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On integrable bounds for differential equations

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In this note* we discuss the relation between two types of boundedness assumptions for right side members of ordinary or partial differential equations. In the weaker assumption, boundedness by an integrable function of one real variable and, in the stronger one, boundedness by a constant is supposed.

We shall prove that one assumption can be reduced to the other. To start with we make the following remark. Let be given a partial differential equation of a finite order k:

$$\frac{\partial u}{\partial t} = f\left(t, x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1^k}, \ldots, \frac{\partial^k u}{\partial x_n^k}\right). \tag{1}$$

We suppose that for $|t| \leqslant a$, $a > 0$, $R = (r_1, \ldots, r_p)$ arbitrary (p being a suitable integer) the following inequalities hold:

$$|f(t, R)| \leqslant F(t), \tag{2}$$

$$|Df(t, R)| \leqslant F(t), \tag{3}$$

where Df denotes suitable derivatives in respect to R and $F(t)$ is an integrable function.

We claim that there exists a mapping

$$s = \varphi(t) \tag{4}$$

of the interval $|t| \leqslant a$ onto the interval $|s| \leqslant \gamma$ ($\gamma = \int_0^a F(u)\,du, \gamma > 0$), transforming (1) into the equation

$$\frac{\partial v}{\partial s} = g\left(s, x_1, \ldots, x_n, v, \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial^k v}{\partial x_n^k}\right), \tag{5}$$

where g has the following properties:

$$|g(s, R)| \leqslant 1, \tag{6}$$

$$|Dg(s, R)| \leqslant 1 \quad \text{for } |s| \leqslant \gamma \text{ almost everywhere.} \tag{7}$$

* I would like to express my gratitude to Professor A. Pliś for suggesting problems considered in this paper and for his valuable advice.

Without loss of generality we can assume that $F(t) \geqslant 1$ and we define

$$\varphi(t) = \int_0^t F(u)\,du \quad (^1).$$

Functions $\varphi(t)$ and its inverse $\varphi^{-1}(s)$ are absolutely continuous on suitable intervals.

We have

$$\frac{d\varphi(t)}{dt} = F(t) \qquad \text{for } |t| \leqslant a \text{ almost everywhere,}$$

$$\frac{d\varphi^{-1}(s)}{ds} = \frac{1}{F(\varphi^{-1}(s))} \qquad \text{for } |s| \leqslant \gamma \text{ almost everywhere.}$$

Function $v(s, X) = u(\varphi^{-1}(s), X)$, $X = (x_1, \ldots, x_m)$, satisfies the equality

$$v_s(s, X) = u_t(\varphi^{-1}(s), X)\frac{d\varphi^{-1}(s)}{ds}.$$

Therefore v satisfies the equation

$$v_s(s, X) = g(s, X, v, \ldots),$$

where

$$g(s, R) = f(\varphi^{-1}(s), R)\frac{d\varphi^{-1}(s)}{ds},$$

$$\left|f(\varphi^{-1}(s), R)\frac{d\varphi^{-1}(s)}{ds}\right| \leqslant F(\varphi^{-1}(s))\frac{1}{F(\varphi^{-1}(s))} = 1,$$

which proves that g is bounded. Analogous estimations may be applied to any derivative Dg. Using this remark, certain theorems on differential equations with right side members bounded by constants can be generalized to the case with right side members bounded by integrable functions. As an example we shall give an application to a system of first order partial differential equations. We shall deal with the following system of differential equations:

$$u_t^i = f^i(t, X, U, U_x^i) \qquad (i = 1, \ldots, m), \tag{8}$$

where

$$U = (u^1, \ldots, u^m), \qquad U_x^i = (u_{x_1}^i, \ldots, u_{x_n}^i)$$

with conditions

$$u^i(a_i, X) = \omega^i(X) \qquad \text{for } X \text{ arbitrary } (i = 1, \ldots, m). \tag{9}$$

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(1) The same mapping has been used by Kasprzyk and Myjak [1].

THEOREM. *Suppose that functions $f^i(t, R)$, $i = 1, \ldots, m$, measurable in t are of class c^1 in $R = (r_1, \ldots, r_{m+2n})$ and satisfy the following inequalities:*

(10)
$$|f^i(t, R)| \leqslant F(t), \quad |f^i_{r_j}(t, \bar{R})| \leqslant F(t),$$
$$|f^i_{r_j}(t, R) - f^i_{r_j}(t, \bar{R})| \leqslant F(t) \sum_{k=1}^{m+2n} |r_k - \bar{r}_k|,$$

where $F(t)$ is an integrable function for $|t| \leqslant a$, R and $\bar{R}$ arbitrary;

(11)
$$\omega^i(X) \text{ are of class } c^1 \text{ in } X;$$

(12)
$$|\omega^i(X)| \leqslant L, \quad |\omega^i_{x_j}(X)| \leqslant L,$$
$$|\omega^i_{x_j}(X) - \omega^i_{x_j}(\bar{X})| \leqslant L \sum_{k=1}^{n} |x_k - \bar{x}_k|$$

for $X = (x_1, \ldots, x_n)$ and $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_n)$ arbitrary $(i = 1, \ldots, m;$ $j = 1, \ldots, n)$.

Then, for an arbitrary system of numbers $a_1, \ldots, a_m$,

(13)
$$|a_i| \leqslant \beta,$$

β being a constant for which

$$\int_0^\beta F(u)\,du = \lambda,$$

where

$$\lambda = \min\left(\gamma, \{4n(1+nL)[(1+mT)^2 + m(1+2L)]\}^{-1}\right)$$

and

$$T = 2L + \min(4\gamma, \tfrac{1}{2}),$$

there exists a solution $U(t, X) = \left(u^1(t, X), \ldots, u^m(t, X)\right)$ absolutely continuous in t, of class c^1 in X, satisfying system (8) *for almost every $t \in [-\beta, \beta]$ and every X, conditions* (9) *and the inequalities*

(14)
$$|u^i(t, X)| \leqslant K, \quad |u^i_{x_j}(t, X)| \leqslant T,$$
$$|u^i_{x_j}(t, X) - u^i_{x_j}(t, \bar{X})| \leqslant (2L+1) \sum_{k=1}^{n} |x_k - \bar{x}_k|.$$

Proof. Applying mapping (4) to system (8) we get

$$v^i_s = g^i(s, X, V, V^i_x), \quad \text{where } V = (v^1, \ldots, v^m),$$
$$V^i_x = (v^i_{x_1}, \ldots, x^i_{x_n}).$$

Functions $g^i(s, R)$ satisfy the conditions

$$|g^i| \leqslant 1, \quad |g^i_{r_j}| \leqslant 1, \tag{16}$$
$$|g^i_{r_j}(s, R) - g^i_{r_j}(s, \bar{R})| \leqslant \sum_{k=1}^{m+2n} |r_k - \bar{r}_k|$$

for almost every $|s| \leqslant \gamma$ and arbitrary R and $\bar{R}$.

Numbers a_i are mapped on b_i and conditions (9) give

$$v^i(b_i, X) = \omega^i(X). \tag{17}$$

Consider now an auxiliary system

$$(v^{ih})_s = g^{ih}(s, X, V^h, V^{ih}_x), \tag{18}$$

where

$$g^{ih}(s, R) = \frac{1}{2h} \int_{s-h}^{s+h} g^i(u, R)\, du;$$
$$v^h = (v^{1h}, \ldots, v^{mh}), \quad V^{ih}_x = (v^{ih}_{x_1}, \ldots, v^{ih}_{x_n}).$$

Functions g^{ih} are continuous in s, of class c^1 in R and satisfy (16). Therefore the theorem of A. Pliś from [2] can be applied. Hence there exists a solution $v^{ih}(s, X)$ of class c^1. Moreover, it satisfies the inequalities

$$|v^{ih}_{x_j}(s, X)| \leqslant T, \quad |v^{ih}_{x_j}(s, X) - v^{ih}_{x_j}(s, \bar{X})| \leqslant (2L+1) \sum_{k=1}^{n} |X_k - \bar{X}_k| \tag{19}$$

for arbitrary $X, \bar{X}$, and for $|s| \leqslant \lambda$. It is easy to see that $|v^{ih}(s, X)| \leqslant L + 2\gamma = K$.

Functions v^{ih} $(i = 1, \ldots, m)$ are equicontinuous and uniformly bounded. By Arzela's lemma there exists a sequence $h_k \to 0$, if $k \to \infty$ such that $v^{ik} = v^{ih_k}$ are uniformly convergent

$$v^{ik}(s, X) \rightrightarrows v^i(s, X) \quad \text{as } k \to \infty \ (i = 1, \ldots, m). \tag{20}$$

Owing to the uniform convergence of functions v^{ik}, and $v^{ik}_{x_j}$ and our assumption that the functions g^i are of class c^1 in R we can estimate the difference as follows:

$$\big|g^{ik}\big(s, X, V^k(s, X), V^{ik}_x(s, X)\big) - g^{ik}\big(s, X, V(s, X), X^i_x(s, X)\big)\big|$$
$$= \frac{1}{2h_k} \left| \int_{s-h_k}^{s+h_k} \big\{g^i\big(\tau, X, V^k(s, X), V^{ik}_x(s, X)\big) - g^i\big(\tau, X, V(s, X), V^i_x(s, X)\big)\big\}\, d\tau \right|$$
$$\leqslant M \left\{ \sum_{i=1}^{m} |v^{ik}(s, X) - v^i(s, X)| + \sum_{j=1}^{n} |v^{ik}_{x_j}(s, X) - v^i_{x_j}(s, X)| \right\} \to 0$$

as $k \to \infty$.

Thus we derived the equality:

$$(21)\quad \begin{aligned} &g^{ik}\big(s, X, V^k(s, X), V_x^{ik}(s, X)\big)\\ &\quad = g^{ik}\big(s, X, V(s, X), V_x^i(s, X)\big) + \varepsilon_1(k, s, X)\\ &\quad = \frac{1}{2h_k}\int_{s-h_k}^{s+h_k} g^i\big(\tau, X, V(s, X), V_x^i(s, X)\big)d\tau + \varepsilon_1(k, s, X),\end{aligned}$$

where $\varepsilon_1(k, s, X) \to 0$ as $k \to \infty$.

In the following we shall show the uniform Lipschitz continuity in X of functions $v_s^i(s, X)$.

Functions g^{ik} are of class c^1 in R and functions v^{ik} and $v_{x_j}^{ik}$ satisfy Lipschitz condition in X. Hence

$$(22)\quad \begin{aligned} &|v_s^{ik}(s, X) - v_s^{ik}(s, \bar{X})|\\ &= \big|g^{ik}\big(s, X, V^k(s, X), V_x^{ik}(s, X)\big) - g^{ik}\big(s, \bar{X}, V^k(s, \bar{X}), V_x^{ik}(s, \bar{X})\big)\big|\\ &\leqslant C\sum_{i=1}^{n}|x_i - \bar{x}_i|,\end{aligned}$$

where $C = M + mM + n(2L+1)$.

The functions $v_s^i(s, X)$ satisfy Lipschitz condition in X too.

Now we show the inequality

$$(23)\qquad |v_{x_j}^i(s, X) - v_{x_j}^i(s, X)| \leqslant C|s - \bar{s}|.$$

For demonstration we use the expression

$$L_h^i = \left|\frac{v^i(s, x_1, \ldots, x_j + h, \ldots, x_n) - v^i(s, x_1, \ldots, x_n)}{h} - \frac{v^i(\bar{s}, x_1, \ldots, x_j + h, \ldots, x_n) - v^i(\bar{s}, x_1, \ldots, x_n)}{h}\right|.$$

By mean value theorem and (22) we have inequality

$$L_h^i \leqslant C|s - \bar{s}|, \qquad L_h^i \to |v_{x_j}^i(s, X) - v_{x_j}^i(\bar{s}, X)| \quad \text{as } h \to 0$$

and (23) is proved.

In the following we estimate the next difference:

$$\frac{1}{2h_k}\left|\int_{s-h_k}^{s+h_k}\big[g^i\big(\tau, X, V(s, X), V_x^i(s, X)\big) - g^i\big(\tau, X, V(\tau, X), V_x^i(\tau, X)\big)\big]d\tau\right|$$

$$\leqslant \frac{1}{2h_k}\int_{s-h_k}^{s+h_k} M\Big(\sum_{k=1}^{m}|v^k(s, X) - v^k(\tau, X)| + \sum_{j=1}^{n}|v_{x_j}^i(s, X) - v_{x_j}^i(\tau, X)|\Big)d\tau \to 0$$

as $h_k \to 0$ in virtue of (23).

The relation

$$(24)\quad \frac{1}{2h_k}\int_{s-h_k}^{s+h_k} g^i\big(\tau, X, V(s, X), V_x^i(s, X)\big)\,d\tau$$

$$= \varepsilon_2(k, s, X)+\frac{1}{2h_k}\int_{s-h_k}^{s+h_k} g^i\big(\tau, X, V(\tau, X), V_x^i(\tau, X)\big)\,d\tau,$$

where $\varepsilon_2(k, s, X)\to 0$ as $k\to\infty$ holds.

Let $\varepsilon(k, s, X) = \varepsilon_1(k, s, X)+\varepsilon_2(k, s, X)$. By (21) and (24) we have

$$(25)\quad g^{ik}\big(s, X, V^k(s, X), V_x^{ik}(s, X)\big)$$

$$= \varepsilon(k, s, X)+\frac{1}{2h_k}\int_{s-h_k}^{s+h_k} g^i\big(\tau, X, V(\tau, X), V_x^i(\tau, X)\big)\,d\tau.$$

The functions $g^i\big(s, X, V(s, X), V_x^i(s, X)\big)$ are measurable in s for any X because $g^i(s, X, q_1, \ldots, q_{m+n})$ are measurable in s as well as Lipschitz continuous in q_i, and $v^i(s, X)$ are Lipschitz continuous in s and X.

Therefore the right side members of (25) tend to $g^i\big(s, X, V(s, X), V_x^i(s, X)\big)$ for almost every s. Hence there exists a system of functions $v^1(s, X), \ldots, v^m(s, X)$ satisfying equations (15) almost everywhere and condition (17). Functions $v^1, \ldots, v^m$ are continuous and of class c^1 in X.

We get in the limit

$$|v^i(s, X)|\leqslant K, \qquad |v_{x_j}^i(s, X)|\leqslant T \qquad (j = 1, \ldots, n),$$

$$|v_{x_j}^i(s, X)-v_{x_j}^i(s, \bar{X})|\leqslant (2L+1)\sum_{k=1}^{n}|x_k-\bar{x}_k|$$

for $|s|\leqslant\gamma$, X, $\bar{X}$ arbitrary.

Put

$$u^i(t, X) = v^i\big(\varphi(t), X\big),$$

$$\begin{aligned} u_t^i &= v_s^i\big(\varphi(t), X\big)\cdot\varphi'(t)\\ &= g^i\Big(\varphi(t), X, V\big(\varphi(t), X\big), V_x^i\big(\varphi(t), X\big)\Big)\cdot\varphi'(t)\\ &= f^i\big(t, X, U(t, X), U_x^i(t, X)\big)\end{aligned}$$

for $|t|\leqslant\beta$ almost everywhere.

Functions $v^i(t, X)$ are absolutely continuous in s, therefore $u^i(t, X) = v^i\big(\varphi(t), X\big)$ are absolutely continuous in t.

Thus the proof is completed.

References

[1] S. Kasprzyk and J. Myjak, *On the existence of solutions of the Floquet problem for ordinary differential equations*, Zeszyty Naukowe U. J. (1967).

[2] A. Pliś, *Generalization of the Cauchy problem for a system of partial differential equations*, Bull. Acad. Polon. Sci. Cl. III, 4 (1956), p. 741-744.

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