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Area methods, extremal problems and extremal domains for pairs of conformal mappings

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Abstract. In the classes of pairs $\{F, G\}$ of functions F and G analytic and univalent in the unit disc and satisfying the condition $F(z)G(\zeta) \neq 1$ there (generating, in the special case, bounded, Bieberbach–Eilenberg and Grunsky–Shah functions) or the condition $F(z)+G(\zeta) \neq 0$ (generating, in the special case, Gel'fer functions), by applying coefficient inequalities of Grunsky–Nehari type, a characterization of extremal pairs maximizing functionals dependent on the derivatives $F^{(n)}$ and $G^{(n)}$, $n = 2, 3$, is given.

1. Introduction, notations and statement of the results. The present paper deals with the class C of pairs $\{F, G\}$ of functions

$$F(z) = a+a_1+a_2 z^2+ \dots, \qquad G(z) = b+b_1 z+b_2 z^2+ \dots$$

univalent in $\Delta = \{z: |z| < 1\}$ and satisfying the condition

$$F(z)G(\zeta) \neq 1$$

for all z, ζ belonging to Δ, and the class D of pairs $\{F, G\}$ of functions

$$F(z) = 1+2a_1 z+2a_2 z^2+ \dots, \qquad G(z) = 1+2b_1 z+2b_2 z^2+ \dots$$

univalent in Δ and satisfying the condition

$$F(z)+G(\zeta) \neq 0$$

for all z, ζ belonging to Δ. These classes, some other classes to them as well as some their subclasses have been intensely investigated since 1934 (cf. e.g., [1]–[3] and [5]–[12]).

Let $\hat{K}$, K, L_ζ, P and Q stand for mappings defined as follows:

$$\hat{K}(z) = [z/(z-1)^2][1+ab-\sqrt{ab}(z+1/z)], \qquad K(z) = z/(z-1)^2,$$

$$L_\zeta(z) = z/[(z-\zeta)(z+\bar{\zeta})], \qquad P(z) = (z-1)/(z+1), \qquad Q(z) = 1/z-z,$$

and let

$$A_n = a_n/a_1, \qquad B_n = b_n/b_1, \qquad n = 2, 3.$$

If z_1, z_2 and $\{F, G\}$ belong to Δ and C, respectively, let

$$\lambda(a, b, z_1, z_2, F, G)$$

$$= \left[\frac{F''(z_1)}{F'(z_1)} + 2\frac{G(z_2)+2\sqrt{b/a}}{1-F(z_1)G(z_2)}F'(z_1)\right](1-|z_1|^2)-2\bar{z}_1, \qquad a, b \neq 0,$$

$$= \left[\frac{F''(z_1)}{F'(z_1)} + 2\frac{G(z_2)+2}{1-F(z_1)G(z_2)}F'(z_1)\right](1-|z_1|^2)-2\bar{z}_1, \qquad a = b = 0,$$

$$v(a, b, z_1, z_2, F, G) = \frac{1-F(z_1)G(z_2)}{b(1-ab)F'(z_1)(1-|z_1|^2)}, \qquad a, b \neq 0,$$

$$= \frac{1-F(z_1)G(z_2)}{F'(z_1)(1-|z_1|^2)}, \qquad a = b = 0,$$

and, if z_1, z_2 and $\{F, G\}$ belong to Δ and D, respectively, let

$$\varphi(z_1, z_2, F, G) = \left[\frac{F''(z_1)}{F'(z_1)} + 4\frac{F'(z_1)}{F(z_1)+G(z_2)}\right](1-|z_1|^2)-2\bar{z}_1,$$

$$\chi(z_1, z_2, F, G) = \left[\{F; z_1\} + 24\frac{F'^2(z_1)}{[F(z_1)+G(z_2)]^2}\right](1-|z_1|^2)^2,$$

$$\psi(z_1, z_2, F, G) = \frac{F(z_1)+G(z_2)}{F'(z_1)(1-z_1^2)},$$

where

$$\{F; z_1\} = \frac{F'''(z_1)}{F'(z_1)} - \frac{3}{2}\left(\frac{F''(z_1)}{F'(z_1)}\right)^2.$$

In this paper we investigate a problem of finding the sharp estimate of the functionals

$$\lambda = |\lambda(a, b, z_1, z_2, F, G)v(a, b, z_1, z_2, F, G)| + |\lambda(b, a, z_2, z_1, G, F)v(b, a, z_2, z_1, G, F)|,$$

where $\{F, G\}$ ranges over C, and

$$\varphi = |\varphi(z_1, z_2, F, G)\psi(z_1, z_2, F, G)| + |\varphi(z_2, z_1, G, F)\psi(z_2, z_1, G, F)|,$$

$$\chi = |\chi(z_1, z_2, F, G)\psi^2(z_1, z_2, F, G)| + |\chi(z_2, z_1, G, F)\psi^2(z_2, z_1, G, F)|,$$

where $\{F, G\}$ ranges over D.

For $\{F^*, G^*\}$ belonging to C, we define

$$F^{**}(z) = \frac{a[1-f(0)g(0)]-(1-ab)f(0)+[1-abf(0)g(0)]f(z)}{1-abf(0)g(0)+\{b[1-f(0)g(0)]-(1-ab)g(0)\}f(z)},$$

(1.1)

$$f(z) = \frac{F^* \circ p(z)-a}{1-bF^* \circ p(z)},$$

$$G^{**}(z) = \frac{b[1-f(0)g(0)]-(1-ab)g(0)+[1-abf(0)g(0)]g(z)}{1-abf(0)g(0)+\{a[1-f(0)g(0)]-(1-ab)f(0)\}g(z)},$$

(1.1′)

$$g(z) = \frac{G^* \circ q(z)-b}{1-aG^* \circ q(z)},$$

and for $\{F^*, G^*\}$ belonging to D, we define

$$F^{**}(z) = \frac{[1-f(0)g(0)][1-f(z)]}{1+f(0)g(0)-2f(0)+[1+f(0)g(0)-2g(0)]f(z)},$$

$$f(z) = \frac{F^* \circ p(z)-1}{F^* \circ p(z)+1},$$

(1.2)

$$G^{**}(z) = \frac{[1-f(0)g(0)][1-g(z)]}{1+f(0)g(0)-2g(0)+[1+f(0)g(0)-2f(0)]g(z)},$$

$$g(z) = \frac{G^* \circ q(z)-1}{G^* \circ q(z)+1},$$

where

$$p(z) = (z-z_1)/(1-\bar{z}_1 z), \qquad q(z) = (z-z_2)/(1-\bar{z}_2 z).$$

The results obtained in this paper will be formulated as follows.

THEOREM 1. *If $\{F, G\}$ belongs to C, then the corresponding λ does not exceed*

$$\lambda^* = 4\big(|v(a, b, z_1, z_2, F, G)| + |v(b, a, z_2, z_1, G, F)|\big).$$

The estimate is sharp for each a and b such that $ab \geqslant 0$. All the extremal pairs are given by formulae (1.1) *and* (1.1′) *with the functions F^*, G^* satisfying the equations* (Figure 1)

$$F^*(z) = \sqrt{\frac{a}{b}}\hat{K}^{-1}\left[\frac{ba_1}{\sqrt{ab}}\frac{1+\sqrt{ab}}{1-\sqrt{ab}}K(e^{\alpha i}z)\right],$$

(1.3)

$$G^*(z) = \sqrt{\frac{b}{a}}\hat{K}^{-1}\left[\frac{ab_1}{\sqrt{ab}}\frac{1+\sqrt{ab}}{1-\sqrt{ab}}K(e^{\beta i}z)\right]$$

when $ab > 0$, where $\operatorname{sgn}\{ba_1\} = \operatorname{sgn}\{ab_1\} = \operatorname{sgn}\{1-\sqrt{ab}\}$, $-\pi < \alpha,\ \beta \leqslant \pi$, *and the equations*

$$F^*(z) = K^{-1}[a_1 K(e^{\alpha i}z)], \qquad G^*(z) = K^{-1}[b_1 K(e^{\beta i}z)]$$

when $a = b = 0$, where $a_1 > 0$, $b_1 > 0$, $-\pi < \alpha, \beta \leqslant \pi$.

THEOREM 2. *If $\{F, G\}$ belongs to D, then the corresponding φ does not exceed*

$$\varphi^* = 2\big(|\psi(z_1, z_2, F, G)| + |\psi(z_2, z_1, G, F)|\big).$$

The estimate is sharp. All the extremal pairs are given by formulae (1.2), *with the functions* F^*, G^* *satisfying the equations* (Figure 2)

(1.4) $$F^*(z) = P^{-1}\{K^1[a_1 K(e^{\alpha i} z)]\}, \quad G^*(z) = P^{-1}\{K^{-1}[b_1 K(e^{\beta i} z)]\},$$

where $a_1 > 0$, $b_1 > 0$, $-\pi < \alpha, \beta \leqslant \pi$.

THEOREM 3. *If* $\{F, G\}$ *belongs to* D, *then the corresponding* χ *does not exceed*

$$\chi^* = 6(|\psi^2(z_1, z_2, F, G)| + |\psi^2(z_2, z_1, G, F)|).$$

The estimate is sharp. All the extremal pairs are given by formulae (1.2), *with that the functions* F^*, G^* *satisfy the equations* (Figure 3)

(1.5) $$F^*(z) = Q^{-1}[4a_1 L_{\zeta_1}(e^{\alpha i} z)], \quad G^*(z) = Q^{-1}[4b_1 L_{\zeta_2}(e^{\beta i} z)],$$

where $a_1 > 0$, $b_1 > 0$, $-\pi < \alpha, \beta \leqslant \pi$, $|\zeta_1| = |\zeta_2| = 1$,

$$A_2 - a_1 = -2\operatorname{Im}\{\zeta_1\} i, \quad |\operatorname{Im}\{\zeta_1\}| \leqslant 1 - a_1,$$

$$B_2 - b_1 = -2\operatorname{Im}\{\zeta_2\} i, \quad |\operatorname{Im}\{\zeta_2\}| \leqslant 1 - b_1.$$

Thus, for every a_1, b_1, *the functions* F^*, G^* *belong to one-parameter families, where the parameters are* a_2, b_2 *or* ζ_1, ζ_2, *respectively.*

The paper ends with a few applications and remarks. In particular, we obtain the sharp estimate for bounded, Bieberbach–Eilenberg, Grunsky–Shah and Gel'fer functions.

2. Proofs. If g_k, $k = 1, 2$, are mappings defined by the formulae

(2.1) $$g_1(w) = \frac{w}{a(1-bw)} - \frac{1}{b(w-a)}, \quad a, b \neq 0,$$
$$= w - 1/w, \quad a = b = 0,$$

when $\{F, G\}$ belongs to C, and

(2.2) $$g_2(w) = -\frac{1}{w-1} - \frac{1}{w+1},$$

when $\{F, G\}$ belongs to D, and if

(2.3) $$\sum_{q=-\infty}^{\infty} c_q^1 z^q = g_1[F(z)], \quad \sum_{q=-\infty}^{\infty} c_q^2 z^q = g_1[1/G(z)],$$
$$= g_2[F(z)], \quad = g_2[-G(z)],$$

where $0 < |z| < 1$, then

(2.4) $$\sum_{q=-\infty}^{\infty} q(|c_q^1|^2 + |c_q^2|^2) \leqslant 0$$

in virtue of (2.4), (2.6), (2.11) and (2.7) of paper [15]. In particular,

$$|c_1^1|^2+|c_1^2|^2 \leqslant |c_{-1}^1|^2+|c_{-1}^2|^2, \tag{2.5}$$

and the equality holds if and only if

$$c_q^1 = c_q^2 = 0, \quad q = 2, 3, \ldots \tag{2.6}$$

and, if $-\pi < \varepsilon, \tau \leqslant \pi$, then

$$\operatorname{Re}\{c_1^1 c_{-1}^1 \exp(\varepsilon i)+c_1^2 c_{-1}^2 \exp(\tau i)\} \leqslant |c_{-1}^1|^2+|c_{-1}^2|^2 \tag{2.7}$$

in virtue of the Cauchy–Schwarz inequality and (2.5), with the equality holding if and only if

$$c_1^1 = \bar{c}_{-1}^1 \exp(-\varepsilon i), \quad c_1^2 = \bar{c}_{-1}^2 \exp(-\tau i). \tag{2.8}$$

Proof of Theorem 1. For $ab \neq 0$, from (2.1) and (2.3) we have

$$\begin{aligned} c_1^1 &= \alpha_3/(ba_1), & c_{-1}^1 &= -1/(ba_1), & c_0^1 &= \alpha_2/(ba_1), \\ c_1^2 &= -\beta_3/(ab_1), & c_{-1}^2 &= 1/(ab_1), & c_0^2 &= -\beta_2/(ab_1), \end{aligned} \tag{2.9}$$

where

$$\alpha_3 = A_3-A_2^2+\frac{b}{a}\frac{a_1^2}{(1-ab)^2}, \quad \beta_3 = B_3-B_2^2+\frac{a}{b}\frac{b_1^2}{(1-ab)^2}.$$

Since $\{F, G\}$ belongs to C if and only if $\{\hat{F}, \hat{G}\}$ belongs to C, where

$$\begin{aligned} \hat{F}(z) &= \frac{\sqrt{a}+f(z)}{1+\sqrt{b}\,f(z)}, & f(z) &= \left[\frac{F(z^2)-a}{1-bF(z^2)}\right]^{1/2}, \\ \hat{G}(z) &= \frac{\sqrt{b}+g(z)}{1+\sqrt{a}\,g(z)}, & g(z) &= \left[\frac{G(z^2)-b}{1-aG(z^2)}\right]^{1/2}, \end{aligned} \tag{2.10}$$

therefore, if $\hat{F}(z) = \hat{a}+\hat{a}_1 z+\hat{a}_2 z^2+\ldots$, $\hat{G}(z) = \hat{b}+\hat{b}_1 z+\hat{b}_2 z^2+\ldots$, then it follows from (2.7) and (2.9) that

$$-\operatorname{Re}\left\{e^{\varepsilon i}\frac{\hat{\alpha}_3}{(\hat{b}\hat{a}_1)^2}+e^{\tau i}\frac{\hat{\beta}_3}{(\hat{a}\hat{b}_1)^2}\right\} \leqslant \frac{1}{|\hat{b}\hat{a}_1|^2}+\frac{1}{|\hat{a}\hat{b}_1|^2}. \tag{2.11}$$

In consequence, from this, (2.10), and because ε, τ are arbitrary we get that

$$\left|A_2+\frac{(b+2\sqrt{b/a})\,a_1}{1-ab}\right||ab_1|+\left|B_2+\frac{(a+2\sqrt{a/b})\,b_1}{1-ab}\right||ba_1| \leqslant 2(|ba_1|+|ab_1|). \tag{2.12}$$

Now, note that in (2.11), along with the extremal pair $\{\hat{F}^*, \hat{G}^*\}$, also the pairs $\{\hat{F}^*_\alpha, \hat{G}^*_\beta\}$, $-\pi < \alpha, \beta \leqslant \pi$, $\hat{F}^*_\alpha = \hat{F}^*(e^{\alpha i}z)$, $\hat{G}^*_\beta(z) = \hat{G}^*(e^{\beta i}z)$ are extremal. So, assume that $\{\hat{F}^*, \hat{G}^*\}$ is an extremal pair for which $\hat{b}\hat{a}_1 > 0$, $\hat{a}\hat{b}_1 > 0$, and let $\varepsilon = \tau = \pi$. Then, by (2.1), (2.3), (2.6), (2.8) and (2.9),

$$\hat{g}_1 \circ \hat{F}^*(z) = \frac{1}{\hat{b}\hat{a}_1}\frac{z^2-1}{z}, \qquad -\hat{g}_1 \circ \frac{1}{\hat{G}^*(z)} = \frac{1}{\hat{a}\hat{b}_1}\frac{z^2-1}{z},$$

whence, by squaring the above equalities and replacing z^2 by z in them, we obtain (1.3).

Moreover, let observe that $\{F^{**}, G^{**}\}$ belongs to C if and only if $\{F^*, G^*\}$ belongs to C, with

$$F^*(z) = \frac{a+f(z)}{1+bf(z)}, \quad f(z) = \frac{F^{**}\circ p^{-1}(z) - F^{**}(z_1)}{1-F^{**}\circ p^{-1}(z)G^{**}(z_2)},$$

$$G^*(z) = \frac{b+g(z)}{1+ag(z)}, \quad g(z) = \frac{G^{**}\circ q^{-1}(z) - G^{**}(z_2)}{1-G^{**}\circ q^{-1}(z)F^{**}(z_1)},$$

which is equivalent to (1.1). From this and (2.12) the inequality $\lambda \leqslant \lambda^*$ follows.

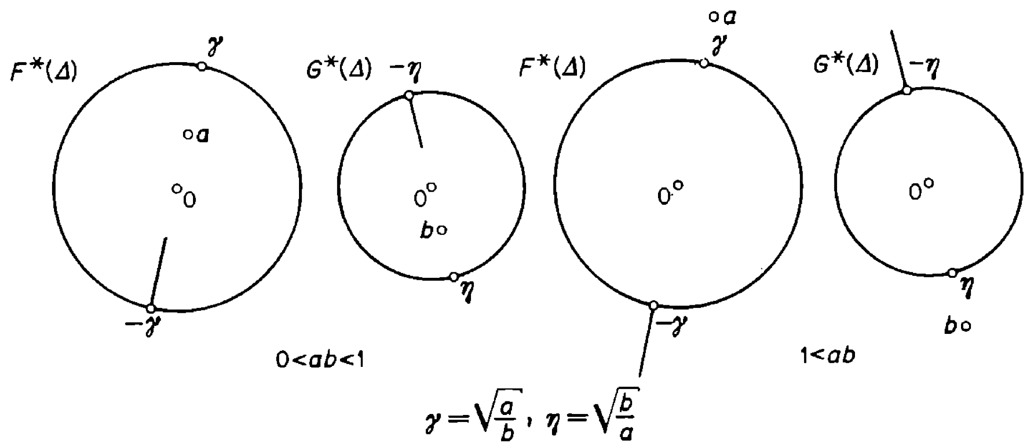


Fig. 1

The case $a = b = 0$ is proved in an analogous way.

Proof of Theorem 2. From (2.2) and (2.3) we have

$$\text{(2.13)}\quad \begin{aligned} c_1^1 &= (A_3 - A_2^2 + a_1^2)/(2a_1), & c_{-1}^1 &= -1/(2a_1), & c_0^1 &= (A_2 - a_1)/(2a_1),\\ c_1^2 &= -(B_3 - B_2^2 + b_1^2)/(2b_1), & c_{-1}^2 &= 1/(2b_1), & c_0^2 &= -(B_2 - b_1)/(2b_1), \end{aligned}$$

and since $\{F, G\}$ belongs to D if and only if $\{\hat{F}, \hat{G}\}$ belongs to D, where

$$\hat{F}(z) = \frac{1+f(z)}{1-f(z)}, \quad f(z) = \left[\frac{F(z^2)-1}{F(z^2)+1}\right]^{1/2},$$
$$\hat{G}(z) = \frac{1+g(z)}{1-g(z)}, \quad g(z) = \left[\frac{G(z^2)-1}{G(z^2)+1}\right]^{1/2}, \tag{2.14}$$

therefore, if $\hat{F}(z) = 1+2\hat{a}_1 z + 2\hat{a}_2 z^2 + \ldots$, $\hat{G}(z) = 1+2\hat{b}_1 z + 2\hat{b}_2 z^2 + \ldots$, then from (2.7) and (2.13) it follows that

$$-\operatorname{Re}\left\{e^{\varepsilon i}\frac{\hat{A}_3-\hat{A}_2^2+\hat{a}_1^2}{\hat{a}_1^2} + e^{\tau i}\frac{\hat{B}_3-\hat{B}_2^2+\hat{b}_1^2}{\hat{b}_1^2}\right\} \leqslant \frac{1}{|\hat{a}_1|^2}+\frac{1}{|\hat{b}_1|^2}. \tag{2.15}$$

From this and (2.14) we obtain the inequality

$$|(A_2+a_1)b_1| + |(B_2+b_1)a_1| \leqslant 2(|a_1|+|b_1|). \tag{2.16}$$

Assume that $\{\hat{F}^*, \hat{G}^*\}$ is an extremal pair in (2.15) with $\hat{a}_1 > 0$, $\hat{b}_1 > 0$, $\varepsilon = \tau = \pi$. Then it follows from (2.1), (2.3), (2.6), (2.8) and (2.13) that

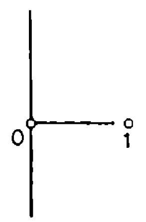
$$g_2[\hat{F}^*(z)] = \frac{1}{2\sqrt{a_1}}\frac{1-z^2}{z}, \qquad -g_2[-\hat{G}^*(z)] = \frac{1}{2\sqrt{b_1}}\frac{1-z^2}{z}.$$

By squaring the above equations and replacing z^2 by z in them, we immediately obtain (1.4).

Since $\{F^{**}, G^{**}\}$ belongs to D if and only if $\{F^*, G^*\}$ belongs to D, where

$$F^*(z) = \frac{1+f(z)}{1-f(z)}, \quad f(z) = \frac{F^{**}\circ p^{-1}(z) - F^{**}(z_1)}{F^{**}\circ p^{-1}(z) + G^{**}(z_2)},$$
$$G^*(z) = \frac{1+g(z)}{1-g(z)}, \quad g(z) = \frac{G^{**}\circ q^{-1}(z) - G^{**}(z_2)}{G^{**}\circ q^{-1}(z) + F^{**}(z_1)}, \tag{2.17}$$

which is equivalent to (1.2), therefore, by applying (1.2) to (2.16), we obtain the inequality $\varphi \leqslant \varphi^*$.



$F^*(\Delta)$, $G^*(\Delta)$

Fig. 2

Proof of Theorem 3. It follows from (2.15) that

$$|(A_3 - A_2^2 + a_1^2)b_1^2| + |(B_3 - B_2^2 + b_1^2)a_1^2| \leqslant |a_1|^2 + |b_1|^2 \tag{2.18}$$

if $\{F, G\}$ belongs to D; by Theorem 9 [16], the equality in (2.18) takes place only for pairs satisfying equations (1.5). The inequality $\chi \leqslant \chi^*$ follows from (2.18) after applying (2.17) and (1.2).

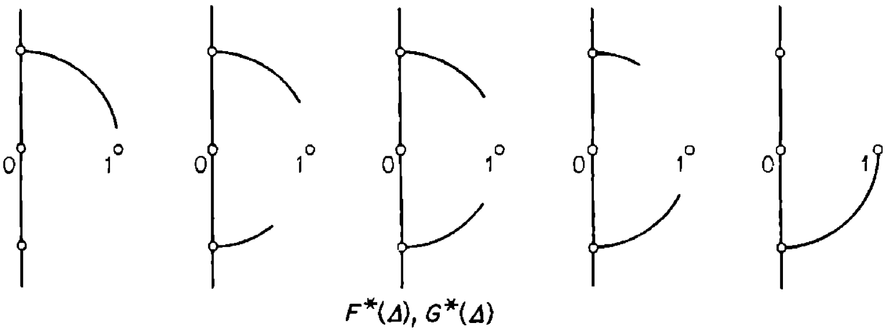


Fig. 3

3. Conclusions. Let S_1 denote the class of all functions $F(z) = b + b_1 z + \ldots + b_n z^n + \ldots$ univalent in Δ and such that $|F(z)| < 1$ for z belonging to Δ. From Theorem 1, if $G = \bar{F}$, $\bar{F}(z) = \overline{F(\bar{z})}$ and $z_1 = \bar{z}_2 = z_0$, we obtain at once the sharp result and all extremal functions in the class S_1.

The function $F(z) = b_1 z + b_2 z^2 + \ldots$, univalent in Δ, is called a *Bieberbach–Eilenberg function*, respectively a *Grunsky–Shah function*, if $F(z)F(\zeta) \neq 1$, respectively $F(z)\overline{F(\zeta)} \neq -1$, for all z, ζ belonging to Δ. From Theorem 1, if $G = F$ and $z_1 = z_2 = z_0$, respectively, $G = -\bar{F}$ and $z_1 = \bar{z}_2 = z_0$, we obtain the sharp estimate and all extremal functions in the class of Bieberbach–Eilenberg functions, and in the class of Grunsky–Shah functions, respectively.

By $S(b_1)$ we denote a subclass of the class S_1 of those functions F for which $b = 0$, $0 < b_1 \leqslant 1$, b_1 – fixed. Since, if F belongs to $S(b_1)$, then also F_α belongs to $S(b_1)$, where $F_\alpha(z) = \exp(-\alpha i) F[\exp(\alpha i) z]$, $-\pi < \alpha \leqslant \pi$, therefore, the well-known sharp estimate in the class $S(B_1)$ now follows from Theorem 1 (see e.g., [13], [14]): $|b_2| \leqslant 2b_1(1-b_1)$.

The function $F(z) = 1 + 2a_1 z + 2a_2 z^2 + \ldots$, univalent in Δ is said to belong to the Gel'fer class if $F(z) + F(\zeta) \neq 0$ for z, ζ belonging to Δ. From Theorem 2 and from the estimate $\chi \leqslant \chi^*$ when $G = F$ and $z_1 = z_2 = z_0$ we obtain the results of Grinšpan and Kolomojceva [4] and Gel'fer [3].

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