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Characterization of some tensor concomitants of the metric tensor and vector fields under restricted groups of transformations

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Introduction. In [3] it is shown that any odd-ranked mixed tensor concomitant of the metric tensor must vanish identically if all non-singular transformations are allowed. However, this result no longer holds if one is restricted to the "proper" transformations (by which we shall mean those with positive Jacobian determinants), the Levi-Civita symbols in V^3 being a counter-example. Here we shall investigate relative tensor concomitants of the metric tensor, its first and second partial derivatives, and of a vector and its first partial derivatives; the results are valid even for restrictions of the above mentioned "proper" transformations. The proofs are purely algebraic, and in that they require no regularity assumptions they extend those found in [5], [6], [10], [11] and [12] concerning Lagrangians and associated field equations.

A basis for the derivation of results herein lies in the following three observations. Firstly, any symmetric matrix with non-zero determinant may be diagonalized [2] (p. 244) to the form $\sigma = \mathrm{diag}[+1, \ldots, +1, -1, \ldots, -1]$ by a matrix with positive determinant; that is, if $G = [g_{ij}]$ is such that $g_{ij} = g_{ji}$ and $\det(G) \neq 0$, then there exists a matrix B with $\det(B) > 0$ such that

$$G = B\sigma B^T, \tag{1}$$

where B^T denotes(1) the transpose of B.

Secondly, if S is any matrix satisfying $S\sigma S^T = \sigma$, we must also have

$$G = B(S\sigma S^T)B^T. \tag{2}$$

(1) Throughout this note, the following notations will be used:
$G = [g_{ij}]$ is a symmetric metric tensor;
$A = [A^i_j] = [\partial x^i / \partial \bar{x}^j]$ is the Jacobian matrix of the transformation $\bar{x}^i \to x^i$;
$\overset{-1}{A} = [\overset{-1}{A^i_j}]$ is the inverse matrix of A;
',' denotes partial differentiation;
'|' denotes covariant differentiation;
M^T denotes the transpose of any matrix M.

We may assume that in (2), $\det(S) > 0$ since if $S\sigma S^T = \sigma$ for diagonal σ, so also $\bar{S}S\sigma S^T\bar{S}^T = \sigma$, where $\bar{S}$ is the identity matrix with one diagonal element replaced by -1.

Finally, if $\bar{\sigma}$ is any square matrix of dimension greater than two such that

$$S\bar{\sigma}S^T = \bar{\sigma} \quad \text{whenever } S\sigma S^T = \sigma \text{ for } \det(S) > 0,$$

then $\bar{\sigma}$ *is a scalar multiple* of σ. More precisely,

BASIC LEMMA. *Let* $\sigma = \mathrm{diag}[\underset{1}{+1}, \ldots, \underset{t}{+1}, -1, \ldots, -1]$, *where* $t = 0, 1, \ldots, n$.

If $S\bar{\sigma}S^T = \bar{\sigma}$ *whenever* $S\sigma S^T = \sigma$ *and* $\det(S) > 0$, *then* $\bar{\sigma} = \tau\sigma$, *where for* $n > 2$, $\tau = [k\delta^i_j]$, *a scalar matrix, whence* $\bar{\sigma} = k\sigma$, *and for* $n = 2$, $\tau = \begin{bmatrix} k & l \\ -l & k \end{bmatrix}$ *for* $t = 0, 2$, *and* $\tau = \begin{bmatrix} k & l \\ l & k \end{bmatrix}$ *for* $t = 1$.

Proof. (The shortened proof below is due to A. Zajtz. The authors also wish to thank H. Davis for his help in the original proof.) The matrices S satisfying $S\sigma S^T = \sigma$ and $\det(S) > 0$ form the proper Lorentz-group $L^+(n, t)$ which, for $t = 0, n$, forms the proper orthogonal group $O^+(n)$. Clearly

$$(*) \qquad C \overset{\mathrm{df}}{=} \bar{\sigma}\sigma^{-1}$$

satisfies

$$(**) \qquad SC = CS$$

since $\bar{\sigma} = S\bar{\sigma}S^T$ and $\sigma^{-1} = (S^T)^{-1}\sigma^{-1}S^{-1}$ implies $\bar{\sigma}\sigma^{-1} = S\bar{\sigma}\sigma^{-1}S^{-1}$, whence (**). Hence C commutes with all of $L^+(n, t)$. Since C commutes with arbitrary even reflections, C must be diagonal for $n > 2$; that is, $C = \mathrm{diag}[c_1, \ldots, c_n]$. If $c_p \neq c_q$, then by (**), $S = [s_{ij}]$ has $s_{pq} = s_{qp} = 0$, and more generally, every S would be the simple sum of two matrices; that is,

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$$

which is certainly not true for $L^+(n, t)$. Hence C is a scalar matrix (for complex matrices this follows directly from the irreducibility of $L^+(n, t)$). For $n = 2$ we have for $t = 0, 2$, $S \in O^+(2)$ while for $t = 1$, $S \in L^+(2, 1)$ and the form of C results from (**).

THEOREM 1. *The only scalar concomitant* $L(g_{ij})$ *of a symmetric metric tensor with* $\det(g_{ij}) \neq 0$ *under the "proper" transformations is a constant (dependent on the signature of the metric).*

Proof. The transformation law of G in co-ordinates x^i to $\bar{G}$ in co-ordinates $\bar{x}^i$ at a point p_0 in V^n may be written in matrix form as $\bar{G} = AGA^T$, where A is evaluated at p_0. The transformation law for the scalar concomitant $L(G)$ may be written $L(G) = L(\bar{G}) = L(AGA^T)$. Hence by (1), $L(G) = L(B\sigma B^T) = L(\sigma)$ at p_0; since this holds for every p_0, the lemma follows.

Similarly, we may prove

THEOREM 2. *Any rank two covariant concomitant of the metric tensor in a space of dimension greater than two must be of the form $T_{ij}(g_{ab}) = kg_{ij}$ relative to the "proper" transformations, k being a constant (dependent on the signature of g_{ij}).*

Proof. Defining the matrix $\bar{\sigma} \overset{\mathrm{df}}{=} [T_{ij}]$, we may write the transformation law of $T_{ij}(g_{ab})$ in matrix form as $\bar{\sigma}(AGA^T) = A\bar{\sigma}(G)A^T$. By (1) we may choose a co-ordinate system in which G has the diagonal form $\sigma = \mathrm{diag}\{-1, \ldots, -1, 1, \ldots, 1\}$ at some given point p_0. But then $\bar{\sigma}(S\sigma S^T) = S\bar{\sigma}(\sigma)S^T$, and the basic lemma implies $\bar{\sigma}(\sigma) = k\sigma$ when $n > 2$. Since this is a tensor relation, it follows that $T_{ij}(g_{ab}) = kg_{ij}$ at p_0 in every co-ordinate system. That k is independent of p_0 follows from Theorem 1.

By applying the basic lemma in the case $n = 2$, one obtains the following

EXTENSION OF THEOREM 2 (S. Gołąb and A. Zajtz). *Any rank two covariant concomitant of the metric tensor in a space of dimension $n = 2$ is of the form*

$$T_{ij}(g_{ab}) = kg_{ij} + l\sqrt{(g)}\, e_{ij}, \quad \textit{where } e = [e_{ij}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The above proofs illustrate the algebraic nature of the proofs of the more general theorems given below.

Results. For the sake of continuity, the results obtained herein are listed below without proofs; the latter will be found in the next section of this note. It is assumed throughout that $\det(g_{ij}) \neq 0$.

1. Any relative tensor concomitant of a symmetric metric tensor having rank both odd and less than the dimension of the space is a null tensor relative to the restricted group of "proper" transformations.

2. Again relative to "proper" transformations, any covariant relative tensor concomitant of a symmetric metric tensor having rank odd and equal to the dimension n of the space is a constant multiple of the Levi-Civita tensor $\varepsilon_{i_1 i_2 \ldots i_n}$.

S. Gołąb has indicated to the authors the following immediate consequence of result 2.

THEOREM (S. Gołąb). *In a space of odd dimension n, there does not exist any (non-trivial) tensor density of odd degree less than n.*

For if $T_{i_1 \ldots i_q}$ were such a tensor, q odd $< n$, it could be "completed" to a tensor $\bar{T}_{i_1 \ldots i_n}$ by multiplications with g_{ij}. But then by result 2,

$$T_{i_1\dots i_q} g_{j_1 r_1} \cdots g_{j_s r_s} = k\varepsilon_{i_1\dots i_n}, \quad \text{where } s = \frac{n-q}{2},$$

and since the left-hand side has at least two symmetric indices while the right is completely skew-symmetric, it follows that $k = 0$, whence $T_{i_1\dots i_q} = 0$.

The following results are also proved algebraically, and they assume the tensor character of the concomitants under only the "proper" transformations.

3. Any relative tensor concomitant of a symmetric metric tensor and its first partial derivatives must be independent of the first partial derivatives.

4. Any relative tensor concomitant of a symmetric metric tensor and its first and second partial derivatives may be written in the form $T^{m_1\dots m_r}_{n_1\dots n_s}(g_{ij}; R_{ijkl})$, r and s being the contravariant and covariant ranks, and R_{ijkl} being the Riemann curvature tensor.

5. Any relative tensor concomitant of a symmetric metric tensor and its first partial derivatives, along with a vector field X_i and its first partial derivatives, may be written in the form $T^{m_1\dots m_r}_{n_1\dots n_s}(X_i; X_{i|j}; g_{ij})$.

6. Any relative tensor concomitant of a symmetric metric tensor and its first and second partial derivatives, along with a vector field X_i and its first partial derivatives, may be written in the form $T^{m_1\dots m_r}_{n_1\dots n_s}(X_i; X_{i|j}; g_{ij}; R_{ijkl})$.

It has been pointed out to the authors by S. Gołąb and A. Zajtz that our proofs for Results 3, 4, 5 and 6 are valid not only for tensor concomitants but *also for all possible objects (geometric or not) of the first class.*

However, we will not pursue this generality.

Proofs of Results. We prove the above results in detail in the following sequence of theorems; Results 1, 2, 3, 5 and 6 are proved respectively in Theorems 3, 4, 5, 9 and 10, while result 4 is a direct consequence of Theorem 8.

THEOREM 3. *Any relative tensor $T^{m_1\dots m_r}_{n_1\dots n_s}(g_{ij})$, covariant of rank s, contravariant of rank r, and of weight p, where $r+s$ is an odd number less than n, the dimension of the space, is a null tensor relative to the restricted group of co-ordinate transformations $x^i \leftrightarrow \bar{x}^i$, $i = 1, \dots, n$ for which $J^{-1} = \det(\partial x^i/\partial \bar{x}^j)$ is positive.*

Proof. For a relative tensor concomitant of weight p, the transformation law is

$$(4) \qquad T^{m_1\dots m_r}_{n_1\dots n_s}(\bar{g}_{ij}) = J^{-p} A^{a_1}_{n_1} \dots A^{a_s}_{n_s} \overset{-1}{A}{}^{m_1}_{b_1} \dots \overset{-1}{A}{}^{m_r}_{b_r} T^{b_1\dots b_r}_{a_1\dots a_s}(g_{ij}).$$

Let B^i_j and S^i_j be transformations as in (1) and (2), viz.

$$(5) \qquad g_{ab} = B^c_a B^d_b \sigma_{cd}, \quad [\sigma_{cd}] = \operatorname{diag}[1, \dots, 1, -1, \dots, -1],$$

$$\sigma_{ab} = S^c_a S^d_b \sigma_{cd} \quad \text{with } \det(S^a_b) = +1, \tag{6}$$

where the last assertion holds in view of the assumption $\det(S^a_b) > 0$. The transformation law (4) then implies

$$T^{m_1 \ldots m_r}_{n_1 \ldots n_s}(g_{ab}) = \det(B^i_j)^p B^{a_1}_{n_1} \ldots B^{a_s}_{n_s} \overset{-1}{B}{}^{m_1}_{b_1} \ldots \overset{-1}{B}{}^{m_r}_{b_r} T^{b_1 \ldots b_r}_{a_1 \ldots a_s}(\sigma_{cd}), \tag{7}$$

$$T^{m_1 \ldots m_r}_{n_1 \ldots n_s}(\sigma_{ab}) = S^{a_1}_{n_1} \ldots S^{a_s}_{n_s} \overset{-1}{S}{}^{m_1}_{b_1} \ldots \overset{-1}{S}{}^{m_r}_{b_r} T^{b_1 \ldots b_r}_{a_1 \ldots a_s}(\sigma_{ab}). \tag{8}$$

Consider the matrix $[S^a_b]$ formed by replacing two distinct diagonal elements in the identity matrix by -1. Clearly $[S^a_b]$ commutes with any diagonal matrix, and $[S^a_b]$ is its own inverse; hence (6) is valid. Specifically, S^a_b is given by

$$S^a_b = \delta^a_b - 2\delta^a_{(M)}\delta^{(M)}_b - 2\delta^a_{(L)}\delta^{(L)}_b \quad \text{for } 1 \leqslant M \neq L \leqslant n,$$

where "( )" indicates no summation; as indicated above $S^a_b = \delta^a_b$ if a, $b \neq M$ or L, $S^a_L = -\delta^a_L$, $S^a_M = -\delta^a_M$, and $S^a_b S^b_c = \delta^a_c$. For this choice of S^a_b, (8) may be written in the more convenient form

$$T^{m_1 \ldots m_r}_{m_{r+1} \ldots m_{r+2}}(\sigma_{ab}) = S^{a_1}_{m_{r+1}} \ldots S^{a_s}_{m_{r+2}} \ldots S^{m_1}_{b_1} \ldots S^{m_r}_{b_r} T^{b_1 \ldots b_r}_{a_1 \ldots a_s}(\sigma_{cd}). \tag{9}$$

Now, given any specific set of indices $\{m_k\}$ $(k = 1, \ldots, r+s)$, since $(r+s) < n$ there exists an integer w in $\{1, \ldots, n\}$ such that $w \notin \{m_k\}$. Further, since $r+s$ is odd, at least one of the $\{m_k\}$, say m_j, occurs an odd number of times in the set $\{m_k\}$. Clearly $m_j \neq w$, and considering the particular δ^a_b of (8) for which $L = w$ and $M = m_j$, we obtain, if m_j occurs only once say,

$$\begin{aligned} T^{m_1 \ldots m_r}_{n_{r+1} \ldots m_j \ldots m_{r+s}}(\sigma_{ab}) &= \delta^{m_1}_{d_1} \ldots \delta^{m_r}_{d_r} \delta^{c_1}_{m_{r+1}} \ldots (-\delta^{c_j}_{m_j}) \ldots (\delta^{c_s}_{m_{r+s}}) T^{d_1 \ldots d_r}_{c_1 \ldots c_s}(\sigma_{ab}) \\ &= -T^{m_1 \ldots m_r}_{m_{r+1} \ldots m_j \ldots m_{r+s}}(\sigma_{ab}) \end{aligned}$$

or

$$T^{m_1 \ldots m_r}_{m_{r+1} \ldots m_{r+s}}(\sigma_{ab}) = 0.$$

Obviously, as long as m_j occurs an odd number of times, we will obtain the same result. Substitution of this result into (7) yields

$$T^{l_1 \ldots l_r}_{k_1 \ldots k_s}(g_{ij}) = 0$$

as required.

THEOREM 4. *Any relative tensor concomitant $T_{m_1 \ldots m_n}(g_{ij})$ having rank n both odd and equal to the dimension n of the space is of the form $T_{m_1 \ldots m_n}(g_{ij}) = a\varepsilon_{m_1 \ldots m_n}$, where a is a constant, $\varepsilon_{m_1 \ldots m_n}$ is the Levi-Civita symbol defined by $\varepsilon_{m_1 \ldots m_n} = \sqrt{|g|}\, e_{m_1 \ldots m_n}$. Here $g \overset{\text{df}}{=} \det(g_{ij})$ and $e_{m_1 \ldots m_n}$ is the permutation symbol.*

Proof. Equations (7) and (8) now become

$$T_{m_1 \ldots m_n}(g_{ij}) = B^{c_1}_{m_1} \ldots B^{c_n}_{m_n} T_{c_1 \ldots c_n}(\sigma_{ij}) \tag{10}$$

and

(11) $$T_{m_1 \dots m_n}(\sigma_{ab}) = \underset{2}{S}{}^{c_1}_{m_1} \dots \underset{2}{S}{}^{c_n}_{m_n} T_{c_1 \dots c_n}(\sigma_{ab}),$$

where $\sigma_{ij} = \mathrm{diag}[+1, \dots, +1, -1, \dots, -1]$. Suppose, for a given set of indices $\{m_k\}$ in (11), one index is repeated. Then there exists an integer w in $\{1, \dots, n\}$ which does not occur in $\{m_k\}$. Further, as in the proof of the previous theorem, at least one integer m_j occurs in $\{m_k\}$ an odd number of times. Since $m_j \neq w$, we may set $L = w$ and $M = m_j$ as before, from which it follows that *the concomitant vanishes if an index is repeated.* Thus the only non-zero components are those for which $\{m_1, \dots, m_n\}$ is some permutation of $\{1, \dots, n\}$.

From this point to (**) it is necessary to consider separately the cases:

(i) g_{ij} is positive definite; that is, $\sigma_{ij} = \delta_{ij}$ for $i, j = 1, \dots, n$;

(ii) g_{ij} is not positive definite; that is, $\sigma_{ij} = \delta_{ij}$ for $i, j = 1, \dots, q$, and $\sigma_{ij} = -\delta_{ij}$ for $i, j = q+1, \dots, n$.

(i) When g_{ij} is positive definite, a second possible δ^a_b is

$$\underset{2}{S}{}^a_b = \delta^a_b - \delta^a_{(M)} \delta^{(M)}_b - \delta^a_{(L)} \delta^{(L)}_b + \delta^a_L \delta^M_b - \delta^a_M \delta^L_b,$$

where L and M are again two fixed integers in $\{1, \dots, n\}$. Given a specific set of indices $\{m_k\}$ for a non-zero component of $T_{m_1 \dots m_n}$, L and M must each occur only once in $\{m_1 \dots m_n\}$. If we suppose $L = m_i$ and $M = m_j$, then $\underset{2}{S}{}^a_b = \delta^a_b$ for $a, b \neq m_i$ or m_j, $\underset{2}{S}{}^a_{m_j} = +\delta^a_{m_i}$, $\underset{2}{S}{}^a_{m_i} = -\delta^a_{m_j}$, $\underset{2}{S}{}^{m_j}_b = -\delta^{m_i}_b$, and $\underset{2}{S}{}^{m_i}_b = \delta^{m_j}_b$. Substitution into (11) yields

$$\begin{aligned} T_{m_1 \dots m_j \dots m_i \dots m_n}(\sigma_{ab}) &= \underset{2}{S}{}^{c_1}_{m_1} \dots \underset{2}{S}{}^{c_j}_{m_j} \dots \underset{2}{S}{}^{c_i}_{m_i} \dots \underset{2}{S}{}^{c_n}_{m_n} T_{c_1 \dots c_n}(\sigma_{ab}) \\ &= \delta^{c_1}_{m_1} \dots \delta^{c_j}_{m_i} \dots (-\delta^{c_i}_{m_j}) \dots \delta^{c_n}_{m_n} T_{c_1 \dots c_n}(\sigma_{ab}) \\ &= -T_{m_1 \dots m_i \dots m_j \dots m_n}(\sigma_{ab}); \end{aligned}$$

that is, $T_{m_1 \dots m_n}(\sigma_{ab})$ is skew-symmetric upon interchange of indices (since we may cover all possible sets of indices simply by letting L and M take on, successively, all possibles pairs in $\{1, \dots, n\}$ and considering the resulting transformations).

Since $T_{m_1 \dots m_n}(\sigma_{ab})$ is skew-symmetric and is non-zero if and only if $(m_1 \dots m_n)$ is some permutation of $(1, \dots, n)$, it must be a constant multiple of the permutation symbol $e_{m_1 \dots m_n}$; that is, $T_{m_1 \dots m_n}(\sigma_{ab}) = a e_{m_1 \dots m_n}$ for g_{ij} positive definite.

(ii) When g_{ij} is not positive definite, the transformation $\underset{2}{S}{}^a_b$ will leave σ_{ab} invariant if and only if either L and M are both in $\{1, \dots, q\}$ or both in $\{q+1, \dots, n\}$. Thus, using $\underset{2}{S}{}^a_b$ as above, we can show only that

$T_{m_1\ldots m_j\ldots m_i\ldots m_n}(\sigma_{ab}) = -T_{m_1\ldots m_i\ldots m_j\ldots m_n}(\sigma_{ab})$ when m_i, m_j are both in $\{1, \ldots, q\}$ or both in $\{q+1, \ldots, n\}$.

We thus consider a third $\underset{3}{S}{}^a_b$ defined by

$$(12)\quad \underset{3}{S}{}^a_b = \delta^a_b - (1-\cosh\theta)\,\delta^a_{(M)}\delta^{(M)}_b - (1-\cosh\theta)\,\delta^a_{(L)}\delta^{(L)}_b + (\delta^a_L\delta^M_b + \delta^a_M\delta^L_b)\sinh\theta$$

which leaves σ_{ab} invariant if and only if either L is in $\{1, \ldots, q\}$ and M is in $\{q+1, \ldots, n\}$ or vice versa. For this transformation, we have $\underset{3}{S}{}^a_b = \delta^a_b$ for $a, b \neq M$ or L,

$$\underset{3}{S}{}^L_b = \delta^L_b - (1-\cosh\theta)\,\delta^L_b + \delta^M_b\sinh\theta,$$

$$\underset{3}{S}{}^M_b = \delta^M_b - (1-\cosh\theta)\,\delta^M_b + \delta^L_b\sinh\theta,\quad \underset{3}{S}{}^a_L = \delta^a_L - (1-\cosh\theta)\,\delta^a_L + \delta^a_M\sinh\theta,$$

and

$$\underset{3}{S}{}^a_M = \delta^a_M - (1-\cosh\theta)\,\delta^a_M + \delta^a_L\sinh\theta.$$

Since any non-zero $T_{m_1\ldots m_n}(\sigma_{ab})$ must be such that $(m_1 \ldots m_n)$ is a permutation of $(1, \ldots, n)$, there will always be some m_i, m_j such that m_i is in $\{1, \ldots, q\}$ and m_j is in $\{q+1, \ldots, n\}$.

Letting $m_i = L$, $m_j = M$ and substituting (12) into (11) we have

$$\begin{aligned}
&T_{m_1\ldots m_i\ldots m_j\ldots m_n}(\sigma_{ab})\\
&= S^{c_1}_{m_1}\ldots S^{c_i}_{m_i}\ldots S^{c_j}_{m_j}\ldots S^{c_n}_{m_n}T_{c_1\ldots c_n}(\sigma_{ab})\\
&= S^{c_1}_{m_1}\ldots S^{c_{i-1}}_{m_{i-1}}\big(\delta^{c_i}_{m_i} - (1-\cosh\theta)\,\delta^{c_i}_{m_i} + \delta^{c_i}_{m_j}\sinh\theta\big)\,\delta^{c_{i+1}}_{m_{i+1}}\ldots\delta^{c_{j-1}}_{m_{j-1}}\times\\
&\quad\times\big(\delta^{c_j}_{m_j} - (1-\cosh\theta)\,\delta^{c_j}_{m_j} + \delta^{c_j}_{m_i}\sinh\theta\big)\,\delta^{c_{j+1}}_{m_{j+1}}\ldots\delta^{c_n}_{m_n}T_{c_1\ldots c_i\ldots c_j\ldots c_n}(\sigma_{ab})\\
&= \big(\delta^{c_i}_{m_i} - (1-\cosh\theta)\,\delta^{c_i}_{m_i} + \delta^{c_i}_{m_j}\sinh\theta\big)\times\\
&\quad\times\big(\delta^{c_j}_{m_j} - (1-\cosh\theta)\,\delta^{c_j}_{m_j} + \delta^{c_j}_{m_i}\sinh\theta\big)\,T_{m_1\ldots m_{i-1}c_i m_{i+1}\ldots m_{j-1}c_j m_{j+1}\ldots m_n}(\sigma_{ab}).
\end{aligned}$$

Upon simplification, this becomes

$$\begin{aligned}
T_{m_1\ldots m_i\ldots m_j\ldots m_n}(\sigma_{ab}) = \big(&\cosh^2\theta\,\delta^{c_i}_{m_i}\delta^{c_j}_{m_j} + \sinh^2\theta\,\delta^{c_i}_{m_j}\delta^{c_j}_{m_i} +\\
&+ (\delta^{c_i}_{m_i}\delta^{c_j}_{m_i} + \delta^{c_i}_{m_j}\delta^{c_j}_{m_j})\sinh\theta\cosh\theta\big)\,T_{m_1\ldots c_i\ldots c_j\ldots m_n}(\sigma_{ab}).
\end{aligned}$$

By summing over c_i and c_j and eliminating all vanishing terms, we obtain

$$T_{m_1\ldots m_i\ldots m_j\ldots m_n}(\sigma_{ab}) = \cosh^2\theta\,T_{m_1\ldots m_i\ldots m_j\ldots m_n}(\sigma_{ab}) + \sinh^2\theta\,T_{m_1\ldots m_j\ldots m_i\ldots m_n}(\sigma_{ab})$$

or

$$(1-\cosh^2\theta)\,T_{m_1\ldots m_i\ldots m_j\ldots m_n}(\sigma_{ab}) = \sinh^2\theta\,T_{m_1\ldots m_j\ldots m_i\ldots m_n}(\sigma_{ab})$$

which yields

$$T_{m_1\ldots m_i\ldots m_j\ldots m_n}(\sigma_{ab}) = -T_{m_1\ldots m_j\ldots m_i\ldots m_n}(\sigma_{ab}).$$

Thus we have

$$T_{m_1\ldots m_i\ldots m_j\ldots m_n}(\sigma_{ab}) = -T_{m_1\ldots m_j\ldots m_i\ldots m_n}(\sigma_{ab})$$

for any two indices m_i and m_j regardless of whether or not g_{ij} is positive definite.

Substitution of $T_{m_1\ldots m_n}(\sigma_{ab}) = ae_{m_1\ldots m_n}$ into (10) yields

$$T_{m_1\ldots m_n}(g_{ij}) = aB^{c_1}_{m_1}\ldots B^{c_n}_{m_n}e_{c_1\ldots c_n} = a\det(B^i_j)e_{m_1\ldots m_n},$$

and by (1) it is readily verified that $\sqrt{|g|} = \det B$, whence the theorem follows. (It should be noted that the above proof holds only at any given point in the space, and conceivably, a could be a scalar function. By Theorem 1, this scalar concomitant of g_{ij} must be constant).

Note. We have, in effect, also determined the result for any mixed relative tensor, since we may always lower the contravariant indices of such a tensor, obtain the solution for this completely covariant tensor, then re-raise the appropriate indices to get the required form.

In the proofs of the remaining theorems we consider a specific point p_0 in V^n, and *transformations* $x^i \leftrightarrow \bar{x}^i$ *for which* $A^i_j|_{p_0} = \delta^i_j$ *at* p_0, *while the higher ordered derivatives* A^i_{jk}, A^i_{jkr}, etc. *are arbitrary at* p_0. Hence any relative tensor $T^{m_1\ldots m_r}_{n_1\ldots n_s}$ remains invariant under the transformations to be considered (since $J = 1$). It is therefore unnecessary to indicate the co- and contravariant nature of the tensor concomitants considered, and we write briefly T^{---}_{---}. Our transformations applied to the metric tensor g_{ij} then become

$$\bar{g}_{ij}|_{p_0} = g_{ij}|_{p_0},$$

$$\bar{g}_{ij,k}|_{p_0} = (A^a_{ik}\delta^b_j + \delta^a_i A^b_{jk})g_{ab} + g_{ij,k}|_{p_0}, \tag{13}$$

$$\begin{aligned}\bar{g}_{ij,kl}|_{p_0} = {} & (A^a_{ikl}\delta^b_j + A^a_{ik}A^b_{jl} + A^a_{il}A^b_{jk} + A^b_{jkl}\delta^a_i)g_{ab} + (A^a_{ik}\delta^b_j + A^b_{jk}\delta^a_i)g_{ab,l} + \\ & + A^a_{il}g_{aj,k} + A^b_{jl}g_{ib,k} + A^c_{kl}g_{ij,c} + g_{ij,kl}|_{p_0}.\end{aligned} \tag{14}$$

THEOREM 5. *Any relative tensor concomitant* $T^{m_1\ldots m_r}_{n_1\ldots n_s}(g_{ij}; g_{ij,k})$, *covariant of order* s *and contravariant of order* r, *and of weight* p, *must be independent of* $g_{ij,k}$ *for any symmetric* g_{ij} *for which* $\det(g_{ij}) \neq 0$.

Proof. We now further demand that $A^i_{jk} = -\Gamma^i_{jk}$ at p_0. By (13) we have

$$\bar{g}_{ij,k} = -\Gamma^a_{ik}g_{aj} - \Gamma^a_{jk}g_{ai} + g_{ij,k} = 0 \quad \text{at } p_0.$$

Thus the transformation law of T^{---}_{---} at p_0 yields $T^{---}_{---}(g_{ij}; g_{ij,k}) = T^{---}_{---}(g_{ij}; 0)$; that is, T^{---}_{---} must be independent of $g_{ij,k}$ at p_0. Since p_0 is any point in V^n the theorem follows.

Note. The above is equivalent to the assertion that it is always possible to transform locally to the normal co-ordinate system in which the Christoffel symbols Γ^i_{jk} vanish at p_0; further, the theorem to follow simply asserts the existence of absolute normal co-ordinates [14]. However, our proofs may be of interest in themselves, since they are purely algebraic.

For other references to Theorems 4 through 10, we note Schouten [13], pp. 163-165, Moór [7], [8], [9] and Zajtz [15] and Lorens [4].

THEOREM 6. *Given an arbitrary point p_0 in V^n it is always possible to choose a co-ordinate transformation $x^i \leftrightarrow \bar{x}^i$ at the point such that $\bar{g}_{ij,kl} = \frac{1}{3}(\bar{R}_{ilkj}+\bar{R}_{iklj})$, provided that* $\det(g_{ij}) \neq 0$.

Proof. We shall assume that we have already performed a transformation to co-ordinates x^i at p_0 such that $g_{ij,k} = 0$ at p_0 in this system; this is always possible as in Theorem 5.

Let us consider transformations for which, at p_0, $A^a_i = \delta^a_i$ and $A^a_{ij} = 0$. The transformation laws (13) and (14), in view of the fact that we assumed $g_{ij,k} = 0$, reduce to $\bar{g}_{ij} = g_{ij}$, $\bar{g}_{ij,k} = 0$, and $\bar{g}_{ij,kl} = A^a_{ikl}g_{aj}+A^a_{jkl}g_{ai}+ + g_{ij,kl}$. Further, suppose that at p_0

$$A^a_{ikl}g_{aj} = \tfrac{1}{6}(g_{kl,ij}+g_{il,jk}+g_{ik,jl})-\tfrac{1}{3}(g_{ij,kl}+g_{jk,il}+g_{jl,ik});$$

this expressions contains the correct symmetries, and the set of equations always possesses a solution since $\det(g_{ij}) \neq 0$. Substitution into (15) yields

$$\begin{aligned}\bar{g}_{ij,kl} &= A^a_{ikl}g_{aj}+A^a_{jkl}g_{ai}+g_{ij,kl}\\ &= \tfrac{1}{6}(2g_{kl,ij}+2g_{ij,kl}-g_{il,jk}-g_{ik,jl}-g_{jl,ik}-g_{jk,il}),\end{aligned}$$

or $\bar{g}_{ij,kl} = \frac{1}{3}(R_{ilkj}+R_{iklj})$ in view of $\Gamma^i_{jk} = 0$. But $\bar{R}_{ijkl} = R_{ijkl}$ at p_0, and we have $\bar{g}_{ij,kl} = \frac{1}{3}(\bar{R}_{ilkj}+\bar{R}_{iklj})$ as required.

THEOREM 7. *Any function $L(R_{iklj}+R_{ilkj})$ may be written in the form $L^*(R_{ijkl})$ for some function L^*, and, conversely, a function $M(R_{ijkl})$ may be written $M^*(R_{iklj}+R_{ilkj})$ for some M^*.*

Proof. The first assertion merely states the obvious fact that a function of $R_{iklj}+R_{ilkj}$ may be considered as a function of R_{ijkl}.

The converse follows immediately from the identity

$$R_{ijkl} = -\tfrac{1}{3}\{(R_{iklj}+R_{ilkj})+(R_{ijlk}+R_{iljk})+(R_{jikl}+R_{jkil})\}$$

which implies that any function of the form $M(R_{ijkl})$ is also a function of the form $M^*(R_{iklj}+R_{ilkj})$.

Consider now a relative tensor concomitant $T^{\dots}_{\dots}(g_{ij}; g_{ij,k}; g_{ij,kl})$; Theorems 5 and 6 imply that we can always transform to a co-ordinate system at p_0 in which $g_{ij,k} = 0$ and $g_{ij,kl} = \frac{1}{3}(R_{ilkj}+R_{iklj})$ via a transformation having $A^i_j = \delta^i_j$ at p_0. Thus, in view of Theorem 7,

$$T^{---}_{---}(g_{ij};\, g_{ij,k};\, g_{ij,kl}) = T^{---}_{---}\{g_{ij};\, 0;\, \tfrac{1}{3}(R_{iklj}+R_{ilkj})\}$$
$$= T^{*---}_{\ \ ---}(g_{ij};\, R_{ijkl})$$

for some functions $T^{*---}_{\ \ ---}$.

This concludes the proof of Result 4, that is,

THEOREM 8. *Any relative tensor concomitant of the form* $T^{m_1\ldots m_r}_{n_1\ldots n_s}(g_{ij};\, g_{ij,k};\, g_{ij,kl})$ *may be expressed in the form* $T^{*m_1\ldots m_r}_{\ n_1\ldots n_s}(g_{ij};\, R_{ijkl})$.

THEOREM 9. *Any relative tensor concomitant* $T^{m_1\ldots m_r}_{n_1\ldots n_s}(X_i;\, X_{i,j};\, g_{ij};\, g_{ij,k})$, *where* X_i *is a vector field and* $X_{i,j} = \partial X_i/\partial x^j$, *is of the form* $T^{*m_1\ldots m_r}_{\ n_1\ldots n_s}(X_i;\, X_{i|j};\, g_{ij})$, $X_{i|j}$ *being the covariant derivative of* X_i *relative to the symmetric metric tensor* g_{ij} *having* $\det(g_{ij}) \neq 0$.

At a specific point p_0 in V^n, the transformation $x^i \leftrightarrow \bar{x}^i$ for which $A^a_i|_{p_0} = \delta^a_i$ and $A^a_{ij}|_{p_0} = -\Gamma^a_{ij}|_{p_0}$ (as in Theorem 5) implies $\bar{X}_i = X_i$, $\bar{X}_{i,j} = X_{i,j} - \Gamma^a_{ij}X_a = X_{i|j}$, and $\bar{g}_{ij,k} = 0$. Thus

$$T^{---}_{---}(X_i;\, X_{i,j};\, g_{ij};\, g_{ij,k}) = T^{---}_{---}(X_i;\, X_{i|j};\, g_{ij};\, 0)$$
$$= T^{*---}_{\ \ ---}(X_i;\, X_{i|j};\, g_{ij})$$

as required.

THEOREM 10. *Any relative tensor* $T^{m_1\ldots m_r}_{n_1\ldots n_s}(X_i;\, X_{i,j};\, g_{ij};\, g_{ij,k};\, g_{ij,kl})$ *is of the form* $T^{*m_1\ldots m_r}_{\ n_1\ldots n_s}(X_i;\, X_{i|j};\, g_{ij};\, R_{ijkl})$, *where* X_i *and* g_{ij} *are as in Theorem* 9 *and* R_{ijkl} *is the Riemann curvature tensor relative to* g_{ij}.

Consider a transformation such that at p_0 in V^n, $A^a_i = \delta^a_i$, $A^a_{ij} = 0$, and

$$A^a_{ikl}g_{aj} = \tfrac{1}{6}(g_{kl,ij}+g_{il,jk}+g_{ik,jl}) - \tfrac{1}{3}(g_{ij,kl}+g_{jk,il}+g_{jl,ik});$$

for such a transformation,

$$\bar{g}_{ij,kl} = \tfrac{1}{3}(R_{iklj}+R_{ilkj}) + M_{ijkl},$$

where M_{ijkl} is a quadratic in the Christoffel symbols Γ^i_{jk}. Also, $\bar{g}_{ij,k} = g_{ij,k}$, $\bar{g}_{ij} = g_{ij}$, $\bar{X}_{i,j} = X_{i,j}$, and $\bar{X}_i = X_i$ at p_0.

Thus we have

$$T^{---}_{---}(X_i;\, X_{i,j};\, g_{ij};\, g_{ij,k};\, g_{ij,kl})$$
$$= T^{---}_{---}\big(X_i;\, X_{i,j};\, g_{ij};\, g_{ij,k};\, \tfrac{1}{3}(R_{iklj}+R_{ilkj})+M_{ijkl}\big).$$

(This differs from the transformation of Theorem 6 only in that Γ^i_{jk} are not assumed zero here; hence the presence of M_{ijkl}.) Since $g_{ij,k}$ may be written in terms of g_{ij} and Γ^i_{jk}, there exist functions $T^{*---}_{\ \ ---}$ such that the above may then be written

$$T^{---}_{---}(X_i;\, X_{i,j};\, g_{ij};\, g_{ij,k};\, g_{ij,kl}) = T^{*---}_{\ \ ---}\big(X_i;\, X_{i,j};\, g_{ij};\, \Gamma^i_{jk};\, (R_{iklj}+R_{ilkj})\big).$$

Consider now a transformation such that $A^a_i = \delta^a_i$ and $A^a_{ij} = -\Gamma^a_{ij}$ at p_0, and A^a_{ijk} are arbitrary. For such a transformation, $\bar{X}_{i,j} = X_{i|j}$ and

$\bar{g}_{ij,k} = 0$; our equation thus reduces to

$$T^{---}_{---}(X_i;\, X_{i,j};\, g_{ij};\, g_{ij,k};\, g_{ij,kl}) = T^{*---}_{\ \ ---}(X_i;\, X_{i,j};\, g_{ij};\, \Gamma^i_{jk};\, R_{iklj}+R_{ilkj})$$
$$= T^{*---}_{\ \ ---}(X_i;\, X_{i|j};\, g_{ij};\, 0;\, R_{iklj}+R_{ilkj}).$$

Thus in view of Theorem 7, there exist functions $T^{**---}_{\ \ \ ---}$ such that

$$T^{---}_{---}(X_i;\, X_{i,j};\, g_{ij};\, g_{ij,k};\, g_{ij,kl}) = T^{**---}_{\ \ \ ---}(X_i;\, X_{i|j};\, g_{ij};\, R_{ijkl}),$$

as required.

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