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An abstract version of the resonance theorem

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Abstract. The paper includes a version of the existence theorem for a nonlinear operator equation and its application to boundary value problems for systems at resonance.

Introduction. In the sixties, there started to appear papers referring to nonlinear vibrations of systems at resonance. From the mathematical point of view, this amounts to studying the boundary value problem

$$Pu - \lambda_0 u = f(u, x), \quad Bu = 0,$$

where P is a linear differential operator acting on a function u defined on an open bounded set $\Omega \subset \mathbf{R}^k$, $x \in \Omega$, f is a nonlinear function, B is a boundary operator, and λ_0 is an eigenvalue of the homogeneous problem, i.e. there is a nontrivial solution of

$$Pu - \lambda_0 u = 0, \quad Bu = 0.$$

The differential operator P can be partial or ordinary (in this last case, Ω is a finite interval). The first and most remarkable work in this direction was that of Landesman and Lazer [4], who considered the case of elliptic and self-adjoint P. Later, a lot of mathematicians studied similar problems using different methods, more or less advanced (cf. [1], [6], [7]). In particular, the method of nonlinear Fredholm mappings and their degree plays an important role ([2], [8]).

The methods used in solving the considered problem seem to be too sophisticated for many natural cases. The proof in [7] is almost elementary, but the problem is specific. In the present paper, we apply only the Schauder Fixed Point Theorem to obtain a rather general result which includes the theorems of Landesman and Lazer [4], [7].

The common part of the assumptions of all resonance theorems is that λ_0 is an isolated eigenvalue and that the eigenspace belonging to λ_0 is finite-dimensional. We have also added some assumption about the form of the inverse operator $(P - \lambda I)^{-1}$ near λ_0.

1. ASSUMPTIONS. (a) Let X, Y, X_1 be Banach spaces, let U be an open interval containing $\lambda_0 \in \mathbf{R}$ and let $D\colon U \to L(X, Y)$ be a continuous mapping taking values in the space of linear bounded operators $X \to Y$ such that $D(\lambda)$ is a linear homeomorphism for $\lambda \neq \lambda_0$ and $D(\lambda_0)$ is a Fredholm operator (obviously, of index zero). Denote by $G(\lambda)$ the inverse operator to $D(\lambda)$, $\lambda \neq \lambda_0$.

(b) Let $J\colon X \to X_1$ be a completely continuous linear embedding. We shall assume that, for $\lambda \neq \lambda_0$, the operator

$$J \circ G(\lambda) = G_0(\lambda) + \sum_{j=1}^{n} c_j(\lambda)\langle u_j(\lambda), \cdot\rangle w_j(\lambda),$$

where $G_0\colon U\setminus\{\lambda_0\} \to L(Y, X_1)$, $u_j\colon U\setminus\{\lambda_0\} \to Y^*$ and $w_j\colon U\setminus\{\lambda_0\} \to X_1$, $j = 1, \ldots, n$, has a continuous extension to U, and that the functions $c_j\colon U\setminus\{\lambda_0\} \to \mathbf{R}$, $j = 1, \ldots, n$, have infinite left-hand and right-hand limits as $\lambda \to \lambda_0$. It is evident that for $\lambda \neq \lambda_0$, $G_0(\lambda)$ is completely continuous as the difference of the completely continuous operator $J \circ G(\lambda)$ and a finite-dimensional operator, and $G_0(\lambda_0)$ $(= \lim_{\lambda \to \lambda_0} G_0(\lambda))$ is completely continuous by continuity of G_0 as a function of λ.

(c) Suppose that

$$\operatorname{Im} D(\lambda_0) = \bigcap_{j=1}^{n} \operatorname{Ker} u_j(\lambda_0),$$

the vectors $w_j(\lambda_0)$, $j = 1, \ldots, n$, are linearly independent and

$$J(\operatorname{Ker} D(\lambda_0)) = \operatorname{Lin}\{w_j(\lambda_0)\colon j = 1, \ldots, n\}.$$

Moreover, let

$$D(\lambda_0) \circ J^{-1} \circ G_0(\lambda_0) y = y \qquad \text{for } y \in \operatorname{Im} D(\lambda_0).$$

This means that the equation $D(\lambda_0)x = y$ has a solution if and only if $\langle u_j(\lambda_0), y\rangle = 0$, $j = 1, \ldots, n$, and then all its solutions are of the form

$$Jx = G_0(\lambda_0)y + \sum_{j=1}^{n} C_j w_j(\lambda_0),$$

where C_j, $j = 1, \ldots, n$, are arbitrary constants.

(d) We shall deal with the nonlinear equation $D(\lambda)J^{-1}x = F(x)$, where $F\colon X_1 \to Y$ is a nonlinear continuous mapping such that $F(X_1)$ is a bounded set. If $\lambda \neq \lambda_0$, then the above equation can be written in the form $x = J \circ G(\lambda)F(x)$. Since $J \circ G(\lambda)$ is completely continuous and $F(X_1)$ is bounded, this last equation is solvable by the Schauder Fixed Point Theorem. When $\lambda = \lambda_0$, the equation $D(\lambda_0)J^{-1}x = F(x)$ is equivalent to the system

$$x = G_0(\lambda_0)F(x) + \sum_{j=1}^{n} C_j w_j(\lambda_0),$$

$$\langle u_j(\lambda_0), F(x)\rangle = 0, \quad j = 1, \ldots, n,$$

where C_j, $j = 1, \ldots, n$, are arbitrary constants.

We need an assumption on the asymptotic behaviour of F on $J(\operatorname{Ker} D(\lambda_0))$. Denote by $[C_1, \ldots, C_n]$ the equivalence class of $(C_1, \ldots, C_n) \in \mathbf{R}^n \setminus \{0\}$ with respect to the relation

$$(C_1, \ldots, C_n) \sim (C'_1, \ldots, C'_n) \Leftrightarrow \exists_{\alpha > 0} \forall_j \; C_j = \alpha C'_j.$$

We shall identify $[C_1, \ldots, C_n]$ with a half-line in X_1 starting from the origin and passing through $\sum_j C_j w_j(\lambda_0)$. The set of all such half-lines covers $J(\operatorname{Ker} D(\lambda_0))$. Suppose that the limits

$$F_{[C_1, \ldots, C_n]} = \lim_{\alpha \to \infty} F(\alpha \sum_{j=1}^{n} C_j w_j(\lambda_0))$$

exist for any $[C_1, \ldots, C_n]$.

Now, we are able to formulate the main theorem.

2. MAIN THEOREM. *Under the above assumptions, if, for each $[C_1, \ldots, C_n]$, there exists j_1 $(1 \leqslant j_1 \leqslant n)$ such that*

$$C_{j_1} \langle u_{j_1}(\lambda_0), F_{[C_1, \ldots, C_n]} \rangle c_{j_1}^+ < 0,$$

where $c_{j_1}^+ = +1$ if $\lim_{\lambda \to \lambda_0^+} c_{j_1}(\lambda) = +\infty$ and $c_{j_1}^+ = -1$ if this limit equals $-\infty$, then the equation

$$D(\lambda_0) J^{-1} x = F(x)$$

has a solution. The same holds if the right-hand limits are replaced by the left-hand ones.

Proof. Suppose that the assumptions of the theorem in the version with the right-hand limits are satisfied, and that $\lambda_k \to \lambda_0^+$. Then, due to the Schauder Fixed Point Theorem, there are $x_k \in X_1$, $k \in \mathbf{N}$, such that

$$x_k = J \circ G(\lambda_k) F(x_k).$$

Assume first that the sequence (x_k) is bounded. Then

$$\sum_{j=1}^{n} c_j(\lambda_k) \langle u_j(\lambda_k), F(x_k) \rangle w_j(\lambda_k), \qquad k \in \mathbf{N},$$

is also bounded as the difference of bounded and relatively compact sequences. Since $w_j(\lambda_k)$, $j = 1, \ldots, n$, are linearly independent (at least for large k), the sequences of real numbers $c_j(\lambda_k) \langle u_j(\lambda_k), F(x_k) \rangle$, $k \in \mathbf{N}$, are bounded for $j = 1, \ldots, n$. Hence, we can pass to convergent subsequences:

$$G_0(\lambda_{k_l}) F(x_{k_l}) \to z \in X_1, \quad c_j(\lambda_{k_l}) \langle u_j(\lambda_{k_l}), F(x_{k_l}) \rangle \to C_j, \quad j = 1, \ldots, n.$$

It follows that $x_{k_l} \to x = z + \sum_{j=1}^{n} C_j w_j(\lambda_0)$. By the continuity of F and u_j and the unboundedness of $c_j(\lambda_{k_l})$, $l \in \mathbf{N}$,

$$\langle u_j(\lambda_0), F(x) \rangle = 0, \quad j = 1, \ldots, n.$$

Therefore

$$x = G_0(\lambda_0)F(x) + \sum_{j=1}^{n} C_j w_j(\lambda_0)$$

and x is a solution of our equation.

Now, we have to show that the sequence (x_k) cannot be unbounded. Suppose the contrary. Then passing to a subsequence (if necessary), we can assume that $\lim\|x_k\| = \infty$. Applying the preceding arguments to the bounded sequence $(\|x_k\|^{-1}x_k)$, we conclude that the sequences

$$c_j(\lambda_k)\|x_k\|^{-1}\langle u_j(\lambda_k),\ F(x_k)\rangle, \quad k \in N,$$

are bounded for each j. Passing to subsequences once more, we obtain

$$c_j(\lambda_{k_l})\|x_{k_l}\|^{-1}\langle u_j(\lambda_{k_l}),\ F(x_{k_l})\rangle \to C_j$$

as $l \to \infty$ for $j = 1, \ldots, n$. On the other hand, $\|x_k\|^{-1}G_0(\lambda_k)F(x_k) \to 0$, so

$$\|x_{k_l}\|^{-1}x_{k_l} \to \sum_{j=1}^{n} C_j w_j(\lambda_0).$$

Hence $F(x_{k_l}) \to F_{[C_1,\ldots,C_n]}$, which implies that

$$\|x_{k_l}\|\, C_j c_j(\lambda_{k_l})^{-1} \to \langle u_j(\lambda_0),\ F_{[C_1,\ldots,C_n]}\rangle$$

for $j = 1, \ldots, n$. But $c_j(\lambda_{k_l}) \to c_j^+ \cdot \infty$, which means that

$$C_j\langle u_j(\lambda_0),\ F_{[C_1,\ldots,C_n]}\rangle c_j^+ \geqslant 0$$

for each j, and this is false for $j = j_1$. The contradiction finishes the proof.

Now, we apply the theorem to boundary value problems for differential equations.

3. Application I. Let $X = \mathring{H}^2(\Omega)$ be the Sobolev space of all functions defined on an open bounded set $\Omega \subset \boldsymbol{R}^k$, having derivatives up to order 2 (in the distribution sense) in $L^2(\Omega)$ and vanishing on the boundary $\partial\Omega$. We assume that $\partial\Omega$ is such that the embedding $J\colon X \to L^2(\Omega) = X_1 = Y$ is completely continuous. Let $D(\lambda) = P - \lambda I$ where P is an elliptic, formally self-adjoint differential operator of order 2 on Ω, and I is the identity map. It is known that $D(\lambda)$ is a linear homeomorphism of X onto Y if λ is not an element of a sequence $\{\lambda_s\colon s = 0, 1, \ldots, n, \ldots\}$, and that the eigenspace corresponding to each eigenvalue λ_s is finite-dimensional. In (λ_s), the eigenvalues are repeated according to their multiplicities.

Choose an eigenfunction w_s corresponding to λ_s in such a way that all the w_s's which correspond to the same eigenvalue are orthonormal with respect to the L^2 scalar product. Due to the Hilbert–Schmidt theory, $\{w_s\colon s = 0, 1, 2, \ldots\}$ is a complete orthonormal system in $L^2(\Omega)$ and for $\lambda \neq \lambda_s$, $s = 0, 1, 2, \ldots$, the unique solution (in $\mathring{H}^2(\Omega)$) of the equation

$$Pu - \lambda u = y$$

is given by the formula

$$u = G(\lambda)y = \sum_{s=0}^{\infty} \frac{(w_s, y)}{\lambda-\lambda_s} w_s.$$

Suppose that λ_0 is a simple eigenvalue, i.e. $\lambda_0 \neq \lambda_s$ for $s \geqslant 1$. We can set in the Main Theorem

$$G_0(\lambda)y = \sum_{s=1}^{\infty} \frac{(w_s, y)}{\lambda-\lambda_s} w_s,$$

$$u_0(\lambda) = (w_0, \cdot), \quad w_0(\lambda) = w_0, \quad c_0(\lambda) = (\lambda-\lambda_0)^{-1}.$$

Let

$$F(u)(x) = f(u, x)$$

where $f\colon \boldsymbol{R} \times \bar{\Omega} \to \boldsymbol{R}$ is a continuous function such that the limits

$$\lim_{u \to -\infty} f(u, x) = f_-(x), \quad \lim_{u \to +\infty} f(u, x) = f_+(x)$$

exist in the L^2-sense where $f_+, f_- \in L^2(\Omega)$.

We look for a solution of the equation $Pu - \lambda_0 u = f(u, x)$ in $\mathring{H}^2(\Omega)$. The following theorem of Landesman and Lazer is a consequence of our result.

THEOREM ([4]). *If the numbers*

$$\alpha = \int_{\{x:\, w_0(x) > 0\}} f_+ w_0 + \int_{\{x:\, w_0(x) < 0\}} f_- w_0,$$

$$\beta = \int_{\{x:\, w_0(x) > 0\}} f_- w_0 + \int_{\{x:\, w_0(x) < 0\}} f_+ w_0$$

have opposite sings then the equation $Pu - \lambda_0 u = f(u, x)$ has a solution in $\mathring{H}^2(\Omega)$.

Proof. When the eigenspace is one-dimensional (as in this case), there are only two half-lines $[+1]$ and $[-1]$. The corresponding limits $F_{[+1]}$ and $F_{[-1]}$ are α and β, respectively. If $\alpha < 0 < \beta$, then we take the right-hand limits $\lambda \to \lambda_0^+$ in the Main Theorem and obtain

$$+|C_1| \alpha \operatorname{sgn}(+\infty) < 0, \quad -|C_1| \beta \operatorname{sgn}(+\infty) < 0.$$

If $\beta < 0 < \alpha$, we take $\lambda \to \lambda_0^-$ to get the assertion. The other assumptions of the Main Theorem are trivially satisfied. ■

One can also study the case of multidimensional eigenspaces by our method. For simplicity, consider the two-dimensional case, and let v_0, w_0 span the eigenspace corresponding to λ_0. Then

$$F_{[C,D]}(x) = \begin{cases} f_+(x) & \text{if } Cv_0(x) + Dw_0(x) > 0, \\ f_-(x) & \text{if } Cv_0(x) + Dw_0(x) < 0. \end{cases}$$

have opposite sings and $(v_0, F_{[C,0]}) \neq 0 \neq (w_0, F_{[0,D]})$. Then the considered equation has a solution in $\mathring{H}^2(\Omega)$. The proof is quite similar to the previous one.

4. Application II. Now, we pass to boundary value problems for ordinary differential equations. One can construct a Green function for such a problem explicitly, even when the operator is not self-adjoint and has an arbitrary order. For this reason we consider the case of ordinary differential equations separately.

Let $a_0, \ldots, a_{m-1}\colon [a, b] \times U \to \boldsymbol{R}$ be continuous functions which depend analytically on $\lambda \in U$ and let

$$D(\lambda)x = \frac{d^m x}{dt^m} + a_{m-1}(t, \lambda)\frac{d^{m-1}x}{dt^{m-1}} + \ldots + a_0(t, \lambda)x.$$

Suppose that $B_1, \ldots, B_m$ are linearly independent functionals of the form

$$B_i(x) = \sum_{j=0}^{m-1}\left[\alpha_{ij}\frac{d^j x(a)}{dt^j} + \beta_{ij}\frac{d^j x(b)}{dt^j}\right].$$

We write shortly $B = (B_1, \ldots, B_m)$. The operators $D(\lambda)$ will be considered on the space X consisting of all C^m-functions x defined on $[a, b]$ such that $Bx = 0$. Let $X_1 = Y = C([a, b])$.

We construct the Green function for the BVP

$$D(\lambda)x = y, \qquad Bx = 0$$

following [3]. Let $\varphi_j(\lambda)$, $j = 1, \ldots, m$, be a C^m-function satisfying the differential equation $D(\lambda)x = 0$ and the initial conditions $\varphi_j^{(i-1)}(\lambda)(a) = l_{ij}$ where $[l_{ij}]_{i,j\leqslant m}$ is an invertible matrix that we shall choose later. The functions $\varphi_1(\lambda), \ldots, \varphi_m(\lambda)$ form a fundamental system of solutions of the homogeneous linear equation for each $\lambda \in U$, and they are analytic functions of λ (see [7]). Hence the matrix $[B_i\varphi_j]$ is analytic, and so is its determinant. We consider the case when λ_0 is an isolated zero of this determinant. This means that the problem $D(\lambda)x = y$, $Bx = 0$ has a unique solution for $\lambda \neq \lambda_0$ and, for $\lambda = \lambda_0$, the homogeneous problem ($y = 0$) has an n-dimensional space of solutions, $1 \leqslant n < m$. Let $n = 1$ for simplicity. Choosing the appropriate matrix $[l_{ij}]$, we can assume that $B_i\varphi_1(\lambda_0) = 0$ for $i = 1, \ldots, m$.

We also need a less natural assumption. For ψ being an analytic function of λ, denote by $\mu(\psi, \lambda_0)$ the multiplicity of the zero of ψ at λ_0; if $\psi(\lambda_0) \neq 0$, then we put $\mu(\psi, \lambda_0) = 0$. Suppose that

$$(*) \qquad \mu(\det[B_i\varphi_j], \lambda_0) = \mu(B_i\varphi_1, \lambda_0)$$

for any $i \leqslant m$.

Define $K\colon [a, b]^2 \times U \to \boldsymbol{R}$ by

$$K(t, s; \lambda) = 0 \qquad \text{for } s > t,$$

$$K(t, s; \lambda) = W(s; \lambda)^{-1} \begin{vmatrix} \varphi_1(\lambda, s) & \dots & \varphi_m(\lambda, s) \\ \dots & \dots & \dots \\ \varphi_1^{(m-2)}(\lambda, s) & \dots & \varphi_m^{(m-2)}(\lambda, s) \\ \varphi_1(\lambda, t) & \dots & \varphi_m(\lambda, t) \end{vmatrix} \quad \text{for } s \leqslant t,$$

where W is the Wrońskian of the fundamental system. If $\lambda \neq \lambda_0$ and $y \in C([a, b])$, then the function

$$\tilde{\psi}(t) = \int_a^b K(t, s; \lambda) y(s) ds$$

satisfies the differential equation $D(\lambda)\tilde{\psi} = y$. Put

$$\tilde{G}(t, s; \lambda) = K(t, s; \lambda) + \sum_{j=1}^{m} h_j(\lambda, s)\varphi_j(\lambda, t)$$

for $\lambda \neq \lambda_0$. The scalar functions h_j will be chosen in such a way that

$$\psi(t) = \int_a^b \tilde{G}(t, s; \lambda) y(s) ds$$

satisfies not only the differential equation $D(\lambda)\psi = y$, but also the condition $B\psi = 0$. This is equivalent to the system of algebraic linear equations

$$\sum_{j=1}^{m} h_j(\lambda, s) B_i \varphi_j(\lambda) = B_i K(\cdot, s; \lambda), \qquad i \leqslant m.$$

Hence, by the Cramer formulas,

$$h_j(\lambda, s) = \left(\det[B_i\varphi_j(\lambda)]\right)^{-1} \sum_{i=1}^{m} (-1)^{i+j+1} \det \tilde{B}_{ij}(\lambda) \times B_i\left(K(\cdot, s; \lambda)\right),$$

where $\tilde{B}_{ij}(\lambda)$ is the matrix obtained from $[B_i\varphi_j(\lambda)]$ by deleting the i-th row and the j-th column. By (*), h_j has a finite limit as $\lambda \to \lambda_0$ for $j > 1$. On the other hand, h_1 does not admit a continuous extension to the whole neighbourhood of λ_0. Thus, one can put in the Main Theorem

$$c_1(\lambda) = \left(\det[B_i\varphi_j(\lambda)]\right)^{-1},$$

$$\langle u_1(\lambda), y\rangle = \int_a^b c_1(\lambda)^{-1} h_1(\lambda, s) y(s) ds,$$

$$w_1(\lambda) = \varphi_1(\lambda).$$

Moreover,

$$G_0(\lambda) y = \int_a^b \tilde{G}(\cdot, s; \lambda) y(s) ds - c_1(\lambda)\langle u_1(\lambda), y\rangle w_1(\lambda).$$

Let $F\colon X_1 \to Y$ be the superposition operator

$$F(x)(t) = f\left(x(t), t\right),$$

where $f\colon \boldsymbol{R}\times[a, b]\to\boldsymbol{R}$ is a bounded continuous function such that the uniform limits

$$\lim_{x\to+\infty} f(x, t) = f_+(t), \qquad \lim_{x\to-\infty} f(x, t) = f_-(t)$$

exist. The limits $\lim_{p\to\pm\infty} F(pw_1(\lambda_0)) = F_{[\pm 1]}$ do not exist in Y, in general. If they exist, they should have the form

$$F_{[+1]}(t) = \begin{cases} f_+(t) & \text{if } w_1(\lambda_0, t) > 0, \\ f_-(t) & \text{if } w_1(\lambda_0, t) < 0, \end{cases}$$

$$F_{[-1]}(t) = \begin{cases} f_-(t) & \text{if } w_1(\lambda_0, t) > 0, \\ f_+(t) & \text{if } w_1(\lambda_0, t) < 0. \end{cases}$$

The functional $u_1(\lambda_0)$ given by an integral can act on the discontinuous functions $E_{[\pm 1]}$ as well. We have

THEOREM. *Under the above assumptions, if the numbers* $\langle u_1(\lambda_0), F_{[+1]}\rangle$ *and* $\langle u_1(\lambda_0), F_{[-1]}\rangle$ *have opposite signs, then the problem*

$$D(\lambda_0)x = F(x), \qquad Bx = 0$$

has a solution in the classical sense.

Proof. The problem will be solved if we show that there exists a continuous function x such that

$$x(t) = \int_a^b \tilde{G}_0(t, s; \lambda_0)F(x(s))ds + Cw_1(\lambda_0, t),$$

$$\langle u_1(\lambda_0), F(x)\rangle = \int_a^b U_1(\lambda_0; s)F(x(s))ds = 0,$$

where $\tilde{G}_0$ is the kernel of the integral operator $G_0(\lambda_0) = \lim_{\lambda\to\lambda_0} G_0(\lambda)$ and U_1 is the kernel of the functional u_1. The zeros of $w_1(\lambda_0)$ are isolated, so we can cut out of $[a, b]$ a small neighbourhood V of the set of zeros in such a way that the numbers

$$\int_{[a,b]\setminus V} U_1(\lambda_0; s)F_{[+1]}(s)ds, \qquad \int_{[a,b]\setminus V} U_1(\lambda_0; s)F_{[-1]}(s)ds$$

have the same signs as $\langle u_1(\lambda_0), F_{[+1]}\rangle$ and $\langle u_1(\lambda_0), F_{[-1]}\rangle$ respectively.

Take a sequence $(V_k)_{k\in\boldsymbol{N}}$ of such neighbourhoods so that their intersection consists of the zeros of $w_1(\lambda_0)$ only. The arguments from the proof of the Main Theorem can be applied to give the existence of a solution of

$$x_k(t) = \int_{[a,b]\setminus V_k} \tilde{G}_0(t, s; \lambda_0)F(x_k(s))ds + C_k w_1(\lambda_0, t),$$

$$\int_{[a,b]\setminus V_k} U_1(\lambda_0; s)F(x_k(s))ds = 0.$$

The sequence (C_k) cannot be unbounded. In fact, if $C_k \to +\infty$ then $C_k^{-1} x_k \to w_1$ and, therefore, $F(x_k) \to F_{[+1]}$ pointwise, so $\langle u_1(\lambda_0), F_{[+1]} \rangle = 0$. The case $C_k \to -\infty$ is similar.

Since (C_k) is bounded, we can pass to subsequences and get $x_{k_l} \to x$, $C_{k_l} \to C$. Obviously,

$$x = G_0(\lambda_0)F(x) + Cw_1(\lambda_0), \quad \langle u_1(\lambda_0), F(x) \rangle = 0,$$

thus x is a solution of the problem considered. ∎

The above considerations become more and more complicated when the dimension of the eigenspace corresponding to λ_0 increases. However, it is possible to get similar results in many special cases.

5. Final remarks. Our method works also for ordinary differential equations in $\boldsymbol{R}^m$.

The assumption that λ_0 is an isolated singular point has not been necessary for the proof of the Main Theorem. In fact, we have only needed to know that λ_0 is a boundary point of the set of regular points. This encourages us to apply our method to boundary value problems for differential equations in unbounded domains. Such problems possess continuous spectra. However, there is an amount of other difficulties: the choice of appropriate spaces and the jumping change of Green functions as $\lambda \to \lambda_0$, for example. These difficulties have not been overcome yet.

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