ANNALES
POLONICI MATHEMATICI
XXXI (1975)

Remark on a conjectured characterization of the sphere

by Rolf Schneider (Berlin)

Abstract. It is proved that a convex body in a three-dimensional Euclidean space whose surface area measures Φ_1, Φ_2 satisfy the inequality $\Phi_2 - 2r\Phi_1 + r^2\Omega \leqslant 0$ (Ω being Lebesgue measure on the unit sphere) with some constant r, and which admits some circular projection, is either a ball or a solid circular cylinder with two half-balls attached to its ends. This generalizes (and simplifies the proof of) a recent result of Koutroufiotis who considered ovaloids of class C^3 with positive Gauss curvature.

It has been conjectured that a sufficiently smooth ovaloid (closed convex surface) in a three-dimensional Euclidean space E^3, whose principal curvatures k_1, k_2 satisfy the inequality

$$(k_1 - c)(k_2 - c) \leqslant 0 \tag{1}$$

with some constant c, must be a sphere. Partial results have been obtained by Aleksandrov [2] (for analytic surfaces and under the additional assumption that $(k_1 - c)(k_2 - c) = 0$ only if both factors are zero), and Münzner [9] (for analytic surfaces) and [8] (for surfaces of revolution). Recently Koutroufiotis [7] has shown that an ovaloid of class C^3 with positive Gauss curvature is necessarily a sphere if (1) holds and if the surface is assumed to admit a circular enveloping cylinder in some direction. The purpose of this note is to show that, if this additional hypothesis is accepted, a very short proof is possible, and a more general result can be obtained.

For an ovaloid with positive Gauss curvature assumption (1) is equivalent to

$$S_2 - 2rS_1 + r^2 \leqslant 0, \tag{2}$$

where $r = c^{-1}$ and S_2, $2S_1$ denote the product, respectively the sum, of the principal radii of curvature of the surface. Here S_2, S_1 are viewed as functions on the unit sphere Σ with its centre at the origin of E^3, where the correspondence between the ovaloid and the sphere is established by the spherical image map. For a unit vector $v \in \Sigma$ we have the well-known formulae (Bonnesen–Fenchel [3], p. 67)

$$(3)\qquad \begin{aligned} \int_{\Sigma} |\langle u, v\rangle| S_2(u)\, d\Omega(u) &= 2A(v),\\ \int_{\Sigma} |\langle u, v\rangle| S_1(u)\, d\Omega(u) &= L(v) \end{aligned}$$

($\langle\ ,\ \rangle$ being the inner product in E^3 and Ω Lebesgue measure on Σ), where $A(v)$ denotes the area of the orthogonal projection of the ovaloid in direction v onto some plane, and $L(v)$ is the length of the boundary curve of that projection. If now the enveloping cylinder of the ovaloid in direction v is circular, say of radius ϱ, it follows from (2) and (3) that

$$0 \geqslant \int_{\Sigma} |\langle u, v\rangle| (S_2 - 2rS_1 + r^2)\, d\Omega(u) = 2\pi(\varrho - r)^2 \geqslant 0,$$

hence $\varrho = r$, and the surface satisfies $S_2 - 2rS_1 + r^2 = 0$, or $(k_1 - c)(k_2 - c) = 0$ throughout. In this case it is well known that the ovaloid must be a sphere.

Let us now turn to the announced generalization. It concerns arbitrary convex bodies. For a convex body K (non-empty, compact, convex point set) in E^3, let $\Phi_2(K, \cdot)$ denote its second and $\Phi_1(K, \cdot)$ its first surface area measure, as introduced by Aleksandrov [1] (§ 2, p. 962 and 966) and Fenchel–Jessen [5] (see also Busemann [4], § 8 and p. 70). Thus Φ_2 and Φ_1 are Borel measures on the unit sphere Σ, which may be characterized by the fact that they depend in a weakly continuous manner on the convex bodies (where the set of convex bodies is equipped with the usual topology induced by the Hausdorff–Blaschke metric, see [3], p. 34), and that

$$\Phi_i(K, \omega) = \int_{\omega} S_i\, d\Omega \quad \text{for each Borel set } \omega \subseteq \Sigma$$

($i = 1, 2$), if the boundary of K is of class C^2 and has positive Gauss curvature. Hence, for such convex bodies, assumption (1) is equivalent to

$$(4)\qquad \Phi_2(K, \cdot) - 2r\Phi_1(K, \cdot) + r^2\Omega \leqslant 0;$$

but the latter inequality makes sense for arbitrary convex bodies. We shall prove the following theorem.

THEOREM. *If a convex body K in E^3 satisfies (4) with some constant r and admits some circular orthogonal projection, then K is either a ball or a solid circular cylinder with two half-balls attached to its ends.*

It has been conjectured by A. D. Aleksandrov (in his lecture at the ICM, Moscow 1966) that this result holds without the assumption on the existence of a circular projection. Clearly our theorem implies the result of Koutroufiotis, since the boundary surface of the "telescoped" ball is only C^1.

To prove the theorem, we first remark that formulae (3) hold for an arbitrary convex body, if $S_i d\Omega$ is replaced by $d\Phi_i$ (see, e.g., [1], II). Therefore assumption (4) implies that any circular projection of the body K has radius r, and that

$$\Phi_2(K,\omega) - 2r\Phi_1(K,\omega) + r^2\Omega(\omega) = 0 \tag{5}$$

for each Borel set $\omega \subseteq \Sigma \setminus \sigma_v$, where $\sigma_v = \{u \in \Sigma \mid \langle u, v\rangle = 0\}$, provided that $v \in \Sigma$ is a direction for which the projection of K is circular. We shall show that (5) holds also for $\omega \subseteq \sigma_v$.

Let Z denote the cylindrical surface of direction v circumscribed about the body K. Thus the (orthogonal) cross-section of Z is a circle of radius r. Let B_r denote a ball of radius r inscribed into Z. Then each convex body $(1-\lambda)K+\lambda B_r$ (where $0 \leqslant \lambda \leqslant 1$) is also inscribed into Z. The geometrical interpretation of the measure Φ_2 tells us that $\Phi_2\big((1-\lambda)\times \times K+\lambda B_r, \sigma_v\big)$ is the surface area of $Z \cap [(1-\lambda)K+\lambda B_r]$. If G is any generating line of the cylinder Z, we have

$$G \cap [(1-\lambda)K+\lambda B_r] = (1-\lambda)(G\cap K) + \lambda(G\cap B_r)$$

(a consequence of [3], p. 31), from which we deduce that

$$\Phi_2\big((1-\lambda)K+\lambda B_r, \sigma_v\big) = (1-\lambda)\Phi_2(K,\sigma_v). \tag{6}$$

On the other hand, for an arbitrary Borel set $\omega \subseteq \Sigma$ the equality

$$\Phi_2\big((1-\lambda)K+\lambda B_r, \omega\big) = (1-\lambda)^2\Phi_2(K,\omega) + 2(1-\lambda)\lambda r\Phi_1(K,\omega) + \lambda^2 r^2\Omega(\omega) \tag{7}$$

holds (a special case of [5], p. 23). Comparison of (6) and (7) yields $\Phi_2(K,\sigma_v) - 2r\Phi_1(K,\sigma_v) = 0$. As $\Omega(\sigma_v) = 0$ and (4) is assumed, we see that (5) is valid for each Borel subset ω of σ_v, and hence for arbitrary Borel sets on Σ. In particular, we may take Σ itself, which means that

$$F(K) - 2rM(K) + 4\pi r^2 = 0. \tag{8}$$

Here $F(K)$ denotes the surface area of K, and $M(K)$ is 2π times its mean width (or "integral of mean curvature").

Finally we consider the function f defined by

$$f(\lambda) = F\big((1-\lambda)K+\lambda B_r\big), \quad 0 \leqslant \lambda \leqslant 1.$$

As K and B_r are inscribed into the same cylinder, the function f is known to be concave (Hadwiger [6], p. 256 and 259). But from the well-known formula

$$f(\lambda) = (1-\lambda)^2F(K) + 2(1-\lambda)\lambda rM(K) + 4\pi\lambda^2r^2$$

and equality (8) we see that f is even linear. As Hadwiger (loc. cit.) has proved, this can only happen if there are numbers $\varrho, \sigma \geqslant 0$ and a segment S parallel to v such that $K + \varrho S$ is a translate of $B_r + \sigma S$. Clearly $\sigma \geqslant \varrho$, and K is a translate of $B_r + (\sigma - \varrho) S$.

Added in proof. That spheres are the only closed analytic surfaces which satisfy (1) has also been proved by A. D. Aleksandrov, *On the curvature of surfaces* (Russian), Vestnik Leningrad. Univ. 21, No. 19 (Ser. Mat. Mch. Astron. No. 4) (1966), p. 5–11.

References

[1] A. D. Aleksandrov, *On the theory of mixed volumes of convex bodies* I, II (Russian), Mat. Sbornik N. S. 2 (1937), p. 947–972, p. 1205–1238.

[2] — *On uniqueness theorems for closed surfaces* (Russian), Doklady Akad. Nauk SSSR 22 (1939), p. 99–102.

[3] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Berlin 1934.

[4] H. Busemann, *Convex surfaces*, New York 1958.

[5] W. Fenchel and B. Jessen, *Mengenfunktionen und konvexe Körper*, Danske Vid. Selsk. Mat.-Fys. Medd. 16, 3 (1938), p. 1–31.

[6] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Berlin–Göttingen–Heidelberg 1957.

[7] D. Koutroufiotis, *On a conjectured characterization of the sphere*, Math. Ann. 205 (1973), p. 211–217.

[8] H. F. Münzner, *Über eine spezielle Klasse von Nabelpunkten und analoge Singularitäten in der zentroaffinen Flächentheorie*, Comment. Math. Helv. 41 (1966/67), p. 88–104.

[9] — *Über Flächen mit einer Weingartenschen Ungleichung*, Math. Zeitschr. 97 (1967), p. 123–139.

TECHNISCHE UNIVERSITÄT BERLIN

Reçu par la Rédaction le 22. 10. 1973