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On some questions concerning determinants

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To the memory of Professor M. Biernacki

1. Various sufficient conditions are known in order that the $n\times n$ matrix

$$A = (a_{ik}) \tag{1.1}$$

should be regular. The germ of them is the idea of making the diagonal-term preponderant and its simplest form is expressed by the theorem of L. Lewy-Minkowski-Hadamard (¹), according to which the inequalities

$$\Delta_\nu \overset{\text{def}}{=} |a_{\nu\nu}| - \sum_{\substack{j=1\\ j\neq\nu}}^{n} |a_{\nu j}| > 0 \qquad (\nu = 1, 2, \ldots, n) \tag{1.2}$$

imply the regularity of A and by the variant, due to Müller (²)

$$|\det A| > \Delta_1\Delta_2 \ldots \Delta_n . \tag{1.3}$$

This idea was transformed and refined in various ways by A. Ostrowski, O. Taussky-Todd (²) and others. In what follows we shall present a sufficient condition for the regularity of A which is of an entirely different character. This will be a simple consequence of the solution of a class of minimum-problems concerning determinants of order n. In order to formulate them conveniently we shall call the elements of the form

$$a_{\nu+h,\nu} \qquad (\nu = 1, 2, \ldots, n-h) \tag{1.4}$$

at a fixed non-negative h as forming "the h^{th} skew-line under the diagonal" and those of the form

$$a_{\nu,\nu+l} \qquad (\nu = 1, 2, \ldots, n-l) \tag{1.5}$$

(¹) For the references see the interesting book of M. Parodi, *La localisation des valeurs caractéristiques des matrices et ses applications*, Gauth. Villars 1959.

(²) See Parodi l. c.

at a fixed non-negative l as forming "the l^{th} skew-line above the diagonal". Now let j be a fixed integer with

$$a \leqslant j \leqslant [(n-1)/2]; \tag{1.6}$$

the positive numbers α and β being given we consider the class $\Gamma_n = \Gamma_n(j, \alpha, \beta)$ of all determinants D_n of order n whose elements are not less than α in the diagonal and in the $2\mu^{\text{th}}$ skew-lines under the diagonal ($\mu = 1, 2, \dots, j$), not less than β in the first skew-line above the diagonal and equal to 0 elsewhere. In other words the class Γ_n should consist of the determinants D_n of order n with

$$a_{\nu+2k,\nu} \geqslant \alpha \qquad (k = 0, 1, 2, \dots, j;\ \nu = 1, 2, \dots, n-2k) \tag{1.7}$$

and

$$a_{\nu,\nu+1} \geqslant \beta \qquad (\nu = 1, 2, \dots, n-1) \tag{1.8}$$

and otherwise 0. Restricting ourselves for the sake of simplicity to the case

$$j = [(n-1)/2]$$

we assert

Theorem I. *If $D_n \in \Gamma_n([(n-1)/2], \alpha, \beta)$, then*

$$D_n \geqslant \frac{\alpha}{\sqrt{\alpha^2+4\beta^2}} \left\{ \left(\frac{\alpha + \sqrt{\alpha^2+4\beta^2}}{2} \right)^n - \left(\frac{\alpha - \sqrt{\alpha^2+4\beta^2}}{2} \right)^n \right\}. \tag{1.9}$$

We have equality in (1.9) *if and only if we have equality in* (1.7) *and* (1.8).

In 2 we shall give the proof of this theorem, in 3 and 4 we shall discuss some further problems which emerge quite naturally after theorem I, and we shall prove theorem II, which throws some light upon the structure of the determinants.

I am much obliged to T. Gallai for his valuable remarks. (For a final remark see the end of the paper.)

2. The proof of theorem I is based on the observation that in D_n owing to the configuration of the 0-elements all terms belonging to odd permutations vanish i.e. D_n consists exclusively of positive terms. This evidently holds for $n = 1, 2, 3$; suppose that it holds for $n \leqslant N-1$ with $N \geqslant 4$, i.e. that for $n \leqslant N-1$ D_n is a sum of positive terms. Expansion according to the first row gives

$$D_N = a_{11} D_{N-1} - a_{12} \begin{vmatrix} 0 & a_{23} & 0 & 0 & \dots \\ a_{31} & a_{33} & a_{34} & 0 & \dots \\ 0 & 0 & a_{44} & a_{45} & \dots \\ a_{51} & a_{53} & 0 & a_{55} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix};$$

expanding again according to the first row, we obtain

$$D_N = a_{11} D_{N-1} + a_{12} a_{23} D_{N-2} \,, \tag{2.1}$$

which owing to the induction hypothesis and the positivity of the a_{ik}'s proves the observation. But then (1.7) and (1.8) give

$$D_N \geqslant a D_{N-1} + \beta^2 D_{N-2} \,. \tag{2.2}$$

Hence if the sequence D_n^* is defined by

$$\begin{gathered} D_1^* = a \,, \qquad D_2^* = a^2 \,, \\ D_n^* = a D_{n-1}^* + \beta^2 D_{n-2}^* \qquad (n \geqslant 3) \,, \end{gathered} \tag{2.3}$$

then evidently we have $D_1 = D_1^*$, $D_2 = D_2^*$ and thus $D_N \geqslant D_N^*$ for $N \geqslant 1$. Since from (2.3)

$$D_N^* = \frac{a}{\sqrt{a^2+4\beta^2}} \left\{ \left(\frac{a+\sqrt{a^2+4\beta^2}}{2} \right)^n - \left(\frac{a-\sqrt{a^2+4\beta^2}}{2} \right)^n \right\},$$

(1.9) is proved. The remark in the theorem concerning the equality-sign follows evidently.

3. The simple proof of theorem I suggests many natural problems. Obviously analogous extremal theorems hold for all Γ_n^*-classes of determinants of order n where the configuration of 0's ensures the vanishing of all terms corresponding to odd permutations. Let us call such a configuration a good one. The first question is the following. The number of non-vanishing terms in $D_n \in \Gamma_n$, as can easily be seen from the proof of the theorem I, is

$$\frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\} \stackrel{\text{def}}{=} A_n \,. \tag{3.1}$$

Is there any good Γ_n^*-class of determinants so that the number of non-vanishing terms in $D_n \in \Gamma_n^*$ is greater than A_n? I thought that the answer to this question is negative; but, as T. Gallai remarked, in the case $n = 7$ (but not for $n \leqslant 6$) this conjecture is certainly false since $A_7 = 13$, but the 0-configuration

$$\begin{vmatrix} \cdot & \cdot & 0 & 0 & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 0 & 0 & \cdot \\ \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & 0 & 0 & \cdot & 0 & \cdot \end{vmatrix}$$

(at the empty places positive elements)

is a good one and the number of non-vanishing terms is, as can easily be checked, 24. Thus the following interesting question arises: which are the good Γ_n^*-classes with the maximum-number of non-vanishing terms? To another question, namely what is the minimum-number of 0's in a good Γ_n^*-class, the answer is, as T. Gallai remarked, $n(n-1)/2$. For the proof it is enough to remark that if there is a non-vanishing term in D_n at all, then after a suitable change of rows and columns one can make all elements in the diagonal non-vanishing and then for all $1 \leqslant i \neq k \leqslant n$ the relation

$$a_{ik}a_{ki} = 0 \tag{3.2}$$

holds, for if for a certain pair $i = i_1$, $k = k_1$ $(i_1 \neq k_1)$ (3.2) is false, then obviously the term

$$a_{i_1k_1}a_{k_1i_1}\prod_{\substack{\nu=1\\ \nu\neq i_1\\ \nu\neq k_1}}^{n}{}' a_{\nu\nu}$$

is a non-vanishing term in D_n belonging to an odd permutation. Hence the minimum-number is $\geqslant n(n-1)/2$; the number of zeros of the "trivial" Γ_n^*-class given by $a_{ik} = 0$ $(i < k)$ is $= n(n-1)/2$.

4. In connection with these questions it is natural to ask a question which throws some light upon the structure of the determinants: what is the maximum M_n of the difference of the number of non-vanishing terms belonging to even resp. odd permutations at various systems of 0's? In comparison with the previous remarks the following theorem is somewhat surprising

THEOREM II. *The inequality*

$$M_n > \frac{1}{2^{n-1}}\sqrt{n!}$$

holds.

At a fixed 0-configuration the (common) difference of the numbers of the non-vanishing terms belonging to even resp. odd permutations is given by the determinant having as elements exclusively 1's at the remaining places. Hence the problem is identical with that of finding the maximal determinant of order n consisting of the elements 0 and 1. Let this maximal determinant be D_n^* and let the maximal one with the elements ± 1 be Δ_n^*; in the latter we can suppose without loss of generality that the first line consists exclusively of 1's. By adding the first row to the k^{th} one as the k^{th} row the elements become 0 or 2; hence factoring out 2 from each row we get

$$\Delta_n^* = 2^{n-1}D_n',$$

where D'_n is a determinant of order n consisting exclusively of 1's and 0's. Thus

$$D^*_n \geqslant \frac{1}{2^{n-1}} \Delta^*_n .$$

But as I have shown [3] together with G. Szekeres among others that the quadratical mean-value of all determinants of order n consiting of ± 1's is $\sqrt{n!}$; since obviously

$$\Delta^*_n \geqslant \sqrt{n!} ,$$

Theorem II is proved.

The proof is evidently an existence proof; it would be desirable to give explicitly a system of 0's with the required property.

Added in proof. (12 Apr. 1962) According to an observation of dr M. Kunkuti the determinants of the form

$$\text{(4.1)} \qquad \begin{vmatrix} a_{11} & -1 & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & -1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,n-1} & -1 \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{n,n-1} & a_{nn} \end{vmatrix}$$

have also the property that $a_{ik} > 0$ $(n \geqslant i \geqslant k \geqslant 1)$ implies the positivity of all non-vanishing expansion-terms. Hence for these determinants a similar extremal-theorem could be formulated.

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[3] *Egy szélsőértékfeladat a determináns-elméletben* (in Hungarian with German abstract), Mat. és Term. Tud. Ért. (1937), p. 796-806.

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