y_1, \ldots, y_n))$ such that their projections $Q(x_0, y_1, \ldots, y_n)$ belong to $\bar{G}_1 - G_2$ and satisfy the relation (3) *(if $\bar{G}_2 \subset G_1$, then $\bar{\Delta}^*$ is a closed domain).*

The set $\bar{\Delta} - \bar{\Delta}^*$ is the union of two sets $\hat{\Delta}_1$ and $\hat{\Delta}_2$, where $\hat{\Delta}_2$ is a set generated just as the set $\bar{\Delta}^*$ by curves C belonging to $\bar{\Delta}$ and issuing from the points of G_2.

2° *If*

$$u(x_0, y_1, \ldots, y_n) = v(x_0, y_1, \ldots, y_n) \quad \text{in} \quad \bar{G}_1 - G_2,$$

$$u(x_0, y_1, \ldots, y_n) \underset{(<)}{>} v(x_0, y_1, \ldots, y_n) \quad \text{in} \quad \bar{G}_0 - \bar{G}_1$$

and

$$u(x_0, y_1, \ldots, y_n) \underset{(>)}{<} v(x_0, y_1, \ldots, y_n) \quad \text{in} \quad G_2,$$

then in the case of $\bar{G}_2 \subset G_1$ we have

$$u(x, y_1, \ldots, y_n) = v(x, y_1, \ldots, y_n) \quad \text{in} \quad \bar{\Delta}^*,$$

$$(6) \qquad u(x, y_1, \ldots, y_n) \underset{(<)}{>} v(x, y_1, \ldots, y_n) \quad \text{in} \quad \hat{\Delta}_1$$

and

$$u(x, y_1, \ldots, y_n) \underset{(>)}{<} v(x, y_1, \ldots, y_n) \quad \text{in} \quad \hat{\Delta}_2,$$

where the sets $\bar{\Delta}^*$, $\hat{\Delta}_1$ and $\hat{\Delta}_2$ are defined in the same manner as in 1°.

In the case where $G_2 \subset G_1$ conclusion 2° *is also valid if we suppose additionally that at any point of the set $\bar{G}_1 - G_2$ the equalities*

$$(7) \qquad u_{y_i}(x_0, y_1, \ldots, y_n) = v_{y_i}(x_0, y_1, \ldots, y_n), \quad i = 1, \ldots, n,$$

are fulfilled.

Remark 1. The curves C in 1° corresponding to solutions u or v may be different but they always form the same domain $\Delta_2 = \hat{\Delta}_2 - F'\hat{\Delta}_2$, just as in the case of Δ_1 in Theorem 5 of [1]. Furthermore $\hat{\Delta}_1 = \bar{\Delta} - (\bar{\Delta}^* + \hat{\Delta}_2)$, where $\Delta_1 = \hat{\Delta}_1 - F'\hat{\Delta}_1$ is also a domain. Observe that the envelope Ω_1 of the domain $\bar{\Delta}^* + \hat{\Delta}_2 - F'(\bar{\Delta}^* + \hat{\Delta}_2)$ generated by curves C issuing from the boundary B_1 and contained in $\bar{\Delta}$ is the common part of the boundaries of $\bar{\Delta}^*$ and $\bar{\Delta}_1$ while the envelope Ω_2 of the domain Δ_2 generated by curves C issuing from the boundary B_2 is the common part of the boundaries of $\bar{\Delta}^*$ and $\bar{\Delta}_2$. (For a precise definition of the envelope Ω see [1], Lemma 2, p. 314.)

Remark 2. In the last case of conclusion 2° of Theorem 1, i.e. $G_2 \subset G_1$, it is sufficient to assume that the relations (7) hold for all the points of $B_1 \cdot B_2$ (if $B_1 \cdot B_2$ is not empty) for which there exist neighbourhoods containing no interior points of $\bar{G}_1 - G_2$.

However, it is easy to see that this assumption is equivalent to condition (7), because the derivatives u_{y_i} and v_{y_i} $(i = 1, \ldots, n)$ are continuous. If $G_1 = G_2$, relation (7) should be supposed at each point of the boundary $B_1 = B_2$, for the set $\bar{G}_1 - G_2 = B_1$ has no interior points.

Remark 3. If we assume in addition that the weak inequality $u \underset{(\leqslant)}{\geqslant} v$ holds for the points of the boundary B_0, then conclusions 1° and 2° of Theorem 1 do not change essentially. In this case inequalities (5) and (6) will be satisfied at all the points of the set $\hat{\Delta}_1$ which are different from those points of B_0 where $u = v$. This follows directly from Theorem 2 of [4].

Proof of Theorem 1. The proof consists of two parts. In the first part (I) we state Theorem 2 of [2]. From this theorem and from assumptions (α), (β) and (δ) of our Theorem 1 we deduce that also assumption (δ) of Theorem 5 of [1] is fulfilled. This assumption (δ) is needed since in [1] we have made use of Lemma 2 of [1] in the proof of Theorem 5. The latter theorem will be used to deduce conclusion 1° of our Theorem 1.

In the second part (II) of the proof assertions 1° and 2° of Theorem 1 are proved. The proof of assertion 2° is reduced to the proof of assertion 1° in the same manner as has been done for 3° in Theorem 1 of [3].

I. Theorem 2 and Lemma 1 of [2] from which Theorem 2 follows are stated here in a slightly modified form.

THEOREM 2. *Let us introduce the following assumptions*:

(α) *The function* $f(x, y_1, \ldots, y_n, z, q_1, \ldots, q_n)$ *and its first derivatives with respect to* $y_1, \ldots, y_n, z, q_1, \ldots, q_n$ *are continuous in a domain* $\mathcal{M}_1$ *the projection of which on the plane* $x, y_1, \ldots, y_n$ *contains the closed domain* $\bar{\Delta}$ *(defined in assumption* (β) *of Theorem* 1).

(β) *The derivatives* f_{y_i}, f_z, f_{q_i} *satisfy in* $\mathcal{M}_1$ *a Lipschitz condition with respect to* $y_1, \ldots, y_n, z, q_1, \ldots, q_n$ *and the inequalities*

$$|f_{q_i}| < M , \quad i = 1, \ldots, n ,$$

are fulfilled.

(γ) *The solution* $z(x, y_1, \ldots, y_n)$ *of equation* (1) *is of class* C^1 *in* $\bar{\Delta}$ *and is generated by characteristics (the remark to assumption* (α) *of Theorem* 1 *concerning the solutions* u *and* v *is here also valid).*

Under these assumptions there exists a domain D_1 *on the plane* $x, y_1, \ldots, y_n$ *such that* $\bar{\Delta} \subset D_1$ *and that the curves* C *are contained in* $\bar{D}_1$ *and reach the boundary of* $\bar{D}_1$ *by their right-hand and left-hand extremities (the domain* $D_1 - \bar{\Delta}$ *is not completely covered by the curves* C).

From Theorem 2 it readily follows that assumption (δ) of Theorem 5 [1] is satisfied.

Theorem 2 results immediately from the following Lemma 1 (see [2]) if we note that property B concerning the curves C follows from Lemma 1 of [1].

LEMMA 1. *Let us assume that a family of curves K, each curve of which is defined by equations*

$$y_i = y_i(x)\,, \quad i = 1, \ldots, n\,,$$

in an interval (a, b), $a < b$, where the functions $y_i(x)$ are continuous (in general the interval (a, b) may be different for every curve of K), has in a set $E \subset E_{n+1}$ the following properties:

A. *Through each point of the closed and bounded domain $\bar{\Delta} \subset E$ there passes exactly one curve of K. The curves of K have no limit points belonging simultaneously to the interior of the domain Δ and to the planes $x = a$ and $x = b$. If a curve issues out from the domain $\bar{\Delta}$ through a point $Q_0(x_0, y_1^0, \ldots, y_n^0)$ of the boundary $F'\Delta$, for $x \to a$ or $x \to b$, then the part of the curve which corresponds to $x \in (a, x_0)$ or $x \in (x_0, b)$ has no points common with $\bar{\Delta}$.*

B. *If a curve K_0 of K passing through a point P_0 is defined in the interval (a, b) and $\langle \alpha, \beta \rangle \subset (a, b)$, then all the curves issuing from the points of a sufficiently small neighbourhood of P_0 exist in the interval $\langle \alpha, \beta \rangle$ and are uniformly convergent in $\langle \alpha, \beta \rangle$ to the curve K_0.*

Under these assumptions there exists a domain $D_1 \subset E_{n+1}$ such that $\bar{\Delta} \subset D_1$ and the curves of K issuing from the points of the closed domain $\bar{\Delta}$ are contained in D_1 and reach the boundary of $\bar{D}_1$ in both directions (the domain D_1 need not be contained entirely in E).

Remark 4. The assumption contained in A that the curves of K have the uniqueness properties is not an essential restriction because, as is easily seen, uniqueness follows from property B.

II.1. The proof of conclusion 1° is reduced in a simple manner to the proof of conclusions 1° and 2° of Theorem 5 of [1]. In particular, from assumptions (3) and (4) it follows that in case 1° the inequality $u \underset{(\leqslant)}{\geqslant} v$ holds in the whole domain $\bar{\Delta}$.

The proof of this proposition is identical with the first part of the proof of Theorem 5 of [1]. Subsequently by using a similar method to that employed in proving Theorem 5 [1] concerning the set Δ_1 one can show that in view of the assumptions of Theorem 1 the domain Δ_2 is uniquely determined both by the curves C which are the projections on the plane $x, y_1, \ldots, y_n$ of the characteristics generating the solution u and by the curves $\bar{C}$ which are the projections of the characteristics generating the solution v.

The proof of the inequality $u \underset{(<)}{>} v$ in the sets $\hat{\Delta}_1$ and $\hat{\Delta}_2$ is identical with that of 1° and 2° of Theorem 5 [1] (p. 325). Part 1° of the latter theorem concerns the fulfillment of the inequality $u \underset{(<)}{>} v$ in the set $\bar{\Delta} - \bar{\Delta}_1$, and part 2°—the same inequality in the set $\bar{\Delta}_1 - \Omega$.

Moreover, as in the proof of the relations $u_{y_i} = v_{y_i}$ in the set $\bar{\Delta}_1$ given in [1] (p. 325), it clearly follows that assumptions (3), the relation $u = v$

in $\bar{G}_1 - G_2$ and the inequality $u \gtrless v$ taking place in the whole set $\bar{\Delta}$ imply the relations $u_{y_i} = v_{y_i}$ $(i = 1, \ldots, n)$ in $\bar{G}_1 - G_2$.

Hence the equality $u = v$ holds in the closed set $\bar{\Delta}^*$, because the curves C and $\bar{C}$ along which $u = v$, issuing from the points of the set $\bar{G}_1 - G_2$, are identical and form in $\bar{\Delta}$ the set $\bar{\Delta}^*$. The set $\bar{\Delta}^*$ is bounded by the envelope Ω_1 on the side of the set $\hat{\Delta}_1$ and by the envelope Ω_2 on the side of $\hat{\Delta}_2$.

The assumption (γ) implies that the envelopes Ω_1 and Ω_2 either are disjoint sets or have a common part or coincide depending on whether the boundary sets B_1 and B_2 are disjoint, have a common part or coincide.

2. In order to prove conclusion 2° observe first that any two solutions u and v generated by characteristics and taking on together with their derivates u_{y_i} and v_{y_i} the same values in the set $\bar{G}_1 - G_2$ coincide in the whole set generated by the curves C issuing from the points of the set $\bar{G}_1 - G_2$ and contained in $\bar{\Delta}$.

Further let us remark that the conditions imposed on the solutions u and v in 2° imply that the derivatives u_{y_i} and v_{y_i} are also equal in the set $\bar{G}_1 - G_2$.

In particular, if $\bar{G}_2 \subset G_1$, the relations $u_{y_i} = v_{y_i}$ in the interior of the closed domain $\bar{G}_1 - G_2$ follow immediately from the relation $u = v$. Hence and from the continuity of the derivatives u_{y_i} and v_{y_i} it follows that the equalities $u_{y_i} = v_{y_i}$ hold also on the boundary of $\bar{G}_1 - G_2$. If $G_2 \subset G_1$ the relations $u_{y_i} = v_{y_i}$ are satisfied by assumption (7). These relations also hold along the curves C issuing from the points of $\bar{G}_1 - G_2$ and therefore they hold in the whole closed domain $\bar{\Delta}^*$.

Hence, since the functions u and v are solutions of equation (1), also the relation $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x}$ holds true in the whole set $\bar{\Delta}^*$.

The sets $\hat{\Delta}_1$ and $\hat{\Delta}_2$ arise in the same way as in case 1° and form together with $\bar{\Delta}^*$ the set $\bar{\Delta}$.

The preceding remarks imply that knowing the solutions u and v we can construct two solutions U and V of equation (1) satisfying conditions (3) and (4) of 1° by setting

$$U(x, Y) \overset{\text{df}}{=} \begin{cases} u(x, Y) & \text{in} \quad \hat{\Delta}_1 + \bar{\Delta}^*, \\ v(x, Y) & \text{in} \quad \hat{\Delta}_2 \end{cases}$$

and

$$V(x, Y) \overset{\text{df}}{=} \begin{cases} v(x, Y) & \text{in} \quad \hat{\Delta}_1 + \bar{\Delta}^*, \\ u(x, Y) & \text{in} \quad \hat{\Delta}_2, \end{cases}$$

where $(x, Y) \overset{\text{df}}{=} (x, y_1, \ldots, y_n)$.

For the solutions U and V assertion 1° holds true, and thus we have:

$$U(x, Y) \underset{(<)}{\geq} V(x, Y) \quad \text{in} \quad \hat{\Delta}_1, \tag{8}$$

$$U(x, Y) = V(x, Y) \quad \text{in} \quad \bar{\Delta}^*, \tag{9}$$

$$U(x, Y) \underset{(>)}{\leq} V(x, Y) \quad \text{in} \quad \hat{\Delta}_2. \tag{10}$$

Moreover, from the definition of the solutions U and V it follows that

$$u(x, Y) = \begin{cases} U(x, Y) & \text{in} \quad \hat{\Delta}_1 + \bar{\Delta}^*, \\ V(x, Y) & \text{in} \quad \hat{\Delta}_2 \end{cases}$$

and

$$v(x, Y) = \begin{cases} V(x, Y) & \text{in} \quad \hat{\Delta}_1 + \bar{\Delta}^*, \\ U(x, Y) & \text{in} \quad \hat{\Delta}_2. \end{cases}$$

Thus, by virtue of (8)-(10) we deduce the validity of assertion 2°.

Remark 5. The initial conditions occurring in Theorem 1 might be prescribed on sets placed in $\bar{G}_0$ in a different manner from that followed above.

For instance, instead of the set $\bar{G}_1 - G_2$ one can select a certain closed set $\bar{G}^* \subset G_0$ dividing in a suitable manner the domain $\bar{G}_0$ into a few separate sets in which mixed inequalities are assumed while in $\bar{G}^*$ the equality $u = v$ is satisfied.

Remark 6. Theorem 1 (and Theorem 5 of [1]) may easily be extended to the case of the whole set D because the number c is selected with the only restriction that $0 < c < a$.

Now in the formulation of Theorem 1 the sets $\bar{\Delta}^*$, $\hat{\Delta}_1$ and $\hat{\Delta}_2$ ought to be replaced by appropriate sets $\bar{\Delta}_0^*$, $\hat{\Delta}_1^0$ and $\hat{\Delta}_2^0$ determined in the same manner as before by the curves C which are now required to be defined in the whole set D. Evidently the sets $\bar{\Delta}^*$, $\hat{\Delta}_1$ and $\hat{\Delta}_2$ should then be contained in the sets $\bar{\Delta}_0^*$, $\hat{\Delta}_1^0$ and $\hat{\Delta}_2^0$ respectively.

The proof of the theorem extended in this way follows from the proof given above for Theorem 1 and from the fact that Theorem 1 holds for any closed domain $\bar{\Delta}$ contained in D.

References

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