ANNALES
POLONICI MATHEMATICI
XXVII (1973)

The radius of conformity of some classes of regular functions

by STANISŁAW WALCZAK (Łódź)

Introduction. Let B denote the family of regular functions

(1) $$f(z) = z + a_2^f z^2 + \ldots + a_n^f z^n + \ldots$$

defined in $|z| < 1$. We denote by r_f the upper bound of those values of r for which $f(z)$ is univalent in $|z| < r$. The number

(2) $$r_B = \inf_{f \in B} r_f$$

will be called the *radius of conformity of the family B*. In [4] r_B has been found in the case when B is the class of bounded function.

This paper is concerned with an investigation of the radius of conformity of some classes of functions (bounded or unbounded) which may be represented in the form of a Stieltjes-integral.

1. Theorems on the radius of conformity. We shall prove the following

THEOREM 1. *If $f \in B$, then $r_f > 0$.*

Proof. For $|z_1| = |z_2| = \varrho < 1$ we have

$$|f(z_1) - f(z_2)| \geqslant |z_1 - z_2| \Big(1 - \sum_{n=2}^{\infty} n |a_n| \varrho^{n-1}\Big).$$

Let ϱ_0 be such that the last term is positive for $\varrho = \varrho_0$. Then the function $f(z)$ is univalent on the circle $|z| = \varrho_0$, hence in the disc $|z| < \varrho_0$. So $r_f \geqslant \varrho_0 > 0$. Another proof of this theorem can be found e.g. in [2].

Let B_1 be any family of regular functions $f(z) = z + a_2^f z^2 + \ldots + a_n^f z^n + \ldots$ defined in $|z| < 1$ such that

(3) (a) for every n there exists $A(n)$ such that $|a_n^f| \leqslant A(n)$,
(b) $\limsup \sqrt[n]{A(n)} < \infty$.

THEOREM 2. *The radius of conformity of the family B_1 is positive.*

Proof. We have, similary as in the proof of Theorem 1,

$$|f(z_1)-f(z_2)| \geqslant |z_1-z_2|\Big(1-\sum_{n=2}^{\infty} nA(n)\varrho^{n-1}\Big).$$

There exists a $\varrho_0 > 0$ independent of f, such that $1-\sum_{n=2}^{\infty} nA(n)\varrho_0^{n-1} > 0$. Therefore $r_{B_1} \geqslant \varrho_0 > 0$.

We denote by M the set of real functions $a(t)$ which are non-decreasing on a certain segment $[a, b]$ and such that $\int_a^b da(t) = 1$. Let E_g be the subclass of the family B_1 consisting of functions of the form

$$f(z) = \int_a^b g(z, t)\, da(t), \tag{4}$$

where $a \in M$, $g(z, t)$ is a function continuous with respect to t and such that for every fixed $t \in [a, b]$ the function $g(\cdot, t) \in B_1$. By Theorem 2 we conclude that $r_{E_g} > 0$.

Consider the functional

$$F(f) = \frac{f(z_1)-f(z_2)}{z_1-z_2}, \tag{5}$$

where $f \in E_g$, $|z_1| = |z_2| = \varrho < 1$, $z_1 \neq z_2$. On account of (4) we have

$$F(f) = \int_a^b \frac{g(z_1, t)-g(z_2, t)}{z_1-z_2}\, da(t), \quad a(t) \in M. \tag{6}$$

The set of values of functional (6) is the convex hull of the curve (cf. e.g. [1])

$$z(t) = z(t, \varrho, \varphi_1, \varphi_2) = \frac{g(\varrho e^{i\varphi_1}, t)-g(\varrho e^{i\varphi_2}, t)}{\varrho e^{i\varphi_1}-\varrho e^{i\varphi_2}}, \tag{7}$$

where $\varrho e^{i\varphi_1} = z_1$, $\varrho e^{i\varphi_2} = z_2$, $a \leqslant t \leqslant b$.

Let $W(\varrho, \varphi_1, \varphi_2)$ denote the convex hull of the curve (7) and Γ its boundary. Let ϱ_0 be the upper bound of those values of ϱ for which for every $\varphi_1, \varphi_2 \in [0, 2\pi)$ with $\varphi_1 \neq \varphi_2$ the closed region $W(\varrho, \varphi_1, \varphi_2)$ does not include the point $F = 0$. From Definition (2) and Theorem 2 it follows that

$$\varrho_0 = r_{E_g} \quad \text{and} \quad r_{E_g} > 0. \tag{8}$$

As an example consider the family of functions $f(z)$ of the form

$$f(z) = \int_{-\pi}^{\pi} (z+e^{it}z^n)\, da(t), \quad a(t) \in M. \tag{8'}$$

In this case the curve (7) is defined by the equation

$$z(t) = 1+e^{it}\varrho^{n-1}\sum_{k=0}^{n-1}\exp i[(n-k-1)\varphi_1+k\varphi_2].$$

The region $W(\varrho,\varphi_1,\varphi_2)$ is the closed disc with centre $z=1$ and radius

$$r = \Big|\varrho^{n-1}\sum_{k=0}^{n-1}\exp i[(n-k-1)\varphi_1+K\varphi_2]\Big|.$$

We have

$$\Big|\varrho^{n-1}\sum_{k=0}^{n-1}\exp i[(n-K-1)\varphi_1+K\varphi_2]\Big| \leqslant n\varrho^{n-1}.$$

Thus for $\varrho<(1/n)^{1/(n-1)}$ the point $F=0$ does not belong to the region $W(\varrho,\varphi_1,\varphi_2)$ for any values of φ_1,φ_2. It is not difficult to notice that the estimation is sharp. Thus the radius of conformity of the family E_g defined by (8′) is $\varrho_0=(1/n)^{1/(n-1)}$.

2. The radius of conformity of the family T. Let T denote the class of regular functions $f(z)=z+a_z^f z^2+\ldots+a_n^f z^n+\ldots$ defined in $|z|<1$ which have real values if and only if z is real. This family has been introduced by Rogosiński [3]. The functions of this class are called *typically-real.* It is known that $\sup_{f\in T}|a_n^f|=n$.

Thus, for B_1 with $A(n)=n$,

$$T\subset B_1 \quad \text{and} \quad r_T>0. \tag{9}$$

In the sequel we shall find the exact value of r_T.

LEMMA 1. *The curve defined by the equation*

$$\mu(t) = (1+2tz_1+z_1^2)(1+2tz_2+z_2^2), \quad z_1=\varrho e^{i\varphi_1}, \quad z_2=\varrho e^{i\varphi_2}, \tag{10}$$
$$\varphi_1,\varphi_2\in(0,\pi),\ \varrho\in(0,1),\ t\in[-1,1],$$

is convex and $\arg\mu(t)$ *is an increasing function.*

Proof. It is not difficult to prove that

$$\frac{d}{dt}\arg\mu'(t) = \frac{\bar{\mu}'\cdot\mu''-\mu'\cdot\bar{\mu}''}{2i}.$$

After detailed calculations we obtain

$$\frac{\bar{\mu}'\cdot\mu''-\mu'\cdot\bar{\mu}''}{2i} = -16(\varrho^3-\varrho^5)(\sin\varphi_1+\sin\varphi_2)<0. \tag{11}$$

Thus the curve (10) is convex. It is casily verified that

$$(12)\qquad \frac{d}{dt}\arg\mu(t) = \frac{2(\varrho-\varrho^3)\sin\varphi_1}{|1+2t\varrho e^{i\varphi_1}+\varrho^2 e^{2i\varphi_1}|^2} + \frac{2(\varrho-\varrho^3)\sin\varphi_2}{|1+2t\varrho e^{i\varphi_2}+\varrho^2 e^{2i\varphi_2}|^2} > 0.$$

Hence also

$$(12')\qquad \frac{\mu'\cdot\bar{\mu}-\bar{\mu}'\cdot\mu}{2i} > 0.$$

From equality (12) we conclude that $\arg\mu(t)$ is an increasing function.

LEMMA 2. *The curve defined by the equation*

$$(13)\qquad z(t) = \frac{1}{\mu(t)}, \quad t\in[-1,1]$$

is convex, $\arg z(t)$ *is a decreasing function and* $\arg z(-1)-\arg z(1) < 2\pi$.

Proof. $\arg\mu(t)$ is an increasing function and therefore $\arg z(t)$ is an increasing function. It can proved that

$$\frac{\bar{z}'\cdot z''-z'\cdot\bar{z}''}{2i} = \frac{1}{|\mu|^6}\left(\frac{\bar{\mu}'\cdot\mu''-\mu'\cdot\bar{\mu}''}{2i}|\mu|^2 + 2\frac{\mu\cdot\bar{\mu}'-\bar{\mu}\cdot\mu'}{2i}|\mu'|^2\right).$$

By (11) and (12) we get

$$\frac{\bar{z}'\cdot z''-z'\cdot\bar{z}''}{2i} < 0.$$

Thus the curve (13) is convex. It is not difficult to notice that the incresement of the argument of the function $z(t)$ in the interval $[-1,1]$ is greater than -2π.

THEOREM 3. $r_T = \sqrt{3-2\sqrt{2}} \approx 0.41\ldots$

Proof. It is known [3] that the family T may be represented by a structural formula of the form

$$f(z) = \int_{-\pi}^{\pi}\frac{z}{1-2z\cos t+z^2}\,da(t), \qquad a(t)\in M.$$

The curve (7) in this case is defined by the formula

$$(14)\qquad z(t) = \frac{1-z_1\cdot z_2}{(1-2z_1\cos t+z_1^2)(1-2z_2\cos t+z_2^2)},$$

$$z_1 = \varrho e^{i\varphi_1},\ z_2 = \varrho e^{i\varphi_2}.$$

The set of points of the curve (14) coincides with that of the curve

$$(14')\qquad z(t) = \frac{1-z_1\cdot z_2}{(1+2tz_1+z_1^2)(1+2tz_2+z_2^2)}, \quad t\in[-1,1].$$

Since the image of the half-circle $|z| = \varrho$, $\operatorname{im} z > 0$ under the transformation $w = f(z)$ is symmetrical to the image of the half-circle $|z| = \varrho$, $\operatorname{im} z < 0$ with respect to the straight line $\operatorname{im} w = 0$, we may assume that $\varphi_1, \varphi_2 \in (0, \pi)$.

By Lemma 2 we conclude that the curve (14) is convex, $\arg z(t)$ is a decreasing function and the incresement of the argument is greater than -2π. Thus the convex hull of this curve is the closed region $W(\varrho, \varphi_1, \varphi_2)$ which does not contain $z = \infty$ and whose boundary is the union of the segment

$$(15) \qquad z_1(\lambda) = \lambda \frac{1 - z_1 \cdot z_2}{(1+z_1)^2(1+z_2)^2} + (1-\lambda)\frac{1 - z_1 \cdot z_2}{(1-z_1)^2(-z_2)^2}, \qquad \lambda \in [0, 1]$$

and the curve (14′) (Fig. 1). By (9), for ϱ sufficiently small the point $z = 0$ does not belong to W. Let ϱ_0 be a number from the interval $(0, 1)$

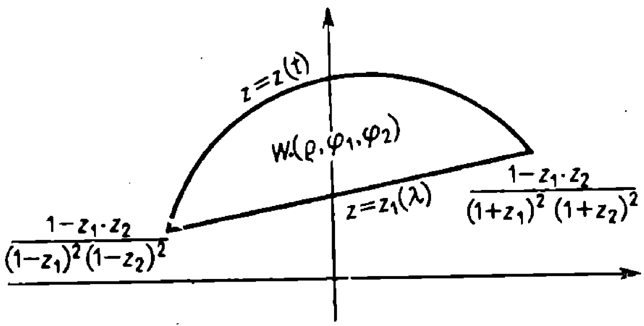

Fig. 1

such that $z = 0$ belongs to Γ for some values φ_1, φ_2. Since $z(t) \neq 0$, the point $z = 0$ belongs to the segment (15) (Fig. 2).

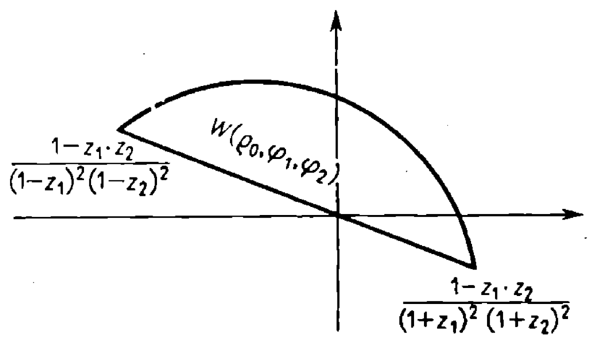

Fig. 2

Thus we have

$$\frac{(1+z_1)^2(1+z_2)^2}{(1-z_1)^2(1-z_2)^2} = a, \qquad a < 0.$$

Hence

$$\frac{(1+z_1)(1+z_2)}{(1-z_1)(1-z_2)} = \beta i, \qquad \beta = \bar{\beta}.$$

Putting $z_1 = \varrho_0 e^{i\varphi_1}$, $z_2 = \varrho_0 e^{i\varphi_2}$ we obtain

$$(16) \qquad \frac{(1-\varrho_0^2+2i\varrho_0\sin\varphi_1)(1-\varrho_0^2+2i\sin\varphi_2)}{(1-\varrho_0 e^{i\varphi_1})(1-\varrho_0 e^{i\varphi_2})^2} = \beta i,$$

$$\operatorname{re}(1-\varrho_0^2+2i\varrho_0\sin\varphi_1)(1-\varrho_0^2+2i\sin\varphi_2)$$
$$= (1-\varrho_0^2)^2 - 4\varrho_0^2\sin\varphi_1\sin\varphi_2 \geqslant (1-\varrho_0^2)^2-4\varrho_0^2.$$

The only root within the interval $(0, 1)$ of the equation

$$(1-\varrho_0^2)^2-4\varrho_0^2 = 0$$

is the number

$$\varrho_0 = \sqrt{3-2\sqrt{2}}.$$

It is easily noticed that for $0 < \varrho < \varrho_0$ the real part of the left-hand side of formula (16) is positive with arbitrary φ_1, φ_2. By (9) we conclude that the point $z = 0$ does not belong to the set W with $\varrho \in (0, \varrho_0)$ for arbitrary values φ_1, φ_2. The function

$$h(z) = \frac{1}{2}\left(\frac{z}{(1-z)^2}+\frac{z}{(1+z)^2}\right)$$

belongs to the family T and $h'(z) = 0$ for $z = i\sqrt{3-2\sqrt{2}}$. Thus $\varrho_0 = \sqrt{3-2\sqrt{2}}$ is the radius of conformity of the family of typically real functions.

Let H denote the family of meromorphic and typically real functions $F(\zeta) = \zeta + \sum_{n=0}^{\infty} a_n/\zeta^n$ defined in $|\zeta| > 1$.

It is not difficult to prove that between the functions of the classes T and H the following relationship

$$F(\zeta) = \frac{1}{f(1/\zeta)}$$

holds.

COROLLARY. *In the region* $|\zeta| > r_H$, *where*

$$r_H = \frac{1}{r_T} = \sqrt{3-2\sqrt{2}}(3+2\sqrt{2}),$$

all functions of the family H *are univalent.*

References

[1] J. Krzyż, *Zbiór zadań z funkcji analitycznych*, rozdz. X.
[2] F. Leja, *Teoria funkcji analitycznych*, Warszawa 1957.
[3] W. Rogosiński, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Mat. Z. 35 (1932), p. 93-121.
[4] S. Valiron, *Fonctions analytiques*, Paris 1954.

Reçu par la Rédaction le 19. 9. 1971