ANNALES
POLONICI MATHEMATICI
XL (1981)

On curves invariant under an axial transform

by D. BRYDAK (Kraków)

Abstract. The theorem about the existence of continuous and strictly increasing solutions of the functional equation $\varphi\{F[x, \varphi(x)]\} = x$, where φ is an unknown function, is proved. There is also given a sufficient and necessary condition for such a solution to be unique.

Let us consider the functional equation

$$\varphi\{F[x, \varphi(x)]\} = x, \tag{1}$$

where φ is an unknown real-valued function of real variable and F is a given real-valued function of two real variables. Equation (1) is the equation of curves invariant under the transform

$$x' = F(x, y), \quad y' = x; \tag{2}$$

solutions of (1) are curves invariant under transform (2). Condition (2) defines an axial transform which carries the x-axis to the y-axis. Curves invariant under other axial transforms have been studied in papers [1]–[4]. A special case of a certain axial transform has also been investigated in [6] and [7]. The transform (2) is the inverse to the transform studied in [3]–[5]. If the function F is invertible with respect to the second variable, then the transform inverse to (2) is given by the formula

$$x = y', \quad y' = G(x, y), \tag{3}$$

where G denotes the function inverse to F with respect to the second variable. The method we are going to apply in the present paper is almost identical to that applied in [5]. However, it is not possible to obtain the results concerning curves invariant under transform (2) directly from the results of [5], because if (2) fulfils the conditions assumed in the present paper, the inverse transform does not satisfy the conditions assumed in [5] and vice versa. But it is possible to get a result similar to that obtained in [5] if the function G is invertible not necessarily strictly increasing with respect to the second variable, as a consequence of the results obtained in this paper. Similarily, the results of [5] can be used to derive some theorems corresponding to the ours but under other conditions.

Let us assume the following

HYPOTHESIS H. 1° Transform (2) is a homeomorphism of the real plane into itself.

2° The function F is strictly increasing with respect to each variable.

3° There exist points a, b with $a < b$ and a function α defined on the interval $[a,b]$ such that $F(a,a) = a$, $F(b,b) = b$, $\alpha(x) < x$ for $x \in (a,b)$ and $F[x, \alpha(x)] = x$ for $x \in [a,b]$.

As a simple consequence of Hypothesis H, we obtain the following

LEMMA 1. *If Hypothesis* H *is fulfilled, then the function* α *is continuous in* $[a,b]$, $\alpha(a) = a$, $\alpha(b) = b$ *and*

$$F(x,y) > x \quad \textit{for } y > \alpha(x),\ x \in (a,b). \tag{4}$$

Let us denote

$$\Phi(x) = F(x,x). \tag{5}$$

The following lemma follows immediately from Hypothesis H and Lemma 1:

LEMMA 2. *If Hypothesis* H *is fulfilled, then the function* Φ *is continuous and strictly increasing in* $[a,b]$ *and*

$$\Phi(x) > x \quad \textit{for } x \in (a,b). \tag{6}$$

In the sequel we shall consider solutions of equation (1) in $[a,b]$ such that their graphs are contained in the set $E = \{(x,y): a \leqslant x \leqslant b, \alpha(x) \leqslant y \leqslant \Phi(x)\}$. Let us denote by L the space of all continuous functions ψ strictly increasing in $[a,b]$, such that $(x, \psi(x)) \in E$ for $x \in [a,b]$, with the usual metric

$$\varphi(\psi_1, \psi_2) = \sup_{[a,b]} |\psi_1(x) - \psi_2(x)|.$$

We define a map $T: L \to L$ by

$$T[\psi](x) = F[x, \psi^{-1}(x)],$$

where ψ^{-1} is the function inverse to ψ. It is easy to see that T is a one-to-one continuous map. Further, let us denote by R the transform given by

$$R(x,y) = (F(x,y), x). \tag{7}$$

We can easily observe that R is a homeomorphism and that

$$R^{-1}(x,y) = (y, F^{-1}(x,y)), \tag{8}$$

where R^{-1} denotes the transform inverse to R, F^{-1} denotes the function inverse to F with respect to the second variable.

LEMMA 3. *If Hypothesis* H *is fulfilled, then for any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for arbitrary* u, v, $a \leqslant u < v \leqslant b$, $v-u \geqslant \varepsilon$, *and for an arbitrary* $\psi \in T(L)$ *we have*

$$\frac{\psi^{-1}(v)-\psi^{-1}(u)}{v-u} \geqslant \delta.$$

Proof. Let us assume that the lemma is false. Then there exist $\varepsilon > 0$, $\psi_n \in T(L)$, u_n, v_n such that $a \leqslant u_n < v_n \leqslant b$, $v_n - u_n \geqslant \varepsilon$ and

$$\frac{\psi_n^{-1}(v_n)-\psi_n^{-1}(u_n)}{v_n-u_n} < \frac{1}{n} \quad \text{for } n = 1, 2, \ldots \tag{9}$$

Since $\psi_n \in T(L)$, then for every $x \in [a, b]$ we have $(\psi_n^{-1}(x), x) \in E$. We may assume the sequences $p_n = (\psi_n^{-1}(u_n), u_n)$ and $q_n = (\psi_n^{-1}(v_n), v_n)$ to be convergent, $\lim_{n\to\infty} p_n = p \in E$ and $\lim_{n\to\infty} q_n = q \in E$, because the set E is compact. Condition (9) implies that the sequences $\psi_n^{-1}(u_n)$ and $\psi_n^{-1}(v_n)$ have the same limit y; thus we may write $p = (y, u)$ and $q = (y, v)$. Let us denote $\bar{p} = R^{-1}(p) = (u, t)$, $\bar{q} = R^{-1}(q) = (v, z)$. The ordinates of points $\bar{p}$ and $\bar{q}$ have to be equal to the abscissae of points p and q, respectively, because of (8). Since the transform R is a homeomorphism and the set E is compact, we have $\bar{p} = \lim_{n\to\infty} R^{-1}(p_n)$, $\bar{q} = \lim_{n\to\infty} R^{-1}(q_n)$ and the transform R^{-1} is defined at $\bar{p}$ and $\bar{q}$. Since $\psi_n \in T(L)$ and $p_n, q_n \in R(E)$, we also have $p, q \in R(E)$. Thus $(y, u) = p = R(\bar{p}) = R(u, t) = (F(u, t), u)$ and $(y, v) = q = R(\bar{q}) = R(v, z) = (F(v, z), v)$, whence $y = F(u, t) = F(v, z)$. Therefore we obtain $t > z$, because the function F is strictly increasing with respect to each variable. The last inequality contradicts the fact that the points $\bar{p}$ and $\bar{q}$ lie on the graph of the strictly increasing function $T^{-1}(\psi_n)$. The above contradiction proves the lemma.

LEMMA 4. *If Hypothesis* H *is fulfilled, then the closure* $\overline{T(L)}$ *of* $T(L)$ *is contained in* L.

Proof. Let $\psi \in \overline{T(L)}$. This means that there exists a sequence ψ_n such that $\psi_n \in T(L)$ for $n = 1, 2, \ldots$ and that

$$\lim_{n\to\infty} \psi_n(x) = \psi(x) \tag{10}$$

uniformly in $[a, b]$. Then the limit ψ is a continuous function and $(x, \psi(x)) \in E$ for $x \in [a, b]$, because $T(L) \subset L$. We are going to prove that ψ if an increasing function.

Let $u, v \in [a, b]$ and $u < v$. It follows from Lemma 3 that for $\varepsilon = v-u$ there exists a $\delta > 0$ such that

$$\frac{\psi_n^{-1}(v)-\psi_n^{-1}(u)}{v-u} \geqslant \delta \quad \text{for } n = 1, 2, \ldots$$

Hence

$$\frac{\psi^{-1}(v)-\psi^{-1}(u)}{v-u} \geqslant \delta,$$

by virtue of (10). Thus the function ψ^{-1} is increasing and, consequently, the function ψ is also increasing.

LEMMA 5. *If Hypothesis* H *is fulfilled, then* $T^2(L)$ *is an equicontinuous set of functions* (T^2 *is the second iterate of* T).

Proof. Let us suppose that the lemma is false. Thus there exists an $\varepsilon > 0$ and sequences $u_n, v_n \in [a, b]$ and $\psi_n \in T^n(L)$, such that

$$0 < v_n - u_n < 1/n \tag{11}$$

and

$$\psi_n(v_n) - \psi_n(u_n) \geqslant \varepsilon. \tag{12}$$

Let us denote $p_n = (\psi_n^{-1}(u_n), u_n)$, $q_n = (\psi_n^{-1}(v_n), v_n)$. Since the set E is compact, we may assume that the sequences p_n and q_n converge to points $p \in E$ and $q \in E$, respectively, and we can put, by (11),

$$p = (u, z), \quad q = (u, y). \tag{13}$$

It follows from (12) that $y \neq z$ and we have, by virtue of (8),

$$\bar{p} = R^{-1}(p) = (z, F^{-1}(u, z)), \quad \bar{q} = R^{-1}(q) = (y, F^{-1}(u, y)). \tag{14}$$

Write $\bar{p} = (u_1, z_1)$ and $\bar{q} = (u_2, z_2)$. It follows from (8), (13) and (14) that

$$R(\bar{p}) = (F(u_1, z_1), u_1) = (u, z), \quad R(\bar{q}) = (F(u_2, z_2), u_2) = (u, y),$$

whence $u = F(u_1, z_1) = F(u_2, z_2)$, $u_1 = z \neq y = u_2$. Since F is a strictly increasing function with respect to each variable, we see that either $u_1 < u_2$ and $z_1 > z_2$, or $u_1 > u_2$ and $z_1 < z_2$. But this contradicts Lemma 3, because $\bar{p} = \lim_{n\to\infty} R^{-1}(p_n)$ and $\bar{q} = \lim_{n\to\infty} R^{-1}(q_n)$ and the points $R^{-1}(p_n)$ and $R^{-1}(q_n)$ lie on the graph of the increasing function $T^{-1}(\psi_n)$ belonging to $T(L)$.

The lemma is thus proved.

Let us put

$$\psi_0(x) = x, \quad \psi_{n+1}(x) = T[\psi_n](x) \quad \text{for } x \in [a, b],\ n = 0, 1, \ldots \tag{15}$$

LEMMA 6. *If Hypothesis* H *is fulfilled, then the sequences* ψ_{2n} *and* ψ_{2n+1} *converge uniformly in* $[a, b]$ *to* $\varphi_2 \in \overline{T(L)}$, $\varphi_1 \in \overline{T(L)}$, *respectively, and* $\varphi_2(x) \leqslant \varphi_1(x)$ *for* $x \in [a, b]$.

Proof. First we prove that

$$\psi_{2n-1}(x) \geqslant \psi_{2n+1}(x) \geqslant \psi_{2n}(x) \geqslant \psi_{2n-2}(x) \quad \text{for } x \in [a, b]. \tag{16}$$

It follows from (5), (6) and from Lemma 2 that $\Phi \in L$. We also have $\Phi^{-1} \in L$. Indeed, if $\Phi^{-1}(x) < \alpha(x)$ for an $x \in [a, b]$, then $x < \Phi[\alpha(x)] = F[\alpha(x), \alpha(x)] < F[x, \alpha(x)] = x$, which contradicts (5) and Hypothesis H. Thus we obtain from (15), (5) and (6) that $\psi_1(x) = \Phi(x) > x = \psi_0(x)$, whence

$$\psi_1(x) = \Phi(x) \geqslant \psi_2(x) = F[x, \Phi^{-1}(x)] < F[x, \alpha(x)] = x = \psi_0(x),$$

and so

$$\psi_0(x) = x = F[x, \alpha(x)] \leqslant F[x, \Phi^{-1}(x)] = \psi_2(x) \leqslant \psi_3(x) = F[x, \psi_2^{-1}(x)].$$

Thus condition (16) holds for $n = 1$. Let us assume that (16) holds for an $n \geqslant 1$. Then $\psi_{2n-1}^{-1}(x) \leqslant \psi_{2n+1}^{-1}(x)$, $\psi_{2n}^{-1}(x) \geqslant \psi_{2n-2}^{-1}(x)$ and $\psi_{2n+1}^{-1}(x) \geqslant \psi_{2n}^{-1}(x)$, in view of (16), whence it follows, by virtue of (15),

$$\psi_{2n}(x) = F[x, \psi_{2n-1}^{-1}(x)] \leqslant F[x, \psi_{2n+1}^{-1}(x)] = \psi_{2n+2}(x),$$
$$\psi_{2n+1}(x) = F[x, \psi_{2n}^{-1}(x)] \geqslant F[x, \psi_{2n+2}^{-1}(x)] = \psi_{2n+3}(x)$$

and

$$\psi_{2n+2}(x) = F[x, \psi_{2n+1}^{-1}(x)] \leqslant F[x, \psi_{2n}^{-1}(x)] = \psi_{2n+1}(x).$$

Hence condition (16) holds for $n+1$ and this ends the inductional proof of condition (16).

It follows from (16) that the sequences $\psi_{2n}(x)$ and $\psi_{2n+1}(x)$ are monotonic and bounded in $[a, b]$; therefore they are convergent in this interval. The convergence is uniform in view of Lemma 5. The limits φ_1 and φ_2 belong to $\overline{T(L)}$, by virtue of Lemma 4, and $\varphi_2(x) \leqslant \varphi_1(x)$ for $x \in [a, b]$, by virtue of (16). Thus the proof of the lemma is completed.

As a simple consequence of the foregoing lemma we obtain the following

LEMMA 7. *If Hypothesis* H *is fulfilled, then*

$$E_0 = \bigcap_{n=0} R^n(E) = \{(x, y)\colon a \leqslant x \leqslant b,\ \varphi_2(x) \leqslant y \leqslant \varphi_1(x)\}$$
$$\subset \operatorname{Int} E \cup \{a, a\} \cup \{b, b\}.$$

Now we are going to study solutions of equation (1). As an immediate consequence of Hypothesis H we obtain the following

LEMMA 8. *Let Hypothesis* H *be fulfilled and let* φ *be a continuous solution of equation* (1) *in* $[a, b]$. *If* $x \in [a, b]$, *then there exists such a point* $t \in [a, b]$ *that* $x = F[t, \varphi(t)]$.

LEMMA 9. *If Hypothesis* H *is fulfilled and* φ *is continuous solution of equation* (1) *in* $[a, b]$ *passing through an inner point of the set* E, *then the function* φ *is strictly increasing in* $[a, b]$ *and* $(x, \varphi(x))$ *is an inner point of* E *for every* $x \in (a, b)$.

Proof. First we are going to show that the function φ has to be one-to-one. Let us suppose the converse. Then there exist points x_1 and x_2 belonging to $[a, b]$ such that $\varphi(x_1) = \varphi(x_2)$. Thus, in view of Lemma 4, there exist $t_1, t_2 \in [a, b]$ such that $x_1 = F[t_1, \varphi(t_1)]$ and $x_2 = F[t_2, \varphi(t_2)]$. Hence we obtain, by virtue of (1), that

$$t_1 = \varphi\{F[t_1, \varphi(t_1)]\} = \varphi(x_1) = \varphi(x_2) = \varphi\{F[t_2, \varphi(t_2)]\} = t_2,$$

whence $\varphi(t_1) = \varphi(t_2)$ and this implies $x_1 = x_2$. Since φ is continuous, it has to be strictly increasing in $[a, b]$, because $\varphi(a) = a < b = \varphi(b)$, by virtue of (1).

Now we are going to prove the second part of the lemma. If φ is a continuous solution of equation (1) in $[a, b]$ for which there exists an inner point x_0 of the interval $[a, b]$ such that the point $(x_0, \varphi(x_0))$ is not an inner point of the set E, then we have either $\varphi(x_0) = \alpha(x_0)$ or $\varphi(x_0) = \Phi(x_0)$. The first possibility cannot happen because

$$x_0 = \varphi\{F[x_0, \varphi(x_0)]\} = \varphi\{F[x_0, \alpha(x_0)]\} = \alpha(x_0) < x_0,$$

by virtue of (1) and Hypothesis H.

Let us suppose that $\varphi(x_0) = \Phi(x_0)$. We obtain from (6) that $\Phi(x_0) > x_0$ and since, by Lemma 2, Φ is a strictly increasing function in $[a, b]$, it follows from (1) that

$$x_0 = \varphi\{F[x_0, \varphi(x_0)]\} = \varphi\{F[x_0, \Phi(x_0)]\} > \varphi(x_0) = \Phi(x_0),$$

because $\Phi(x_0) > \alpha(x_0)$, by virtue of Hypothesis H. We thus obtain a contradiction with Lemma 1, which ends the proof.

LEMMA 10. *If Hypothesis* H *is fulfilled and* $(x_0, y_0) \in \operatorname{Int} E$, *then for an arbitrary continuous and strictly increasing function* φ_0, *satisfying the condition*

$$\varphi_0^{-1}(x_0) = y_0, \quad \varphi_0^{-1}[F(x_0, y_0)] = x_0 \tag{17}$$

and defined in the interval $[x_0, F(x_0, y_0)]$,

$$\varphi(x) = \varphi_0(x) \quad \textit{for } x \in [x_0, F(x_0, y_0)]. \tag{18}$$

This solution is continuous and strictly increasing in $[x_0, b]$.

Proof. Let us put

$$y_1 = F(x_0, y_0), \quad x_1 = F(x_0, y_1),$$
$$y_{n+1} = F(x_n, y_{n+1}), \quad x_{n+1} = F(x_n, x_{n+1}).$$

It follows from Hypothesis H and Lemma 1 that the sequences x_n and y_n are strictly increasing and that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = b.$$

In fact, since $(x_0, y_0)\in E$, then we see from Lemma 1 that

$$b \geqslant x_{n+1} \geqslant x_n > a, \quad b \geqslant y_{n+1} \geqslant y_n > a. \tag{19}$$

Let us put $\lim_{n\to\infty} x_n = \bar{x}$, $\lim_{n\to\infty} y_n = \bar{y}$. It follows from (19) that these limits exist, because the sequences x_n and y_n are increasing and bounded, and that $\bar{x} \leqslant b$, $\bar{y} \leqslant b$. By the continuity of F we get

$$a < \bar{y} = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(\bar{x}, \bar{y}),$$

$$a < \bar{x} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_{n+1}) = F(\bar{x}, \bar{y}).$$

Therefore $\bar{x} = \bar{y}$ and $F(\bar{x}, \bar{y}) = \bar{x} > a$. Since the set E is compact, we have $(a, a) \neq (\bar{x}, \bar{y})\in E$, whence $\bar{x} = b$.

Let us define a function ψ in $[x_0, b]$ by

$$\psi(x) = \begin{cases} \psi_n(x) & \text{for } x\in[x_n, x_{n+1}), \\ b & \text{for } x = b, \end{cases} \tag{20}$$

where ψ_n is defined by (15). It follows easily from (20) and (15) that $x = \psi_{n+1}^{-1}\{F[x, \psi_n^{-1}(x)]\}$ and, consequently, the function $\varphi = \psi^{-1}$ satisfies equation (1). It is easy to verify, by virtue of (20) and Lemma 9, that the function φ is continuous and strictly increasing in $[x_0, b]$.

If the point (x_0, y_0) belongs to E_0, we can extend the function φ (defined above) to the left up to the point a, by the same kind of reasoning as that applied in [5]; but this is impossible if (x_0, y_0) is in $E\backslash E_0$.

As a result of our considerations, we obtain the following

THEOREM. *If Hypothesis* H *is fulfilled, then equation* (1) *has a continuous and strictly increasing solution in the interval* $[a, b]$. *This solution is unique if and only if* $\varphi_1(x) = \varphi_2(x)$.

It is easy to see that this solution is contained between φ_1 and φ_2; thus if $\varphi_1(x) = \varphi_2(x)$, then the solution is unique.

Since a curve invariant under transform (2) is automatically invariant under the inverse transform (3), and vice versa, the above theorem is still valid if we assume that the function F is strictly decreasing with respect to the second variable, because then the function G fulfils the assumptions of the theorem proved in [5]. It is enough to take

$$\beta(x) = \Phi^{-1}(x) \quad \text{and} \quad \Omega = \{(x, y)\colon a \leqslant x \leqslant b,\ \Phi^{-1}(a) \leqslant y \leqslant x\}.$$

Conversely, if the function G in [5] is strictly decreasing with respect to the second variable, then the inverse of the transform considered in [5] fulfils Hypothesis H. It is enough to take $\alpha(x) = G(x, x)$.

References

[1] D. Brydak, *Sur une équation fonctionnelle* (I), Ann. Polon. Math. 15 (1964), p. 237–251.

[2] — *Sur une équation fonctionnelle* (II), ibidem 21 (1968), p. 1–13.

[3] M. K. Fort, Jr., *Continuous solutions of a functional equation*, ibidem 13 (1963), p. 205–211.

[4] M. Kuczma, *On monotonic solutions of a functional equation* (I), ibidem 9 (1960), p. 295–297.

[5] — *On monotonic solutions of a functional equation* (II), ibidem 10 (1961), p. 161–166.

[6] K. Kuratowski, *Sur une équation fonctionnelle*, Sprawozdania z posiedzeń Tow. Nauk. Warszawskiego 22 (1929), p. 160–161.

[7] R. Wagner, *Eindeutige Lösungen der Funktionalgleichung* $f[x+f(x)] = f(x)$, Elem. Math. 14 (1959), p. 73–79.

Reçu par la Rédaction le 20.11.1976