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Toeplitz operators for hypodirichlet algebras

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Abstract. In this paper we prove some results for Toeplitz operators defined in the context of hypodirichlet algebras. We also give an application of these results to Toeplitz operators in multiply connected domains in the plane.

1. Let $A \subset C(X)$ be a hypodirichlet algebra, i.e., $\mathrm{Re}A$ has a finite codimension in $C_R(X)$ and the linear span of $\log|A^{-1}|$ is dense in $C_R(X)$. If A is hypodirichlet on X, then every $\xi \in \mathrm{Sp}A$ (spectrum of A) has a finite-dimensional set of representing measures M_ξ and has a unique logmodular measure $m \in M_\xi$, [7].

Denote by $L^2(m)$ the standard Hilbert space of all complex m-square integrable functions on X. We define the Hardy space $H^2(m)$ as the closure of A in $L^2(m)$. For $\varphi \in L^\infty(m)$ an m-essentially bounded function we define the Toeplitz operator in $H^2(m)$ by

$$T_\varphi f = P(\varphi \cdot f),$$

where $P\colon L^2(m) \to H^2(m)$ is the orthogonal projection.

Now we will characterize the C^*-algebras generated by the families $\{T_\varphi\}_{\varphi\in L^\infty(m)}$ and $\{T_\varphi\}_{\varphi\in C(X)}$. To this purpose we apply the method developed by Douglas [6]. This method reduces the above characterization to the problem of computing the joint approximate point spectrum $\sigma_\pi(T_{\varphi_1}, \ldots, T_{\varphi_p})$ of a p-tuple of operators T_{φ_i}, $i = 1, \ldots, p$. Let M_{L^∞} stand for the spectrum of $L^\infty(m)$. Denote by

$$\wedge\colon L^\infty(m) \to C(M_{L^\infty})$$

the Gelfand isomorphism. Let $H^\infty(m) = \{\varphi \in L^\infty(m),\ \varphi \cdot H^2(m) \subset H^2(m)\}$. It is easy to prove that $H^\infty(m)$ is equal to the weak-$*$ closure of A in $L^\infty(m)$. Moreover, using the results of [2] one can prove that

(a) $M_{L^\infty} = \partial H^\infty(m)$, where $\partial H^\infty(m)$ is the Shilov boundary of $H^\infty(m)$,

(b) $\widehat{H^\infty(m)}$ separates the points of M_{L^∞}.

We will need these facts later. Now we state the following

LEMMA 1. *If* $\varphi_i \in H^\infty(m)$, $i = 1, \ldots, p$, *then*

$$\sigma_\pi(T_{\varphi_1}, \ldots, T_{\varphi_p}) = \{(\hat{\varphi}_1(m), \ldots, \hat{\varphi}_p(m)), m \in M_{L^\infty}\}.$$

Proof. The inclusion $\sigma_\pi(T_{\varphi_1}, \ldots, T_{\varphi_p}) \subset \{(\hat{\varphi}_1(m), \ldots, \hat{\varphi}_p(m)), m \in M_{L^\infty}\}$ is obvious. To prove the converse inclusion assume that $\varphi_i(\mathring{m}) = 0$ for $i = 1, \ldots, p$. For $\varepsilon > 0$ denote by $U_{\mathring{m}}$ the neighborhood of $\mathring{m}$ defined by the conditions

$$|\hat{\varphi}_i(m)| < \varepsilon \quad \text{for } m \in U_{\mathring{m}} \text{ and } i = 1, \ldots, p.$$

Let $\hat{m}$ be the measure on M_{L^∞} induced by m, i.e.,

$$\hat{m}(\hat{f}) = \int f dm \quad \text{for } f \in L^\infty(m).$$

Let $\hat{v} \geqslant 0$ be in $C(M_{L^\infty})$ and assume that $\int \hat{v} d\hat{m} = 1$ and $\hat{v}(m) = 0$ for $m \notin U_{\mathring{m}}$. Now we have

$$\int_X |\varphi_i|^2 v dm = \int_{M_{L^\infty}} |\hat{\varphi}_i|^2 \hat{v} d\hat{m} \leqslant \varepsilon^2 \quad \text{for } i = 1, \ldots, p.$$

But for $k = 1, 2, \ldots$ the functions $g_k = v + 1/k$ are positive on X, thus by Theorem 10,3 of [2] there are $f_k \in H^2(m)$ such that $|f_k|^2 = g_k$. Denoting $M = \sup_{1 \leqslant i \leqslant p} (\|\varphi_i\|_\infty)$ we get

$$\int |\varphi_i|^2 |f_k|^2 dm \leqslant \varepsilon^2 + 1/k \cdot M.$$

Since $\int |f_k|^2 dm = 1/k \cdot m(X) + 1$, the above inequality proves that $(0, \ldots, 0) \in \sigma_\pi(T_{\varphi_1}, \ldots, T_{\varphi_p})$ and the proof is complete.

By the same method one can prove the next lemma:

LEMMA 2. *If* $\varphi_i \in A$, $i = 1, \ldots, p$, *then*

$$\sigma_\pi(T_{\varphi_1}, \ldots, T_{\varphi_p}) = \{(\varphi_1(X), \ldots, \varphi_p(X)), x \in X\}.$$

Proof. A similar reasoning to that in the proof of Lemma 1.

Now we will apply the above lemmas to characterize the C^*-algebra generated by $\{T_\varphi\}_{\varphi \in L^\infty}$. Indeed, as we know, $\widehat{H^\infty(m)}$ separates the points of M_{L^∞}. Thus the set $\{\varphi\bar{\psi}\}$, $\varphi, \psi \in H^\infty(m)$ is linearly dense in $L^\infty(m)$ by the Stone–Weierstrass theorem. Since $\|T_\varphi\| \leqslant \|\varphi\|_\infty$ for $\varphi \in L^\infty(m)$, it follows that the C^*-algebra generated by $\{T_\varphi\}_{\varphi \in L^\infty}$ is equal to the C^*-algebra generated by $\{T_\varphi\}_{\varphi \in H^\infty(m)}$. Next, note that each operator T_φ for $\varphi \in H^\infty(m)$ is subnormal. Thus applying the theorem of Bunce [6] and Lemma 1 we get

THEOREM 1. *Let $A \subset C(X)$ be a hypodirichlet algebra. Denote by $\mathscr{C}$ the C^*-algebra generated by the family $\{T_\varphi\}_{\varphi \in L^\infty(m)}$. If J denotes the commutator ideal for $\mathscr{C}$, then the following short sequence*

$$(0) \to J \xrightarrow{i} \mathscr{C} \xrightarrow{\varrho} L^\infty(m) \to (0)$$

is exact, where i is the inclusion, ϱ is a $*$-homomorphism and $\varrho(T_\varphi) = \varphi$.

Proof. It is enough to note that $\sigma_\pi(\{T_\varphi\}_{H^\infty})$ is homeomorphic to M_{L^∞}; and this follows from Lemma 1 and the above remarks.

In the same way using Lemma 2 one can prove the next theorem.

THEOREM 2. *Let $A \subset C(X)$ be a hypodirichlet algebra. Denote by $\mathscr{C}_1$ the C^*-algebra generated by the family $\{T_\varphi\}_{\varphi\in C(X)}$. If J_1 denotes the commutator ideal for $\mathscr{C}_1$, then the following short sequence*

$$(0)\to J_1 \xrightarrow{i_1} \mathscr{C}_1 \xrightarrow{\varrho_1} C(X)\to(0)$$

is exact, where i_1 is the inclusion, ϱ_1 is a $*$-homomorphism and $\varrho_1(T_\varphi) = \varphi$.

One can extend the above theorems to the matrix case by the same method as in [6]. We will not formulate the full generalizations. Instead, we will give two simple corollaries; compare [6], p. 16. Let M_n be the algebra of all complex $n\times n$ matrices. Let us recall that for any C^*-algebra A, the tensor product $A\otimes M_n$ is naturally isomorphic with the C^*-algebra of all $n\times n$ matrices with elements of A. Let $L^\infty_{M_n}(m)$ be the C^*-algebra of all m-essentially bounded functions on X with values in M_n. For any $\Phi \in L^\infty_{M_n}(m)$ denote by L_Φ the operator of multiplication in $L^2(m)\otimes C^n$. Let T_Φ be the matrix Toeplitz operator in $H^2(m)\otimes C^n$ associated with Φ. We have the following corollary; compare [6].

COROLLARY 1. *If $\Phi\in L^\infty_{M_n}(m)$, then $\sigma(L_\Phi)\subset\sigma_e(T_\Phi)$, where $\sigma_e(T_\Phi)$ is the essential spectrum of T_Φ.*

Proof. Let $\mathscr{C}_{M_n}$ be the C^*-algebra generated by $\{T_\Phi\}_{\Phi\in L^\infty_{M_n}(m)}$. We identify $\mathscr{C}_{M_n}$ with $\mathscr{C}\otimes M_n$ (notations as in Theorem 1). Denote by

$$\pi\colon \mathscr{C}_{M_n}\to\mathscr{C}_{M_n}/\mathscr{K}\big(H^2_{C^n}(m)\big)\ (^1)$$

the canonical projection. For $\Phi\in L^\infty_{M_n}(m)$ the operator T_Φ is Fredholm iff $\pi(T_\Phi)$ is invertible in $\mathscr{C}_{M_n}/\mathscr{K}\big(H^2_{C^n}(m)\big)$. Therefore, if $0\notin\sigma_e(T_\Phi)$, then $\pi(T_\Phi)$ is invertible in $\mathscr{C}_{M_n}/\mathscr{K}\big(H^2_{C^n}(m)\big)$ and, what is more, it is invertible in $\mathscr{C}_{M_n}/I\otimes M_n$. But $\mathscr{C}_{M_n}/I\otimes M_n$ is isometrically isomorphic to $L^\infty_{M_n}(m)$, and so Φ is invertible in $L^\infty_{M_n}(m)$ and $0\notin\sigma(L_\Phi)$.

COROLLARY 2. *If $\Phi\in L^\infty_{M_n}(m)$, then T_Φ is a compact or quasi-nilpotent operator iff $\Phi = 0$.*

Proof. In both cases the essential spectrum of T_Φ, $\sigma_e(T_\Phi) = \{0\}$; hence $\Phi = 0$.

2. Now we will give an application to the Toeplitz operators in the Hardy space generated by $A = R(X)|_{\partial X}$, where $X\subset C$ is compact and $C\setminus X$ has finitely many components. $R(X)$ denotes here the Banach algebra of all functions on X which can be uniformly approximated on X by rational functions with poles outside X. It is well known [7] that A

(^{1}) $\mathscr{K}(H^2_{C^n}(m))$ denotes the ideal of compact operators in $H^2_{C^n}(m) = H\ (m)\otimes C^n$.

is a hypodirichlet algebra on ∂X. Let $\mathring{z} \in X$ and let m be the logmodular measure on ∂X for the functional $R(X) \ni f \to f(\mathring{z})$. Obviously, $H^2(m)$ does not contain real functions different from constants. Therefore it is easy to prove that the C^*-algebra $\mathscr{C}$ generated by $\{T_\varphi\}_{\varphi \in C(\partial X)}$ is irreducible. Next, note that $\sigma(T_z) = X$ and 1 is a cyclic vector for T_z. By the result of Berger–Shaw, the commutator $T_z^* T_z - T_z T_z^*$ is compact ([3], Theorem 1). It follows that the commutator ideal J for $\mathscr{C}$ is equal to the ideal of compact operators $\mathscr{K}(H^2(m))$ in $H^2(m)$. Indeed, since $\mathscr{C} \cap \mathscr{K}(H^2(m)) \neq \{0\}$ and $\mathscr{C}$ is irreducible, $\mathscr{C} \supset \mathscr{K}(H^2(m))$, [5], Theorem 5.39. Denote by

$$\pi: \mathscr{C} \to \mathscr{C}/\mathscr{K}(H^2(m))$$

the canonical projection. The element $\pi(T_z)$ is normal in $\mathscr{C}/\mathscr{K}(H^2(m))$.

Since $\pi(\varphi)$ is generated (as a C^*-algebra) by $\pi(T_z)$, it is commutative. Thus $J \subset \mathscr{K}(H^2(m))$. But $\mathscr{K}(H^2(m))$ cannot contain non-trivial closed ideals and so $\mathscr{K}(H^2(m)) = J$. On the other hand, we know that $\sigma_\pi(T_z) = \partial X$; hence applying Theorem 2 we get

THEOREM 3. *Let $X \subset C$ be a compact set. Assume that $C \setminus X$ has finitely many components. There exists a $*$-homomorphism ϱ from $\mathscr{C}$ onto $C(\partial X)$ such that the following short sequence*

$$(0) \to \mathscr{K}(H^2(m)) \xrightarrow{i} \mathscr{C} \xrightarrow{\varrho} C(\partial X) \to (0)$$

is exact and $\varrho(T_\varphi) = \varphi$.

Remark 1. The above theorem was proved by Abrahamse in the case ∂X is equal to a finite number of analytic Jordan curves [1].

As an immediate consequence of Theorem 3 we get the equality $\sigma_e(T_\varphi) = \varphi(\partial X)$. How to find $\sigma_e(T_\psi)$ for $\psi \in L^\infty(m)$? By the results of Clancey–Morrel [4] and the Reduction Theorem of Abrahamse [1], p. 282, one can describe $\sigma_e(T_\psi)$ for $\psi = \chi_E \cdot g$, where χ_E is the characteristic function of $E \subset \partial X$, $g \in C(\partial X)$ and ∂X is a finite union of analytic Jordan curves.

Remark 2. Note, by the way that the set of invertible Toeplitz operators T_φ with continuous φ is norm-dense in the set of Fredholm Toeplitz operators of index zero. This follows by the result of Widom [8], Corollary, since for $f \neq 0$, $g \neq 0$ in $H^2(m)$ the product $fg \neq 0$ in $H^1(m)$. The last fact can be easly proved using the characterization of an invariant subspace for $R(X)$ in $H^1(m)$, given in [2].

We conclude with an application of localization technique for T_φ with $\varphi \in L^\infty(m)$ (first stated and applied by Douglas). In the case $\partial X = \partial D$, the closed unit circle, Douglas applied localization technique and obtained a local criterium on $\varphi \in L^\infty(m)$, which guarantees that T_φ is Fredholm. It turns out that it is possible to extend this technique to our situation. We shall not explain this in detail; the interested reader should consult [5], p. 196–199. We only point out that this possibility follows from the facts which we now describe.

(i) For any $\lambda \in \partial X$, $\partial F_\lambda \overset{\mathrm{df}}{=} \{\xi \in M_{H^\infty(m)},\ \hat{z}(\xi) = \lambda\} \cap M_{L^\infty(m)} \neq \{\emptyset\}$, where $\wedge$: $L^\infty(m) \to C(M_{L^\infty})$ is the Gelfand transform.

(ii) The algebra $\mathscr{C}_1/\mathscr{K}(H^2(m))$ is contained in the centre of $\mathscr{C}/\mathscr{K}(H^2(m))$, where we have used the notations from Theorems 1 and 2.

Ad (i). To prove (i), note that for any $\lambda \in \partial X$ the functional ξ_λ: $R(X) \ni f \to f(\lambda)$ extends to a complex multiplicative linear functional η_λ on $L^\infty(m)$ (recall that the Shilov boundary of $R(X)$ is equal to ∂X!). Thus

$$\hat{z}(\eta_\lambda) = \hat{z}(\xi_\lambda) = \xi_\lambda(z) = \lambda, \quad \text{i.e., } \partial F_\lambda \neq \{\emptyset\}.$$

Ad (ii). Since. $R(X)$ is hypodirichlet and for any $\varphi \in C(\partial X)$ the operator $T_\varphi^* \cdot T_\varphi - T_\varphi T_\varphi^*$ is compact, one can check that

$$(*) \qquad T_\psi T_\varphi - T_{\psi\varphi} \text{ is compact for } \psi \in L^\infty(m) \text{ and } \varphi \in C(\partial X).$$

If $a \in \mathscr{C}_1/\mathscr{K}(H^2(m))$ and $b \in \mathscr{C}/\mathscr{K}(H^2(m))$, then we can write $a = \pi(T_\varphi)$, $b = \pi(T_\psi + V)$, where $\psi \in L^\infty(m)$, $\varphi \in C(\partial X)$ and $V \in J$ (commutator ideal for $\mathscr{C}$). Now using $(*)$ we get

$$\begin{aligned} ab = \pi(T_\varphi)\pi(T_\psi + V) = \pi\big(T_\varphi(T_\psi + V)\big) &= \pi(T_\psi T_\varphi + VT_\varphi) \\ &= \pi\big((T_\psi + V)T_\varphi\big) = ba, \end{aligned}$$

which proves (ii).

The following corollary is a generalization of Corollaries 4.7 and 4.8 of Douglas [6]. The proofs are the same as in Douglas' work.

COROLLARY 3. *If $\varphi \in L^\infty(m)$, then T_φ is a Fredholm operator if and only if for each $\lambda \in \partial X$ there exists $\psi \in L^\infty(m)$ such that T_ψ is a Fredholm operator and $\hat{\psi} - \hat{\varphi}|_{\partial F_\lambda} = 0$.*

COROLLARY 4. *If $\varphi \in L^\infty(m)$ and for each $\lambda \in \partial X$ there exists an open disc K such that $\lambda \in K \cap \partial X$ and 0 is not contained in the closed convex hull of $\varphi(K \cap \partial X)$, then T_φ is a Fredholm operator.*

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