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On a non-linear convolution equation occurring in the theory of water percolation

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Abstract. We study the one-dimensional non-linear equation

$$u^a = K*u+L \quad (a > 1),$$

where K and L vanish on the half-line $(-\infty, 0)$, K is a non-decreasing function having a jump at the origin and $L(x)/x^{1/(a-1)}$ is a non-decreasing convex function.

We give some theorems concerning the existence and uniqueness of solutions u such that $u(x) = 0$ for $x \leqslant 0$ and $u(x) > 0$ for $x > 0$.

For $a = 2$ we show a dependence of solutions of our equation on the function L.

Suppose we are considering water percolation from a cylindrical reservoir into the surrounding unsaturated region. Describing this phenomena in the Boussinesq model (see [1]) we are led to the one-dimensional non-linear integral equation (see [2]), which can be brought to the form

$$u^2 = K*u+L, \tag{1}$$

or

$$u^2 = K*u, \tag{1'}$$

if we seek only approximative solutions of the problem. Here K, L are known smooth functions depending on physical parameters. The unknown function u describes the water table in a suitably introduced coordinate system. This is the reason why, from the physical point of view, non-negative solutions of (1) or (1′) vanishing identically in the interval $(-\infty, 0]$ are most interesting. Equation (1) may be considered independently of its physical meaning. One may ask for its properties in the case when K, L are arbitrary locally summable functions or, more generally, distributions such that the convolution on the right-hand side is well defined. Paper [5] was an initial step in this direction. Now we are going to consider the more general equation

$$u^a = K*u+L, \quad \text{where } a > 1. \tag{2}$$

1. Properties of non-negative solutions. We study equation (2) on the whole real line R. We suppose that the function K satisfies the following conditions:

(i) K is a non-decreasing function on R,

(ii) $K(x) = 0$ for $x < 0$,

(iii) the limit $\lim\limits_{x\to 0^+} K(x) = g$ is a positive number (see [5]).

Moreover, we suppose that

(iv) $L(x) = 0$ for $x < 0$,

(v) $S(x) = L(x)/x^{1/(a-1)}$ is a non-decreasing convex function for $x > 0$,

(vi) $\lim\limits_{x\to 0^+} S(x) = 0$.

We denote by M_0 the set of all measurable functions f on R such that $f(x) = 0$ for $x \leqslant 0$ and $f(x) > 0$ for $x > 0$. By Q_0 we denote the subset of all continuous functions $u \in M_0$.

THEOREM 1. *If $u \in M_0$ is a solution of* (2), *then u is a non-decreasing function belonging to the set Q_0.*

Proof. Since $K*u$ (see [5]) and, by condition (v), also L are non-decreasing functions, then u^a is a non-decreasing function. Hence u is a non-decreasing function.

By conditions (iv), (v), (vi) L is a continuous function on R. Since $K*u$ is continuous (see [5]), it follows that u is a continuous function.

THEOREM 2. *If $u \in Q_0$ is a solution of* (2), *then*

$$(3)\qquad \left[\frac{a-1}{a}gx\right]^{1/(a-1)} \leqslant u(x) \leqslant \left[\int\limits_0^x K(\tau)d\tau + \left[\frac{a}{(a-1)g}\right]^{1/(a-1)} S(x)\right]^{1/(a-1)}$$

for $x > 0$.

Proof. The first inequality of (3) can be obtained just as in [5]. We have, by Theorem 1,

$$u^a(x) \leqslant u(x)\left[\int\limits_0^x K(\tau)d\tau + \frac{L(x)}{u(x)}\right] \quad \text{for } x > 0.$$

Then, by the first inequality of (3), we get

$$u^a(x) \leqslant u(x)\left[\int\limits_0^x K(\tau)d\tau + \left[\frac{a}{(a-1)g}\right]^{1/(a-1)} S(x)\right],$$

from which we obtain

$$u^{a-1}(x) \leqslant \int\limits_0^x K(\tau)d\tau + \left[\frac{a}{(a-1)g}\right]^{1/(a-1)} S(x).$$

Hence we get the second inequality of (3).

2. Existence and uniqueness of non-negative solutions. For a function $f \in Q_0$ we denote by $T(f)$ the function

$$T(f) = (K * f + L)^{1/a}. \tag{4}$$

Let P be the set of all functions belonging to the set Q_0 and satisfying inequality (3).

LEMMA 1. *The operator T transforms P into P.*

Proof. Let

$$F(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ \left[\dfrac{a-1}{a} gx\right]^{1/(a-1)} & \text{for } x > 0, \end{cases} \tag{5}$$

$$G(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ \left[\displaystyle\int_0^x K(\tau)\,d\tau + \left[\frac{a}{(a-1)g}\right]^{1/(a-1)} S(x)\right]^{1/(a-1)} & \text{for } x > 0. \end{cases} \tag{6}$$

Using conditions (i)–(vi) we get

$$T(F)(x) \geqslant F(x) \quad \text{and} \quad T(G)(x) \leqslant G(x)$$

for all $x \in R$, from which, just as it was done in Lemma 4 in [5], we obtain the proof.

It follows from property (iii) that for every $b > 0$ there exists $c > 0$ such that $c < b$ and $K(c)/g < a$.

Let P^b be the set of all functions from Q_0 restricted to $(-\infty, b]$ and satisfying inequality (3) on $(0, b]$.

For any $f_1, f_2 \in P^b$ we can define

$$\varrho_b(f_1, f_2) = \sup_{0<\tau\leqslant b} \frac{|f_1(\tau) - f_2(\tau)|}{e^{\beta\tau} r(\tau)}, \tag{7}$$

where

$$\beta = \frac{1}{g} \sup_{c\leqslant\tau\leqslant b} \frac{K(\tau) - g}{\tau} \quad \text{and} \quad r(x) = G(x) - F(x).$$

LEMMA 2. *The function ϱ_b is a metric in P^b and P^b is a complete metric space.*

This lemma can be proved like Lemma 5 in [5].

LEMMA 3. *For every $x \geqslant 0$*

$$\frac{a-1}{a} x r(x) \geqslant \int_0^x r(\tau)\,d\tau. \tag{8}$$

Proof. It follows from conditions (iv)–(vi) that

$$S(x) = \int_0^x W(\tau)\,d\tau,$$

where W is a non-negative non-decreasing function (see [6]). Hence, by (6), we obtain

$$G(x) = \int_0^x Z(\tau)\,d\tau,$$

where Z is a non-negative non-decreasing function.

Let

$$l(x) = \frac{a-1}{a} x r(x) - \int_0^x r(\tau)\,d\tau.$$

Then

$$l'(x) = \frac{1}{a}[G(x)]^{2-a}\Big[xZ(x) - \int_0^x Z(\tau)\,d\tau\Big]$$

for almost all $x \geqslant 0$. From the inequality

$$\int_0^x Z(\tau)\,d\tau \leqslant Z(x)x \quad \text{for } x \geqslant 0$$

we get

$$l'(x) \geqslant 0 \quad \text{for almost all } x \geqslant 0.$$

We have $l(x) \geqslant 0$ for $x \geqslant 0$, because l is absolutely continuous (see [4]). This implies inequality (8).

LEMMA 4. *For* $f_1, f_2 \in P^b$

$$\varrho_b\big(T(f_2), T(f_1)\big) \leqslant \frac{K(c)}{ag}\,\varrho_a(f_2, f_1). \tag{9}$$

Proof. We get, by the Lagrange theorem,

$$|T(f_2)(x) - T(f_1)(x)| \leqslant \frac{1}{a}\,\frac{|K*(f_2-f_1)(x)|}{\big[\min\big(T(f_2)(x), T(f_1)(x)\big)\big]^{a-1}} \quad \text{for } x > 0,$$

from which, by Lemma 1, we obtain

$$|T(f_2)(x) - T(f_1)(x)| \leqslant \frac{1}{(a-1)gx}\,K*|f_2 - f_1|(x).$$

From the last inequality we get

$$|T(f_2)(x) - T(f_1)(x)| \leqslant \frac{1}{(a-1)gx}\int_0^x K(x-\tau)e^{\beta\tau} r(\tau)\,d\tau\,\varrho_b(f_2, f_1).$$

This inequality can be written as

$$|T(f_2)(x)-T(f_1)(x)| \leqslant \frac{e^{\beta x}}{(a-1)gx}\int_0^x K(\tau)e^{-\beta\tau}r(x-\tau)\,d\tau\,\varrho_b(f_2,f_1)$$

(see [5]). Since for $x \in [0, b]$ we have

$$K(x)e^{-\beta x} \leqslant K(c) \tag{10}$$

(see Lemma 7 in [5]), we obtain

$$|T(f_2)(x)-T(f_1)(x)| \leqslant \frac{e^{\beta x}K(c)}{(a-1)gx}\int_0^x r(\tau)\,d\tau\,\varrho_b(f_2,f_1).$$

Applying inequality (8) we can write

$$|T(f_2)(x)-T(f_1)(x)| \leqslant \frac{K(c)}{ag}e^{\beta x}r(x)\,\varrho_b(f_2,f_1).$$

The lemma is proved.

THEOREM 3. *Equation* (2) *has a unique solution in the set* M_0; *this solution belongs to* Q_0.

Proof. The operator T is, by Lemma 4, a contraction on the complete metric space P^b. Using the Banach contraction theorem (see [3]) we infer that T has a unique fixed point in P^b for every $b > 0$. This implies that T has a unique fixed point in P.

3. Some estimates. For applications it is often useful to investigate solutions of the homogeneous equation

$$u^a = K*u. \tag{11}$$

Now we give an estimation for the solution of this equation.

LEMMA 5. *If* $u \in M_0$ *is the solution of* (11), *then*

$$u(x) \leqslant \left[\frac{a-1}{a}\int_0^x K(\tau)\,d\tau\right]^{1/(a-1)} \quad \textit{for } x \geqslant 0. \tag{12}$$

This lemma can be proved like Theorem 5 in [5].

Let $a = 2$. We assume conditions (iv)–(vi) for functions L_j $(j = 1, 2)$.

Let u_j be the solution of the equation

$$u^2 = K*u+L_j \quad (j = 1, 2). \tag{13}$$

We have, by Theorem 2,

$$\tfrac{1}{2}gx \leqslant u_j(x) \leqslant \int_0^x \left[K(\tau)+\max\left(W_1(\tau),\, W_2(\tau)\right)\right]d\tau \quad (j = 1, 2),$$

where W_1, W_2 are non-decreasing non-negative functions and

$$S_j(x) = \int_0^x W_j(\tau)\,d\tau \qquad (j = 1, 2)$$

(see [6]). We can show the following dependence of solutions of (13) on the function L.

LEMMA 6. *For* $b > 0$

$$\left[1-\frac{K(c)}{2g}\right]\sup_{0<\tau\leqslant b}\frac{|u_1(\tau)-u_2(\tau)|}{e^{\beta\tau}h(\tau)} \leqslant \frac{1}{g}\sup_{0<\tau\leqslant b}\frac{|L_1(\tau)-L_2(\tau)|}{e^{\beta\tau}\tau h(\tau)}, \tag{14}$$

where $h(x) = \int_0^x \left[K(\tau)-\tfrac{1}{2}g+\max\left(W_1(\tau), W_2(\tau)\right)\right]d\tau$.

Proof. For $x \in (0, b]$, by Lemma 1, we have

$$|u_1(x)-u_2(x)| \leqslant \frac{1}{gx}\left[\int_0^x K(x-\tau)|u_1(\tau)-u_2(\tau)|\,d\tau + |L_1(x)-L_2(x)|\right].$$

Hence we get

$$|u_1(x)-u_2(x)| \leqslant \sup_{0<\tau\leqslant b}\frac{|u_1(\tau)-u_2(\tau)|}{e^{\beta\tau}h(\tau)}\,\frac{1}{gx}\int_0^x K(x-\tau)e^{\beta\tau}h(\tau)\,d\tau + \frac{1}{gx}|L_1(x)-L_2(x)|.$$

Applying (10) we obtain

$$|u_1(x)-u_2(x)| \leqslant \sup_{0<\tau\leqslant b}\frac{|u_1(\tau)-u_2(\tau)|}{e^{\beta\tau}h(\tau)}\,\frac{K(c)}{2g}e^{\beta x}h(x) + \frac{1}{gx}|L_1(x)-L_2(x)|.$$

Since $a = 2$, the number

$$\sup_{0<\tau\leqslant b}\frac{|L_1(\tau)-L_2(\tau)|}{e^{\beta\tau}\tau h(\tau)} \quad \text{is finite.}$$

We can write

$$\sup_{0<\tau\leqslant b}\frac{|u_1(\tau)-u_2(\tau)|}{e^{\beta\tau}h(\tau)} \leqslant \frac{K(c)}{2g}\sup_{0<\tau\leqslant b}\frac{|u_1(\tau)-u_2(\tau)|}{e^{\beta\tau}h(\tau)} + \frac{1}{g}\sup_{0<\tau\leqslant b}\frac{|L_1(\tau)-L_2(\tau)|}{e^{\beta\tau}\tau h(\tau)},$$

from which we get (14).

COROLLARY. *From* (14) *we obtain*

$$\sup_{0<\tau\leqslant b}|u_1(\tau)-u_2(\tau)| \leqslant \frac{2e^{\beta b}}{2g-K(c)}\left[\int_0^b [K(\tau)-\tfrac{1}{2}g]\,d\tau + bB\right]\sup_{0<\tau\leqslant b}\frac{|L_1(\tau)-L_2(\tau)|}{e^{\beta\tau}\tau\int_0^\tau [K(s)-\tfrac{1}{2}g]\,ds}, \tag{15}$$

where $B = \max\left(W_1(b), W_2(b)\right)$.

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