An S^1-equivariant degree and the Fuller index

by Grzegorz Dylawerski, Kazimierz Gęba, Jerzy Jodel
and Wacław Marzantowicz (Gdańsk)

Abstract. In this paper, we define a homotopy invariant for S^1-equivariant maps φ: $(U, \partial U)$ $\to (V, V\backslash\{0\})$, where V is a representation of S^1 and $U \subset V \oplus R$ is open and bounded. We call this invariant the S^1-degree. An infinite-dimensional generalization of the S^1-degree is also given. As an application we give a description of the Fuller index in terms of the S^1-degree.

Introduction. In 1965 F. B. Fuller introduced an invariant of a flow, called now the *Fuller index*. Roughly speaking, the Fuller index counts the algebraic number of periodic orbits of flow. It is natural to ask whether the Fuller index can be defined as a homotopy invariant of a map defined by the flow on a function space. This led to a construction of a new homotopy invariant for S^1-equivariant maps. We call this invariant the S^1-*degree* despite the fact that it is defined for maps with domain in the $(n+1)$-dimensional euclidean space and range in the n-dimensional euclidean space.

In this paper, we first construct the S^1-degree in the finite-dimensional case. More precisely, we assume that there is given a pair (V, ϱ), where V is a finite-dimensional linear space over R and ϱ: $S^1 \to \mathrm{GL}(V)$ is a continuous homomorphism into the group of all linear automorphisms of V. We consider continuous maps f: $X \to V$ such that X is an invariant subset of $V \oplus R$ (i.e. $(x, \lambda) \in X$, $g \in S^1$ imply $(\varrho(g) \cdot x, \lambda) \in X$) and f is equivariant (i.e. $f(\varrho(g) \cdot x, \lambda)$ $= \varrho(g) f(x, \lambda)$ for all $g \in S^1$, $(x, \lambda) \in X$). Suppose $\Omega \subset V \oplus R$ is bounded, invariant, $\bar{\Omega} \subset X$ and $f(x, \lambda) \neq 0$ for $(x, \lambda) \in \partial\Omega$; then we define $\mathrm{Deg}(f, \Omega) = \{\alpha_r\}$, where $r \in \{0\} \cup N$, $\alpha_0 \in Z_2$, $\alpha_r \in Z$, $\alpha_r = 0$ for almost all r; Z denotes the group of integers and $Z_2 = Z/2Z$.

In order to give at least a very rough idea of our approach we will sketch the definition of $\mathrm{Deg}(f, \Omega)$ for f satisfying additional assumptions A.1–A.3 below. In some sense, equivariant maps f satisfying these assumptions are "generic" for our construction. First we assume

A.1. *There exists* $a \in \Omega$ *such that* $f^{-1}(0) = \{\varrho(g)a: g \in S^1\}$.

Recall that the subgroup of S^1 defined by $G_a = \{g \in S^1: \varrho(g) \cdot a = a\}$ is called the *isotropy group* of a. Since ϱ is continuous, G_a is either finite or S^1. Our second assumption is

A.2. G_a is finite.

Note that since G_a is a finite subgroup of S^1 there exists $k \in N$ such that $G_a = Z_k$, the subgroup of S^1 consisting of k-th roots of 1. Without loss of generality we may assume that (V, ϱ) is orthogonal, i.e. all $\varrho(g)$ are orthogonal with respect to a scalar product in V. Let $W = \{x \in V : \varrho(g) \cdot x = x \text{ for } g \in G_a\}$. Take the direct sum decomposition $V = W \oplus W^{\perp}$. Writing points of $V \oplus R$ $= W \oplus W^{\perp} \oplus R$ in the form (x, y, λ), we have

$$f(x, y, \lambda) = (f_1(x, y, \lambda), f_2(x, y, \lambda)) \in W \oplus W^{\perp}.$$

Our last additional assumption is

A.3. $f_2(x, y, \lambda) = y$ for all $(x, y, \lambda) \in \Omega$.

Now assume that D^p, $p = $ the dimension of W, is a sufficiently small closed disc contained in $\Omega \cap (W \oplus R)$ and transversal to the orbit $Ga = M$ at a. Orienting D^p in a suitable way we identify it with the standard unit disc in W. Then, using the classical Brouwer degree we define an integer $\deg(\varphi, D^p)$, where φ denotes the restriction of f. Finally, we let $\text{Deg}(f, \Omega) = \{\alpha_r\}$, where $\alpha_k = \deg(\varphi, D^p)$ and $\alpha_r = 0$ for $r \neq k$.

In Section 1 we introduce notations, recall basic facts concerning finite-dimensional representations of S^1 and formulate the main theorem (Theorem 1.2) which lists the most important properties of our degree. Sections 2 and 3 are devoted to the proof of Theorem 1.2. First, we construct our degree under the assumption that the set Ω consists of points with the same isotropy group (Section 2). For maps of an arbitrary Ω, we define the S^1-degree in Section 3. In Section 4 we discuss the case of differentiable equivariant maps; assuming that 0 is a regular value of f we derive formulas for the S^1-degree. In Sections 5 and 6, following the classical Leray-Schauder theory, we consider an infinite-dimensional generalization of our degree. In Sections 7 and 8 we discuss the relationship between our degree and the Fuller index.

The idea of defining a (generalized) degree for S^1-equivariant maps is not new. In [6] Dancer constructs such a degree for S^1-equivariant gradient maps. The relationship between these two degrees will be discussed in a separate paper [10] by Dylawerski. Another version of S^1-equivariant degree is defined in Ize-Massabó-Vignoli [14] (the authors use cohomological obstruction theory).

It should also be noted that the methods used in the paper closely relate it to the work of Rubinsztein [19] and tom Dieck [7].

1. Degree for S^1-equivariant maps. For $r > 0$ we let

$$D^n(r) = \{x \in R^n : |x| \leqslant r\}, \quad S^{n-1}(r) = \{x \in R^n : |x| = r\},$$

$$\overset{\circ}{B}{}^n(r) = \{x \in R^n : |x| < r\}.$$

We also let

$$D^n = D^n(1), \quad S^{n-1} = S^{n-1}(1), \quad B^n = B^n(1).$$

For $m \in N$, we let $Z_m = Z/mZ$; we often use the identification

$$Z_m = \{g \in S^1: g = e^{i\theta}, \ \theta = 2\pi j/m, \ j = 0, 1, \ldots, m-1\}.$$

We begin by recalling some terminology and facts concerning group actions and group representations. Let G denote a compact Lie group and let $e \in G$ be its neutral element.

We say that G *acts* on a topological space X (or X is a *G-space*) if there is a continuous map $\mu: G \times X \to X$ such that

(i) $\mu(e, x) = x$,

(ii) $\mu(g_1, \mu(g_2, x)) = \mu(g_1 g_2, x)$,

for all $g_1, g_2 \in G$ and $x \in X$. In what follows, for simplicity of notation, we write $gx = \mu(g, x)$. For a given $x \in X$ the subgroup $G_x = \{g \in G, \ gx = x\}$ is called the *isotropy group* of x and the set $Gx = \{gx, \ g \in G\}$ is called the *orbit* of x. Two points $x, y \in X$ are of the same *orbit type* if there exists $g \in G$ such that $G_x = g^{-1} G_y g$.

Given a subgroup $H \subset G$ we let

$$X^H = \{x \in X: hx = x \text{ for all } h \in H\} = \{x \in X: H \subset G_x\},$$

$$X_H = \{x \in X: G_x = H\}.$$

Let X, Y be two G-spaces. We say that a map $f: X \to Y$ is *G-equivariant* if $f(gx) = gf(x)$ for all $x \in X$, $g \in G$. We say that f is a *G-map* if it is equivariant and continuous. We say that a continuous map $h: X \times [0, 1] \to Y$ is a *G-homotopy* between two G-maps $f_0, f_1: X \to Y$ if $h(x, 0) = f_0(x)$, $h(x, 1) = f_1(x)$ and $h(gx, t) = gh(x, t)$ for all $g \in G$, $x \in X$, $t \in [0, 1]$. The set of all G-maps from X into Y will be denoted by $\text{Map}_G(X, Y)$. We also let $[X, Y]_G$ denote the set of all G-homotopy classes of G-maps from X into Y.

A *representation* of G is a pair $V = (V_0, \varrho)$, where V_0 is a finite-dimensional, real linear space and $\varrho: G \to \text{GL}(V_0)$ is a continuous homomorphism from G into the group of all linear automorphisms of V_0. Note that if $V = (V_0, \varrho)$ is a representation of G, then letting $gv = \varrho(g)(v)$ we obtain a G-action of G. Moreover, $g(\alpha v + \beta w) = \alpha(gv) + \beta(gw)$ for all $g \in G$, $v, w \in V_0$ and $\alpha, \beta \in R$, i.e. the action is linear. It is evident that any linear G-action on V_0 defines a representation. We often do not distinguish between V and V_0 using the same letter for a representation and the underlying linear space. Two representations V and W of G are *equivalent* if there exists an equivariant linear isomorphism $T: V \to W$. Given two representations V and W of G we denote by $V \oplus W$ the *direct sum* of V and W, i.e. the direct sum of linear spaces with the linear group action defined by $g(v, w) = (gv, gw)$.

Throughout the rest of the paper we assume $G = S^1 = \{z \in C: |z| = 1\}$. For $m \in N$ define $\varrho^m: S^1 \to \text{GL}(2, R)$ by

$$\varrho^m(e^{i\theta}) = \begin{bmatrix} \cos m\theta & -\sin m\theta \\ \sin m\theta & \cos m\theta \end{bmatrix}, \quad 0 \leqslant \theta \leqslant 2\pi.$$

For $k, m \in N$ we denote by $R[k, m]$ the direct sum of k copies of (R^2, ϱ^m); we also denote by $R[k, 0]$ the trivial k-dimensional representation of S^1. The following classical result gives a complete classification up to equivalence of finite-dimensional representations of S^1 (Adams [1]).

1.1. Theorem. *If V is a representation of S^1 then there exist finite sequences $\{k_i\}, \{m_i\}$ satisfying*

$$(*) \qquad m_i \in \{0\} \cup N, \qquad k_i \in N, \qquad 1 \leqslant i \leqslant r, \qquad m_1 < \ldots < m_r,$$

such that V is equivalent to $\bigoplus R[k_i, m_i]$. Moreover, the equivalence class of V is uniquely determined by $\{m_i\}, \{k_i\}$ satisfying $(*)$.

Suppose now that V is a representation of S^1. Let Ω be an open bounded invariant subset of $V \oplus R$. Suppose further that $f \colon \bar{\Omega} \to V$ is a G-map such that $f(\partial \Omega) \subset V \setminus \{0\}$. Our construction, carried over in Sections 2 and 3, assigns to each closed subgroup $H \subset S^1$ an element

$$\deg_H(f, \Omega) \in \begin{cases} Z & \text{if } H \text{ is finite,} \\ Z_2 & \text{if } H = S^1. \end{cases}$$

We denote by $\mathscr{A}_0$ the free abelian group generated by N and let $\mathscr{A} = Z_2 \oplus \mathscr{A}_0$. Note that $\alpha \in \mathscr{A}$ means $\alpha = \{\alpha_r\}$, where $\alpha_0 \in Z_2$ and $\alpha_r \in Z$ for $r \in N$, and $\alpha_r = 0$ for almost all r. Using the fact that there is a one-to-one correspondence between N and the family of all proper closed subgroups of S^1, we define $\mathrm{Deg}(f, \Omega) = \{\alpha_r\} \in \mathscr{A}$ by $\alpha_0 = \deg_{S^1}(f, \Omega)$ and $\alpha_r = \deg_H(f, \Omega)$ if $r = |H|$.

We are now in a position to state our main result.

1.2. Theorem. *Let V run through representations of S^1, Ω through the family of all open bounded invariant subsets of $V \oplus R$, and $f \colon X \to V$ through G-maps such that X is invariant, $\bar{\Omega} \subset X$ and $f(\partial \Omega) \subset V \setminus \{0\}$. Then there exists an $\mathscr{A}$-valued function $\mathrm{Deg}(f, \Omega) = \{\deg_H(f, \Omega)\}$, called the S^1-degree, satisfying the following conditions:*

(a) *If $\deg_H(f, \Omega) \neq 0$ then $f^{-1}(0) \cap \Omega^H \neq \emptyset$.*

(b) *If $\Omega_0 \subset \Omega$ is open, invariant and $f^{-1}(0) \cap \Omega \subset \Omega_0$ then $\mathrm{Deg}(f, \Omega) = \mathrm{Deg}(f, \Omega_0)$.*

(c) *If Ω_1, Ω_2 are open invariant subsets of Ω such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$ then $\mathrm{Deg}(f, \Omega) = \mathrm{Deg}(f, \Omega_1) + \mathrm{Deg}(f, \Omega_2)$.*

(d) *If $h \colon (\bar{\Omega} \times [0, 1], \partial \Omega \times [0, 1]) \to (V, V \setminus \{0\})$ is a G-homotopy then $\mathrm{Deg}(h_0, \Omega) = \mathrm{Deg}(h_1, \Omega)$.*

(e) *Suppose W is another representation of $G = S^1$ and let U be an open bounded invariant subset of W such that $0 \in U$. Define $F \colon U \times \bar{\Omega} \to W \oplus V$ by $F(x, y) = (x, f(y))$. Then $\mathrm{Deg}(F, U \times \Omega) = \mathrm{Deg}(f, \Omega)$.*

2. Proof of a special case of Theorem 1.2. First we recall basic properties of the topological (Brouwer) degree in R^n (cf. Amann–Weiss [2], Chow–Hale [4], Dugundji–Granas [9]).

Let U be an open bounded subset of R^n. If $f\colon X \to R^n$ is a continuous map such that $\bar{U} \subset X$ and $f(\partial U) \subset R^n\setminus\{0\}$ then there is defined an integer $\deg(f, U)$, called the *degree* of f with respect to U. All the constructions and results of Sections 2, 3 and 4 are based on the following properties of degree.

(I) (*Existence of solutions*). If $\deg(f, U) \neq 0$ then there exists $x_0 \in U$ such that $f(x_0) = 0$.

(II) (*Excision*). If U_0 is an open subset of R^n such that $U_0 \subset U$ and $f^{-1}(0) \cap U = f^{-1}(0) \cap U_0$ then $\deg(f, U_0) = \deg(f, U)$.

(III) (*Additivity*). If U_1, U_2 are open disjoint subsets of U such that $f^{-1}(0) \cap U \subset U_1 \cup U_2$ then $\deg(f, U) = \deg(f, U_1) + \deg(f, U_2)$.

(IV) (*Homotopy*). If $h\colon \bar{U} \times [0, 1] \to R^n$ is a continuous map such that $h^{-1}(0) \subset U \times [0, 1]$ then $\deg(h_0, U) = \deg(h_1, U)$, where $h_i(x) = h(x, i)$ for $x \in U$, $i = 0, 1$.

(V) (*Contraction*). Let $W = U \times (-1, 1) \subset R^n \times R = R^{n+1}$ and define $F\colon W \to R^{n+1}$ by $F(x, t) = (f(x), t)$. Then $\deg(F, W) = \deg(f, U)$.

(VI) (*Diffeomorphism invariance*). Suppose U_0 is an open subset of R^n and $d\colon U_0 \to R^n$ is an orientation preserving diffeomorphism such that $U \subset d(U_0)$. Then $\deg(f \circ d, d^{-1}(U)) = \deg(f, U)$.

Note that (I)–(IV) are the well-known fundamental properties of degree (see Amann–Weiss [2]). Properties (V) and (VI) are easy consequences of the differential interpretation of degree (see Chow–Hale [4]).

For the rest of this section we fix an orthogonal representation V of $G = S^1$. We let n be the dimension of the real space V. If the action of G is not involved we do not distinguish between V and R^n. For an invariant open bounded subset $\Omega \subset V \oplus R$ we denote by $C_G(\bar{\Omega})$ the linear normed space of all G-maps $f\colon \bar{\Omega} \to V$ with the standard sup norm. If X is a closed invariant subset of $\bar{\Omega}$ we let

$$C_G(\bar{\Omega}, X) = \{f \in C_G(\bar{\Omega})\colon f(X) \subset V\setminus\{0\}\}.$$

Clearly $C_G(\bar{\Omega}, X)$ is an open subset of $C_G(\bar{\Omega})$. We say that $f_0, f_1 \in C_G(\bar{\Omega}, X)$ are *G-homotopic* in $C_G(\bar{\Omega}, X)$ if there exists a G-homotopy $h\colon \bar{\Omega} \times [0, 1] \to V$ such that $h(X \times [0, 1]) \subset V\setminus\{0\}$ and $h_0 = f_0$, $h_1 = f_1$.

In order to make our proofs more readable, we introduce some notational conventions. Suppose W is a finite-dimensional linear space over R and $A\colon W \to W$ is a linear automorphism. Choose an orientation of W and let

$$\operatorname{sgn} A = \begin{cases} +1 & \text{if } A \text{ preserves orientation,} \\ -1 & \text{if } A \text{ reverses orientation,} \end{cases}$$

Note that $\operatorname{sgn} A$ does not depend on the choice of orientation of W.

If $\Phi\colon W \oplus R \to W$ is a linear map such that $\Phi(W \oplus R) = W$ then $\operatorname{Ker} \Phi$ is a one-dimensional linear subspace of $W \oplus R$. If $v \in \operatorname{Ker} \Phi$ and $v \neq 0$ choose

a linear functional $\varphi\colon W\oplus R\to R$ such that $\varphi(v)=1$. Define $\Lambda\colon W\oplus R$ $\to W\oplus R$ by $\Lambda(x)=(\Phi(x),\varphi(x))$. Evidently Λ is an automorphism. We let

$$\mathrm{sgn}(\Phi,v)=\mathrm{sgn}\,\Lambda;$$

clearly this definition is correct, i.e. independent of the choice of φ.

Suppose $a\in V\oplus R$ and $G_a\neq S^1$. Then $M=Ga$ is a one-dimensional submanifold of $V\oplus R$. Set $v=\eta'(0)$, where $\eta(t)=g(t)\cdot a$, $g(t)=\exp(2\pi it)$. Thus v is tangent to M at a and determines an orientation of M. Let

$$N=\{x\in V\oplus R\colon \langle x,v\rangle=0\};$$

we call N the *normal space* at a. Note that $a\in N$. We say that a map $A\colon N\to V$ is a *normal map* at a if it is a linear isomorphism and $\mathrm{sgn}(\Phi_A,v)=1$, where $\Phi_A=A\circ P_N$ and P_N denotes the orthogonal projection of $V\oplus R$ onto N.

The following proposition is an immediate consequence of the Slice Theorem (see Bredon [3]).

2.1. PROPOSITION. *Let* $B_N(a,\varepsilon)=\{x\in N\colon |x-a|<\varepsilon\}$ *and let* $U=GB_N(a,\varepsilon)$. *There exists* ε_0 *such that if* $\varepsilon<\varepsilon_0$ *then* U *is a tube about* Ga *and* $B_N(a,\varepsilon)$ *is a slice for* U, *and consequently satisfies the following condition:*

$$(*)\qquad x=g_1y_1=g_2y_2,\qquad x\in U,\qquad y_1,y_2\in B_N(a,\varepsilon),\qquad g_1,g_2\in G$$

imply $g_1(g_2)^{-1}\in G_a$.

2.2. DEFINITION. Let $a\in V\oplus R$ and $G_a\neq G$. We say that a continuous map $\varphi\colon B^n(2)\to V\oplus R$ is a *slice map* at a if

(a) $\varphi(0)=a$,

(b) there exists $\varepsilon>0$ such that $B_N(a,\varepsilon)$ is a slice and $\varphi(B^n(2))\subset B_N(a,\varepsilon)$,

(c) there exists a normal map at a, $A\colon N\to V$, such that $\varphi(x)=a+A^{-1}(x)$.

Denote by $H=\{H_1,\ldots,H_q\}$ the family of all closed subgroups of S^1 such that $V_H\neq\varnothing$ and let $K=H_1\cap\ldots\cap H_q$, $k=|K|$, i.e. K is the smallest isotropy group of V. First we assume $K\neq S^1$ and postpone the case $K=S^1$ to the end of this section.

2.3. ASSUMPTION. *Throughout the rest of this section we assume that* $\Omega\subset V\oplus R$ *is an invariant open bounded set and* $(\bar\Omega)_K=\bar\Omega$.

Let $C(D^n)$ be the linear normed space of all continuous maps $f\colon D^n\to R^n$ with the sup norm and let

$$C(D^n,S^{n-1})=\{f\in C(D^n)\colon f(S^{n-1})\subset R^n\backslash\{0\}\}.$$

We will later need the following lemma.

2.4. LEMMA. $C(D^n,S^{n-1})$ *is an open dense subset of* $C(D^n)$.

Proof. Clearly $C(D^n,S^{n-1})$ is open in $C(D^n)$. To prove it is dense, choose $f\in C(D^n)$ and $\varepsilon>0$. It is well known that the set of C^1-mappings is dense in

$C(D^n)$. Hence there exists $f_0 \in C(D^n)$ such that $\|f - f_0\| < \varepsilon$ and f_0 extends to a C^1-map $F\colon U \to R^n$, where U is an open subset containing D^n. Using Sard's theorem choose a regular value $y_0 \in R^n$ of F such that $|y_0| < \varepsilon$ and define $F_1\colon U \to R^n$ by $F_1(x) = F(x) - y_0$. Choose $r_0 > 1$ such that $D^n(r_0) \subset U$. Since 0 is a regular value of F_1, there exists r, $1 \leqslant r < r_0$, such that $|F_1(rx) - F_1(x)| < \varepsilon$ for all $x \in D^n$ and $F_1^{-1}(0) \cap S^{n-1}(r) = \varnothing$. Define f_1 by $f_1(x) = f_1(rx)$. Then $f_1^{-1}(0) \cap S^{n-1} = \varnothing$, $\|f_1 - f\| < 3\varepsilon$ and the conclusion follows.

We frequently use the following simple observation.

2.5. LEMMA. *Suppose* $f_0 \in C_G(\bar{\Omega}, X)$ *and let* $\varepsilon = \inf\{\|f_0(x)\|\colon x \in X\}$. *If* $f \in C_G(\bar{\Omega})$ *and* $\|f - f_0\| < \varepsilon$ *then* $f \in C_G(\bar{\Omega}, X)$ *and* f *is* G-homotopic to f_0 in $C_G(\bar{\Omega}, X)$.

Proof. Clearly $f \in C_G(\bar{\Omega}, X)$. The formula $h(x, t) = (1 - t)f_0(x) + tf(x)$ defines the required homotopy.

2.6. LEMMA. *Let* $\varphi\colon B^n(2) \to \Omega$ *be a slice map. If* $F \in C(D^n)$ *then there exists* $f \in C_G(\bar{\Omega})$ *such that* $\|f\| = \|F\|$ *and* $F(x) = f(\varphi(x))$ *for all* $x \in D^n$

Proof. Extend F to a continuous map $F_1\colon B^n(2) \to R^n$ such that $F_1(x) = 0$ outside a compact subset of $B^n(2)$ and $\sup\{|F_1(x)|\colon x \in B^n(2)\} = \|F\|$.

Since φ is a slice map there exists $B_N(a, \varepsilon)$ such that $\varphi(B^n(2)) \subset B_N(a, \varepsilon)$. Set $U = G\varphi(B^n(2))$. If $x \in U$ then there exists $g \in G = S^1$ such that $gx \in B_N(a, \varepsilon)$, and we let $f(x) = g^{-1}F_1(\varphi^{-1}(gx))$; if $x \in \bar{\Omega} \backslash U$ we let $f(x) = 0$. In view of 2.1 and 2.2 it is easy to see that f is the required map.

2.7. DEFINITION. We say that an open invariant subset $\Omega_0 \subset \Omega$ is *elementary* if there exists a finite family $\{\Omega_1, \ldots, \Omega_r\}$ of open invariant subsets of Ω such that

(*) $\Omega_0 \subset \Omega_1 \cup \ldots \cup \Omega_r$,

(**) $\Omega_i \cap \Omega_j = \varnothing$ for $i \neq j$,

(***) for each i, $1 \leqslant i \leqslant r$, there is a slice map $\varphi_i\colon B^n(2) \to \Omega$ such that $\Omega_i \subset G\varphi_i(B^n)$.

We call $f \in C_G(\bar{\Omega}, \partial\Omega)$ an *elementary* G-map if there exists an elementary subset $\Omega_0 \subset \Omega$ such that $f^{-1}(0) \subset \Omega_0$. Similarly we say that a G-equivariant homotopy $h\colon (\bar{\Omega}, \partial\Omega) \to (V, V \backslash \{0\})$ is *elementary* if there exists an elementary subset $\Omega_0 \subset \Omega$ such that $h^{-1}(0) \subset \Omega_0 \times [0, 1]$.

2.8. LEMMA. *If* $f_0 \in C_G(\bar{\Omega}, \partial\Omega)$ *is an elementary* G-map then there exists $\varepsilon > 0$ *such that* $f \in C_G(\bar{\Omega}, \partial\Omega)$ *and* $\|f_0 - f\| < \varepsilon$ *imply that* f *is elementary and* h *defined by* $h(x, t) = (1 - t)f_0(x) + tf(x)$ *is an elementary* G-homotopy *between* f_0 *and* f.

Proof. Let Ω_0 be an elementary subset of Ω such that $f_0^{-1}(0) \subset \Omega_0$ and let $\varepsilon = \inf\{\|f_0(x)\|\colon x \in \Omega \backslash \Omega_0\}$. It is evident that ε has the required property.

Assume now that $f \in C_G(\bar{\Omega}, \partial\Omega)$ is an elementary G-map and $\{\Omega_i\}$, $\{\varphi_i\}$ satisfy the conditions of Definition 2.7. Let $U_i = \varphi_i^{-1}(\Omega_i)$ and $F_i = f \circ \varphi_i$: $U_i \to R^n$. Clearly $F_i^{-1}(0)$ is a compact subset of U_i and thus $\deg(F_i, U_i)$ is well defined for $i = 1, \ldots, r$. Define

$$\deg_K(f, \Omega) = \deg(F_1, U_1) + \quad + \deg(F_r, U_r).$$

From properties (I)–(VI) of the classical topological degree it follows at once that the definition of $\deg_K(f, \Omega)$ is independent of the choice of $\{\Omega_i\}$ and $\{\varphi_i\}$.

The following observation is a direct consequence of our definition of $\deg_K(f, \Omega)$.

2.9. Remark. If h_t: $(\bar{\Omega}, \partial\Omega) \to (V, V\backslash\{0\})$ is an elementary G-homotopy then $\deg_K(h_0, \Omega) = \deg_K(h_1, \Omega)$.

2.10. Lemma. Let φ: $B^n(2) \to \Omega$ be a slice map and let $\Omega_0 = G\varphi(B^n)$. Then $C_G(\bar{\Omega}, \partial\Omega \cup \partial\Omega_0)$ is a dense subset od $C_G(\bar{\Omega}, \partial\Omega)$.

Proof. Suppose $f \in C_G(\bar{\Omega}, \partial\Omega)$ and choose $0 < \varepsilon < \inf\{|f(x)|: x \in \partial\Omega\}$. Let F denote the map defined by $F(x) = f(\varphi(x))$; clearly $F \in C(D^n)$. By 2.4 there exists $F_1 \in C(D^n, S^{n-1})$ such that $\|F - F_1\| < \varepsilon$ in $C(D^n)$. By 2.6 there exists $f_0 \in C_G(\bar{\Omega})$ such that $\|f_0\| < \varepsilon$ and $f_0(\varphi(x)) = F(x) - F_1(x)$ for all $x \in D^n$. Let $f_1(x) = f(x) - f_0(x)$. Since $\|f - f_1\| = \|f_0\| < \varepsilon$, $f_1 \in C_G(\bar{\Omega}, \partial\Omega)$. Moreover, $f_1(\varphi(x)) = F_1(x)$ for all $x \in D^n$; hence $f_1 \in C_G(\bar{\Omega}, \partial\Omega \cup \partial\Omega_0)$. Therefore $C_G(\bar{\Omega}, \partial\Omega \cup \partial\Omega_0)$ is dense in $C_G(\bar{\Omega}, \partial\Omega)$ and the proof is complete.

To define our degree in the general case we need the following two results.

2.11. Proposition. The set of all elementary G-maps is an open and dense subset of $C_G(\bar{\Omega}, \partial\Omega)$.

Proof. From 2.8 it follows at once that the set of all elementary G-maps is an open subset of $C_G(\bar{\Omega}, \partial\Omega)$. To prove it is dense assume $f_0 \in C_G(\bar{\Omega}, \partial\Omega)$. There exists a finite family of slice maps φ_i: $B^n(2) \to \Omega$, $i = 1, \ldots, r$, such that $\{G\varphi_i(B^n)\}$ covers $f_0^{-1}(0)$. Let $U_i = G\varphi_i(B^n)$. Take ε such that $0 < \varepsilon < \inf\{|f_0(x)|: x \in \bar{\Omega}\backslash(U_1 \cup \ldots \cup U_r)\}$. Since an intersection of open and dense subsets is open and dense, Lemma 2.10 implies that $C_G(\bar{\Omega}, \partial\Omega \cup \partial U_1 \cup \ldots \cup \partial U_r)$ is an open and dense subset of $C_G(\bar{\Omega}, \partial\Omega)$. Therefore there exists $f \in C_G(\bar{\Omega}, \partial\Omega \cup \partial U_1 \cup \ldots \cup \partial U_r)$ such that $\|f_0 - f\| < \varepsilon$. Let $\Omega_i = U_i\backslash(\bar{U}_1 \cup \ldots \cup \bar{U}_{i-1})$, $\Omega_1 = U_1$. It is evident that the Ω_i are mutually disjoint and $f^{-1}(0) \subset \Omega_1 \cup \ldots \cup \Omega_r$. Thus the proof is complete.

2.12. Proposition. Suppose $f_0, f_1 \in C_G(\bar{\Omega}, \partial\Omega)$ are two elementary G-maps which are G-homotopic in $C_G(\bar{\Omega}, \partial\Omega)$. Then $\deg_K(f_0, \Omega) = \deg_K(f_1, \Omega)$.

The proof of 2.12 will be based on the following technical lemma.

2.13. Lemma. Suppose h: $(\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V\backslash\{0\})$ is a G-homotopy and $\{U_i\}$, $1 \leqslant i \leqslant r$, is a finite family of open invariant subsets of Ω such that

(∗) $h^{-1}(0) \subset (U_1 \cup \ldots \cup U_r) \times [0, 1]$,

(∗∗) *there exist slice maps* φ_i: $B^n(2) \to \Omega$, *such that* $U_i \subset G\varphi_i(B^n)$ *for* $i = 1, \ldots, r$,

(∗∗∗) $h_0, h_1 \in C_G(\bar{\Omega}, \partial\Omega \cup \partial U_1 \cup \ldots \cup \partial U_r)$ *are elementary.*

Then $\deg_K(h_0, \Omega) = \deg_K(h_1, \Omega)$.

Proof. The proof is by induction on r. Property (IV) of the classical degree implies that the lemma is true for $r = 1$. Suppose now it is true for $r-1$ and let $X = \Pi(h^{-1}(0))$, where Π: $\bar{\Omega} \times [0, 1] \to \bar{\Omega}$ denotes the projection. Clearly X is a compact subset of $U_1 \cup \ldots \cup U_r$. Let $U = U_2 \cup \ldots \cup U_r$, $\Omega_0 = U_1 \backslash U$ and $\Omega_1 = U \backslash \bar{U}_1$. Let θ: $\bar{\Omega} \to [0, 1]$ be a continuous function such that

(i) $\theta(gx) = \theta(x)$ for all $x \in \bar{\Omega}$, $g \in G$,

(ii) $\theta(x) = 1$ for $x \in X \cap \bar{\Omega}_0$,

(iii) $\theta(x) = 0$ for $x \in \bar{\Omega} \backslash U_1$.

For $(x, t) \in \bar{\Omega} \times [0, 1]$ set $h^{(1)}(x, t) = h(x, \theta(x) \cdot t)$, $h^{(2)}(x, t) = h(x, \theta(x) + t(1 - \theta(x)))$ and $f(x) = h^{(1)}(x, 1) = h^{(2)}(x, 0)$. From the definition of θ it follows that f is elementary and $f \in C_G(\bar{\Omega}, \partial\Omega \cup \partial\Omega_0 \cup \partial\Omega_1)$. Since $h^{(1)}$ determines an elementary G-homotopy between h_0 and f in $C_G(\bar{\Omega}, \bar{\Omega} \backslash (U_1 \cup \Omega_1))$ and $U_1 \cap \Omega_1 = \emptyset$, Remark 2.9 implies $\deg_K(h_0, U_1) = \deg_K(f, U_1)$.

Moreover, $h^{(1)}(x, t) = f(x) = h_0(x)$ for all $(x, t) \in \bar{\Omega}_1 \times [0, 1]$. Thus

$$\deg_K(h_0, \Omega) = \deg_K(h_0, U_1) + \deg_K(h_0, \Omega_1)$$

$$= \deg_K(f, U_1) + \deg_K(f, \Omega_1) = \deg_K(f, \Omega).$$

Since $h^{(2)}$ determines a G-homotopy between f and h_1 in $C_G(\bar{\Omega}, \bar{\Omega} \backslash (\Omega_0 \cup U))$ the inductive hypothesis implies $\deg_K(f, U) = \deg_K(h_1, U)$, hence

$$\deg_K(f, \Omega) = \deg_K(f, \Omega_0) + \deg_K(f, U)$$

$$= \deg_K(h_1, \Omega_0) + \deg_K(h_1, U) = \deg_K(h_1, \Omega).$$

Therefore $\deg_K(h_0, \Omega) = \deg_K(h_1, \Omega)$ and the proof of 2.13 is complete.

Proof of Proposition 2.12. Suppose h is a G-homotopy between two elementary G-maps $f_0, f_1 \in C_G(\bar{\Omega}, \partial\Omega)$. Let $X = \Pi(h^{-1}(0))$, where Π denotes the projection of $\bar{\Omega} \times [0, 1]$ onto $\bar{\Omega}$. Clearly X is a compact subset of Ω. There exists a family of slice maps φ_i: $B^n(2) \to \Omega$, $i = 1, \ldots, r$, such that $\{G\varphi_i(B^n)\}$ covers X. Let $U_i = G\varphi_i(B^n)$. By 2.10, $C_G(\bar{\Omega}, \partial\Omega \cup \partial U_1 \cup \ldots \cup \partial U_r)$ is open and dense in $C_G(\bar{\Omega}, \partial\Omega)$. By 2.8 there exist $F_0, F_1 \in C_G(\bar{\Omega}, \partial\Omega \cup \partial U_1 \cup \ldots \cup \partial U_r)$ such that

$$(1-t)f_0 + tF_0, \ (1-t)f_1 + tF_1 \in C_G(\bar{\Omega}, \bar{\Omega} \backslash (U_1 \cup \ldots \cup U_r))$$

and are elementary for all $t \in [0, 1]$. Let

$$H(x, t) = \begin{cases} 3t \cdot f_0(x) + (1-3t) \cdot F_0(x) & \text{for } 0 \leqslant t \leqslant \frac{1}{3}, \\ h(x, 3t-1) & \text{for } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ (3-3t) \cdot f_1(x) + (3t-2) \cdot F_1(x) & \text{for } \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

Since H satisfies the assumptions of 2.13 we have $\deg_K(F_0, \Omega) = \deg_K(F_1, \Omega)$. On the other hand, by 2.9, $\deg_K(f_0, \Omega) = \deg_K(F_0, \Omega)$ and $\deg_K(f_1, \Omega) = \deg_K(F_1, \Omega)$. Therefore $\deg_K(f_0, \Omega) = \deg_K(f_1, \Omega)$ and the proof is complete.

Now, our goal is to extend the definition of $\deg_K(f, \Omega)$ from elementary G-maps to all $f \in C_G(\bar{\Omega}, \partial\Omega)$. To begin with, let $f \in C_G(\bar{\Omega}, \partial\Omega)$ and $\varepsilon = \inf\{|f(x)|: x \in \partial\Omega\}$. By 2.11 there exists an elementary G-map $f_0 \in C_G(\bar{\Omega}, \partial\Omega)$ such that $\|f-f_0\| < \varepsilon$. We define

$$\deg_K(f, \Omega) = \deg_K(f_0, \Omega).$$

To see that our definition is independent of the choice of f_0 choose another elementary G-map f_1 such that $\|f-f_1\| < \varepsilon$. Then, by 2.5, both f_0 and f_1 are homotopic to f in $C_G(\bar{\Omega}, \partial\Omega)$. Therefore f_0 and f_1 are two elementary G-maps which are homotopic in $C_G(\bar{\Omega}, \partial\Omega)$ and by 2.12, $\deg_K(f_1, \Omega) = \deg_K(f_0, \Omega)$. From the definition of $\deg_K(f, \Omega)$ and Properties (I)–(VI) of the classical degree we obtain the following result.

2.14. Theorem. *Suppose that V is a representation of S^1. Let K denote the smallest isotropy group of V and assume $K \neq S^1$. Let Ω run through the family of all open bounded invariant subsets of $V \oplus R$ such that $\Omega_K = \Omega$, and $f: X \to V$ through G-maps such that X is invariant, $\bar{\Omega} \subset X$ and $f(\partial\Omega) \subset V\backslash\{0\}$. Then there exists a Z-valued function $\deg_K(f, \Omega)$ satisfying the following conditions:*

(a) *If $\deg_K(f, \Omega) \neq 0$ then there exists $x \in \Omega$ such that $f(x) = 0$.*

(b) *If $\Omega_0 \subset \Omega$ is open, invariant and $f^{-1}(0) \cap \Omega \subset \Omega_0$ then $\deg_K(f, \Omega_0) = \deg_K(f, \Omega)$.*

(c) *If Ω_1, Ω_2 are open invariant subsets of Ω such that $\Omega_1 \cap \Omega_2 = \varnothing$ and $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$ then $\deg_K(f, \Omega) = \deg_K(f, \Omega_0) + \deg_K(f, \Omega_1)$.*

(d) *If $h: (\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V\backslash\{0\})$ is a G-homotopy then $\deg_K(h_0, \Omega) = \deg_K(h_1, \Omega)$.*

(e) *Suppose W is another representation of S^1 such that K is the smallest isotropy group of $V \oplus W$ and let U be an open bounded invariant subset of W such that $0 \in U$. Define $F: \bar{U} \times \bar{\Omega} \to W \oplus V$ by $F(x, y) = (x, f(y))$. Then $\deg_K(F, U \times \Omega) = \deg_K(f, \Omega)$.*

(f) *Suppose f is of class C^1 on Ω, 0 is a regular value of f and $f^{-1}(0) = Ga$. Let $v = \eta'(0)$, where $\eta(t) = g(t) \cdot a$, $g(t) = \exp(2\pi it)$, be the tangent vector to Ga at a. Then $\deg_K(f, \Omega) = \mathrm{sgn}(Df(a), v)$.*

Finally, we briefly discuss the case $K = S^1$. Thus we assume that the smallest isotropy group equals $G = S^1$, hence V is a trivial representation of S^1

Therefore we let $V = R^n$ and consider a continuous map $f: (\bar{\Omega}, \partial\Omega) \to (R^n, R^n\setminus\{0\})$, where Ω is an open bounded subset of R^{n+1}. Let $\overline{R^n}$ denote the one-point compactification of R^n. It can easily be shown that there exists $\bar{f}: \overline{R^{n+1}} \to \overline{R^n}$ which extends f and $\bar{f}(\overline{R^{n+1}}\setminus\Omega) \subset \overline{R^n}\setminus\{0\}$. Moreover, the homotopy class of $\bar{f}$ depends only on the homotopy class of f. Since $\overline{R^n}$ is homeomorphic to S^n we may identify $[\overline{R^{n+1}}, \overline{R^n}]$ with the $(n+1)$th homotopy group of S^n. Recall that the suspension homomorphism determines a homomorphism $\Sigma: \pi_{n+1}(S^n) \to Z_2$, and that Σ is an isomorphism for $n \geqslant 3$. Define $\deg_G(f, \Omega) = \Sigma([\bar{f}]) \in Z_2$. The following theorem follows easily from the results proved in Gęba–Massabó–Vignoli [12].

2.15. THEOREM. *Suppose that V, $\dim V \geqslant 1$, is a trivial representation of $G = S^1$. Let Ω run through the family of all open bounded invariant subsets of $V \oplus R$ and $f: X \to V$ through continuous maps such that $\Omega \subset X$ and $f(\partial\Omega) \subset V\setminus\{0\}$. Then there exists a Z_2-valued function $\deg_G(f, \Omega)$ satisfying the following conditions:*

(a) *If $\deg_G(f, \Omega) \neq 0$ then there exists $x \in \Omega$ such that $f(x) = 0$.*

(b) *If $\Omega_0 \subset \Omega$ is open and $f^{-1}(0) \cap \Omega \subset \Omega_0$ then $\deg_G(f, \Omega_0) = \deg_G(f, \Omega)$.*

(c) *If Ω_1, Ω_2 are open subsets of Ω such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$ then $\deg_G(f, \Omega) = \deg_G(f, \Omega_0) + \deg_G(f, \Omega_1)$.*

(d) *If $h: (\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V\setminus\{0\})$ is a homotopy then $\deg_G(h_0, \Omega) = \deg_G(h_1, \Omega)$.*

(e) *Suppose W is another trivial representation of S^1 and let U be an open bounded subset of W such that $0 \in U$. Define $F: U \times \bar{\Omega} \to W \oplus V$ by $F(x, y) = (x, f(y))$. Then $\deg_G(F, U \times \Omega) = \deg_G(f, \Omega)$.*

3. Proof of the general case of Theorem 1.2.

In this section, using the results of the preceding section, we complete the proof of Theorem 1.2. In view of 1.1 we may assume that

$$(*) \qquad V = R[k_0, 0] \oplus R[k_1, m_1] \oplus \ldots \oplus R[k_r, m_r]$$

is a fixed representation of $G = S^1$. In what follows we identify the linear space V with R^n, where $n = k_0 + 2k_1 + \quad + 2k_r$. This determines an orientation of V. Moreover, with respect to the standard scalar product in R^n, V is an orthogonal representation of S^1 ($\langle x, y \rangle = \langle gx, gy \rangle$ for all $g \in S^1$ and $x, y \in V$).

Let $H = \{H_1, \ldots, H_k\}$ denote the family of all closed subgroups S^1 such that $V_H \neq \emptyset$. If $H \in H$ then V^H (resp. $(V \oplus R)^H$) is a G-invariant linear space of V (resp. $V \oplus R$). Note also that $(V \oplus R)^H = V^H \oplus R$ for all $H \in H$. Since V and $V \oplus R$ are orthogonal representations of S^1 we have direct sum decompositions $V = V^H \oplus (V^H)^\perp$, $V \oplus R = (V \oplus R)^H \oplus (V^H)^\perp$. Let $P^H: V \oplus R \to (V^H)^\perp$ denote the orthogonal projection. Note that $(V^H)^\perp$ is a G-invariant subspace of V

In this section we still use all the notations of Sections 1 and 2.

Assume now that $\Omega \subset V \oplus R$ is G-invariant, open and bounded.

3.1. Definition. We say that a map $f \in C_G(\bar{\Omega}, \partial\Omega)$ is *normal* if there exists $\eta > 0$ such that $|P^H x| < \eta$ implies $P^H x = P^H f(x)$ for all $x \in \bar{\Omega}$ and $H \in H$. Similarly we say that a G-homotopy h: $(\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V\backslash\{0\})$ is *normal* if there exists $\eta > 0$ such that $|P^H x| < \eta$ implies $P^H h(x, t) = P^H x$ for all $(x, t) \in \bar{\Omega} \times [0, 1]$ and $H \in H$.

The following fact is an immediate consequence of Definition 3.1 and the observation that Ω_H is open in V^H.

3.2. Corollary. *Suppose* $f \in C_G(\bar{\Omega}, \partial\Omega)$ *is normal. For* $H \in H$ *let* f_H: $\bar{\Omega}_H \to V^H$ *denote the restriction. Then* $f_H^{-1}(0)$ *is a compact subset of* Ω_H. *Similarly, if* h: $(\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V\backslash\{0\})$ *is a normal homotopy and* h_H: $\bar{\Omega}_H \times [0, 1] \to V^H$ *denotes the restriction then* $h_H^{-1}(0)$ *is a compact subset of* $\Omega_H \times [0, 1]$.

3.3. Proposition. *Let* Ω *be an open bounded G-invariant subset of* $V \oplus R$.

(*) *For any* $f \in C_G(\bar{\Omega}, \partial\Omega)$ *there exists* $f_0 \in C_G(\bar{\Omega}, \partial\Omega)$ *such that* f_0 *is normal and G-homotopic to* f.

(**) *If* $f_0, f_1 \in C_G(\bar{\Omega}, \partial\Omega)$ *are two G-homotopic normal maps then there exists a normal homotopy between* f_0 *and* f_1.

In the proof of 3.3 we will use the following lemmas.

3.4. Lemma. *For every* $\varepsilon > 0$ *and* $H \in H$ *there exists a G-equivariant map* r: $V \oplus R \to V \oplus R$ *such that*

(*) $|r(x) - x| < \varepsilon$ *for all* $x \in V \oplus R$,

(**) *if* $x \in V \oplus R$ *and* $|P^H(x)| \leqslant \varepsilon$ *then* $r(x) \in V^H \oplus R$.

Proof. Define θ: $V \oplus R \to [0, 1]$ by

$$\theta(x) = \begin{cases} 0 & \text{if } |P^H(x)| \leqslant \varepsilon, \\ (|P^H(x)| - \varepsilon)\varepsilon^{-1} & \text{if } \varepsilon < |P^H(x)| < 2\varepsilon, \\ 1 & \text{if } |P^H(x)| \geqslant 2\varepsilon. \end{cases}$$

Define r by $r(x) = x - P^H(x) + \theta(x)P^H(x)$. It is easy to check that r has the required properties.

3.5. Lemma. *For every* $\delta > 0$ *there exist* $\eta > 0$ *and a G-map* Γ: $V \oplus R \to V \oplus R$ *such that*

(*) $|\Gamma(x) - x| \leqslant \delta$ *for all* $x \in V \oplus R$,

(**) *if* $H \in H$, $x \in V \oplus R$ *and* $|P^H x| \leqslant \eta$ *then* $\Gamma(x) \in V^H \oplus R$.

Proof. For $H_i \in H = \{H_1, \ldots, H_k\}$ we set $P_i = P^{H_i}$. Given $\delta > 0$ set $\delta_i = 2^{-i}\delta$. From the preceding lemma it follows that for each i, $1 \leqslant i \leqslant k$, there exists r_i: $V \oplus R \to V \oplus R$ such that $|r_i(x) - x| \leqslant \delta_i$ for all $x \in V \oplus R$ and $|P_i(x)| \leqslant \delta_i$ implies $r_i(x) \in V^{H_i} \oplus R$. Set $\Gamma = r_1 \circ \ldots \circ r_k$. An easy computation shows that Γ has all the required properties.

Proof of 3.3. First, to prove (∗) suppose f: $(\bar{\Omega}, \partial\Omega) \to (V, V\backslash\{0\})$ is a G-map. Choose $\varepsilon > 0$ such that $|f(x)| \geqslant 3\varepsilon$ for all $x \in \partial\Omega$. Let $D = \{x \in V \oplus R: |x| \leqslant R\}$ be a closed disc such that $\Omega \subset \{x \in V \oplus R: |x| \leqslant R - \varepsilon\}$. By the Tietze–Gleason extension theorem (Bredon [3]) there exists a G-map F: $D \to V$ which extends f. Since F is uniformly continuous, there exists δ, $0 < \delta \leqslant \varepsilon$, such that $x, y \in D$ and $|x - y| < \delta$ imply $|F(x) - F(y)| < \varepsilon$. Applying Lemma 3.5 we get Γ: $V \oplus R \to V \oplus R$ and $\eta > 0$ which satisfy conditions (∗) and (∗∗) of that lemma. Since $\delta \leqslant \varepsilon$ and $|\Gamma(x) - x| < \delta$, we have $\Gamma(\Omega) \subset D$. Let Q: $V \oplus R \to V$ denote the projection map. For $(x, t) \in \bar{\Omega} \times I$ let

$$h(x, t) = F((1 - t)x + t\Gamma(x)) + tQ(x - \Gamma(x)).$$

If $(x, t) \in \partial\Omega \times I$ then $|x - \Gamma(x)| \leqslant \delta \leqslant \varepsilon$, which implies

$$|Q(x - \Gamma(x))| \leqslant \varepsilon \quad \text{and} \quad |((1 - t)x + t\Gamma(x)) - x| = |t(\Gamma(x) - x)| \leqslant \delta$$

and so $|F((1 - t)x + t\Gamma(x)) - F(x)| \leqslant \varepsilon$. Thus for $(x, t) \in \partial\Omega \times [0, 1]$ we have

$$|h(x, t)| = |h(x, t) - f(x) + f(x)|$$

$$\geqslant |f(x)| - |F((1 - t)x + t\Gamma(x)) - F(x)| - |Q(x - \Gamma(x))| \geqslant \varepsilon.$$

Therefore h: $(\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V\backslash\{0\})$ is a G-homotopy. Let $f_0 = h_1$.

If for some $H \in \mathcal{H}$ and $x \in \bar{\Omega}$, $|P^H(x)| \leqslant \eta$, then $\Gamma(x) \in V^H \oplus R$, hence $F(\Gamma(x)) \in V^H$ and thus $P^H f_0(x) = P^H(F(\Gamma(x)) + x - \Gamma(x)) = P^H(x)$. Therefore f_0 is normal and the proof of (∗) is complete.

To prove (∗∗) assume h: $(\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V\backslash\{0\})$ is a G-homotopy such that $f_0 = h_0$ and $f_1 = h_1$ are normal. By the definition of a normal map

(a) there exists $\eta_0 > 0$ such that $P^H(f_i(x)) = P^H(x)$ for all $x \in \bar{\Omega}$ such that $P^H(x) \leqslant \eta_0$ and $H \in \mathcal{H}$ and $i = 0, 1$.

Choose $\varepsilon > 0$ such that $|h(x, t)| \geqslant 3\varepsilon$ for all $(x, t) \in \partial\Omega \times [0, 1]$. Arguing as in the proof of (∗) we extend h to a G-map Φ: $D \times [0, 1] \to V$ and choose $\delta > 0$ such that $|x - y| + |t - s| < \delta$ implies $|\Phi(x, t) - \Phi(y, s)| < \varepsilon$. Applying Lemma 3.5 we obtain $\eta > 0$ and Γ satisfying conditions (∗) and (∗∗) of that lemma.

For $(x, t) \in \Omega \times [0, 1]$ set $h^*(x, t) = \Phi(\Gamma x, t) + Q(x - \Gamma x)$, $h^{(0)}(x, t) = (1 - t)f_0(x) + th^*(x, 0)$, $h^{(1)}(x, t) = tf_1(x) + (1 - t)h^*(x, 1)$. Evidently

$$h^{(0)}, h^*, h^{(1)}: (\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V\backslash\{0\})$$

are G-homotopies. Set $\eta_1 = \min\{\eta_0, \eta\}$. From the definition of h^* it follows at once that h^* is a normal G-homotopy. The definition of $h^{(0)}$, $h^{(1)}$ and condition (a) imply $P^H h^{(i)}(x, t) = P^H x$ if $|P^H x| < \eta_1$, $x \in \bar{\Omega}$, $H \in \mathcal{H}$, $i = 0, 1$. Therefore $h^{(0)}$, $h^{(1)}$ are normal G-homotopies. The composition of the normal homotopies

$h^{(0)}$, h^*, $h^{(1)}$ defines a normal homotopy between f_0 and f_1. The proof of 3.3 is complete.

In view of 3.2 the following definition is correct.

3.6. Definition. Suppose $f \in C_G(\bar{\Omega}, \partial\Omega)$ is normal. For each $H \in H$ choose an open invariant subset U_H such that $f_H^{-1}(0) \subset U_H \subset \bar{U}_H \subset \Omega_H$. Then $\deg_H(f_H, U_H)$ is well defined and independent of the choice of U_H. Define

$$\deg_H(f, \Omega) = \begin{cases} \deg_H(f_H, U_H) & \text{for } H \in H, \\ 0 & \text{for } H \notin H, \end{cases}$$

and $\mathrm{Deg}(f, \Omega) = \{\deg_H(f, \Omega)\}$.

Consider now the general case and take $f \in C_G(\bar{\Omega}, \partial\Omega)$ not necessarily normal. By Corollary 3.2 there exists $f_0 \in C_G(\bar{\Omega}, \partial\Omega)$ normal and G-homotopic to f. Moreover, if $f_1 \in C_G(\bar{\Omega}, \partial\Omega)$ is another normal map homotopic to f then there exists a normal homotopy between f_0 and f_1. Hence, by 2.14 and 2.15, $\mathrm{Deg}(f_0, \Omega) = \mathrm{Deg}(f_1, \Omega)$. Therefore the following definition is correct.

3.7. Definition. For $f \in C_G(\bar{\Omega}, \partial\Omega)$ let $\mathrm{Deg}(f, \Omega) = \mathrm{Deg}(f_0, \Omega)$, where $f_0 \in C_G(\bar{\Omega}, \partial\Omega)$ is normal and G-homotopic to f.

Proof of Theorem 1.2. To prove (a) suppose that $f \in C_G(\bar{\Omega}, \partial\Omega)$ and $H \in H$ are such that $f(x) \neq 0$ for all $x \in \Omega^H$. Let $\varepsilon = \inf\{|f(x)|: x \in \bar{\Omega}^H\}$. The arguments used in the proof of condition (*) of 3.3 show that there exists a normal map f_0 which is homotopic in $C_G(\bar{\Omega}, \partial\Omega)$ to f and $|f(x) - f_0(x)| < \varepsilon$ for all $x \in \bar{\Omega}$. Thus $f_0(x) \neq 0$ for all $x \in \bar{\Omega}_H$. Therefore $\deg_H(f, \Omega) = \deg_H(f_0, \Omega) = 0$ and the proof of (a) is complete.

(b), (c) and (e) are easy consequences of 2.14(b), 2.15(b), 2.14(c), 2.15(c), 2.14(e) and 2.15(e). Property (d) follows from Definition 3.6, 2.14(d), 2.15(d) and condition (**) of 3.3. Thus the proof of Theorem 1.2 is complete.

4. Computations of the S^1-degree. In this section we prove some formulas which express the S^1-degree of equivariant map in terms of its derivative.

We keep the notations of the preceding sections. As before we assume that Ω is an open invariant bounded subset of $V \oplus R$, where V is an orthogonal representation of $G = S^1$ of the form described by condition (*) given at the beginning of Section 3. Suppose $f \in C_G(\bar{\Omega}, \partial\Omega)$ and let $f_0: \Omega \to V$ denote the restriction of f. We assume further that f_0 is of class C^1 and 0 is a regular value of f_0. In this case $f^{-1}(0) = M_1 \cup \ldots \cup M_k$, where each M_i is diffeomorphic to S^1 and $M_i \cap M_j = \emptyset$ for $i \neq j$. For each i choose an open invariant subset $\Omega_i \subset \Omega$ such that $M_i \subset \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Clearly $\mathrm{Deg}(f, \Omega) = \mathrm{Deg}(f, \Omega_1) + \ldots + \mathrm{Deg}(f, \Omega_k)$. Therefore throughout the rest of this section we assume $k = 1$, i.e. we assume $f^{-1}(0) = M$ is connected, hence diffeomorphic to S^1. We also let K denote the isotropy group of points in M. We discuss separately the following two cases:

Case 1. $K \neq S^1$, $K = Z_m$.
Case 2. $K = S^1$.

Having in mind our applications of the equivariant degree to differential equations, we first discuss Case 1. Choose $a \in M$ and let $A = Df(a)$. Differentiating, for $g \in K$, the equality $f(ga) = gf(a)$ we see that $A: V \oplus R \to V$ is a K-equivariant linear map. By the Schur Lemma, the direct sum decompositions

$$V \oplus R = (V^K \oplus R) \oplus (V^K)^\perp, \qquad V = V^K \oplus (V^K)^\perp$$

give a decomposition of A of the form

$$A = \begin{bmatrix} A^K & 0 \\ 0 & A^\perp \end{bmatrix},$$

where $A^K: V^K \oplus R \to V^K$, $A^\perp: (V^K)^\perp \to (V^K)^\perp$.

Since $M = Ga$ is diffeomorphic to S^1/K, it is an oriented 1-dimensional submanifold of Ω. Set $v = \eta'(0)$, where $\eta(t) = g(t)a$, $g(t) = \exp(2\pi i t)$. Clearly $v \in V^K \oplus R$. Let $N^K = \{x \in V^K \oplus R: \langle x, v \rangle = 0\}$, i.e. N^K is the linear subspace of $V^K \oplus R$ which is normal to M at a.

The following theorem describes the S^1-degree of the map f in question.

4.1. Theorem. *Suppose* $f \in C_G(\bar\Omega, \partial\Omega)$ *satisfies all the assumptions of Case 1.*

(i) *If* $\det A^\perp > 0$ *then*

$$\deg_H(f, \Omega) = \begin{cases} \operatorname{sgn}(A^K, v) & \text{if } H = K = Z_m, \\ 0 & \text{if } H \neq K. \end{cases}$$

(ii) *If* $\det A^\perp < 0$ *then* $m = 2\mu$ *is even and*

$$\deg_H(f, \Omega) = \begin{cases} \operatorname{sgn}(A^K, v) & \text{if } H = K = Z_m, \\ -\operatorname{sgn}(A^K, v) & \text{if } H = Z_\mu, \\ 0 & \text{if } H \neq Z_m, Z_\mu. \end{cases}$$

In the proof of Theorem 4.1 we will need the following two facts. The first is essentially a version of the Slice Theorem (see Bredon [3]).

4.2. Proposition. *Assume* $\Omega = GB$, *where* B *is an open sufficiently small disc in the space* $N = \{x \in V \oplus R: \langle x, v \rangle = 0\}$ *with center* a. *Then every* K-*equivariant map* $\varphi \in C_K(B, \partial B)$ (*every* K-*equivariant homotopy* $\chi \in C_K(B \times [0, 1], \partial B \times [0, 1])$) *determines a* G-*equivariant map* $f \in C_G(\bar\Omega, \partial\Omega)$ (*G*-*equivariant homotopy* $h \in C_G(\bar\Omega \times [0, 1], \partial\Omega \times [0, 1])$) *by*

$$f(g \cdot x) = g \cdot \varphi(x) \qquad (h(g \cdot x, t) = g \cdot \chi(x, t)),$$

where $x \in B$, $g \in G = S^1$

4.3. Lemma. *Suppose that* W *is a representation of* $K = Z_m$ *such that* $W^K = \{0\}$ *and let* $\operatorname{Aut}_K W$ *denote the group of all* K-*equivariant linear automorphisms of* W.

(∗) *If* $A \in \mathrm{Aut}_K(W)$ *and* $\det A > 0$ *then* A *belongs to the connected component of* $\mathrm{Aut}_K W$ *containing the identity.*

(∗∗) *If* $A \in \mathrm{Aut}_K W$ *and* $\det A < 0$ *then there exist subrepresentations* W_1, W_2 *of* W *such that* $W = W_1 \oplus W_2$, W_1 *is one-dimensional and* A *belongs to the connected component of* J, *where (with respect to the above decomposition)*

$$J = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

The above lemma is a consequence of some standard facts in group representation theory. For the sake of completeness we will sketch the proof of it after proving Theorem 4.1.

Proof of Theorem 4.1. Notice that without loss of generality we may assume that $\Omega = GB$, where B denotes a sufficiently small disc in N with center at a. Recall that $A = Df(a)$ is a K-equivariant linear map and $A_{|N}: N \to V$ is a K-equivariant isomorphism. Hence the map $\varphi: B \to V$ defined by $\varphi(x) = A(x-a)$ belongs to $C_K(B, \partial B)$. If the disc B is sufficiently small then the linear homotopy $\chi(x, t) = t f_{|N}(x) + (1-t)\varphi(x)$ is zero only for $x = a$. Applying Lemma 4.3 to $A^\perp \in \mathrm{Aut}_K((V^K)^\perp)$ we obtain a continuous path $\eta: [0, 1] \to \mathrm{Aut}_K((V^K)^\perp)$ such that $\eta(0) = A^\perp$ and $\eta(1) = A_1$, where A_1 is either I or J. Define $\varphi^\wedge \in C_K(B, \partial B)$ by $\varphi^\wedge(x) = A_*(x-a)$, where, with respect to the direct sum decompositions $V \oplus R = (V^K \oplus R) \oplus (V^K)^\perp$ and $V = V^K \oplus (V^K)^\perp$,

$$A_* = \begin{bmatrix} A^K & 0 \\ 0 & A_1 \end{bmatrix}.$$

Define $f_1 \in C_G(\bar{\Omega}, \partial\Omega)$ by $f_1(\theta x) = \theta \varphi^\wedge(x)$, where $x \in B$, $\theta \in G = S^1$. Proposition 4.2 implies that f_1 is homotopic in $C_G(\bar{\Omega}, \partial\Omega)$ to f. If $\det A^\perp > 0$ then the first part of our theorem follows easily from 1.2(e).

If $\det A^\perp < 0$ then by 4.3 there exists a direct sum decomposition $W = (V^K)^\perp = W_1 \oplus W_2$ such that W_1 is one-dimensional and $A_1 = J$. Let $\Omega_0 = \Omega \cap (V^K \oplus R)$ and, for $i = 1, 2$, let U_i denote the unit disc in W_i. Without loss of generality we may assume $\Omega = \Omega_0 \times U_1 \times U_2$. Let $F: \bar{\Omega}_0 \times \bar{U}_1 \to V^K \oplus W_1$ denote the restriction of f_1. Writing points of $(V^K \oplus R) \oplus W_1$ in the form $x = (y, z)$, we have $F(y, z) = (f_1(y), -z)$. Let $\psi: W_1 \to W_1$ be an odd C^1-function such that $\psi(z) = z$ for $\|z\| \leq \frac{1}{4}$, $\psi(z) = -z$ for $\|z\| \geq \frac{1}{2}$, $\psi^{-1}(0) = \{-\tau, 0, \tau\}$ and $\psi'(-\tau) = \psi'(\tau) = -1$. The formula $\psi_t(z) = (1-t)\psi(z) + t(-x)$ gives a homotopy between $-I: W_1 \to W_1$ and ψ. Define $F_1: \bar{\Omega}_0 \times \bar{U}_1 \to V^K \oplus W_1$ by $F_1(y, z) = (f_1(y), \psi(z))$. From the definition of F_1 and 4.2 it follows at once that $F_1 \in C_G(\bar{\Omega}_0 \times \bar{U}_1, \partial(\Omega_0 \times U_1))$ and F_1 is homotopic to F in $C_G(\bar{\Omega}_0 \times \bar{U}_1, \partial(\Omega_0 \times U_1))$. Obviously $(F_1)^{-1}(0) = M \cup M_1$, where $M_1 = Gb$ and $b = (a, \tau) \in \Omega_0 \times U_1$ has isotropy group Z_μ. Applying the first part of our theorem to F_1 we obtain

$$\deg_H(F, \Omega_0 \times U_1) = \begin{cases} \text{sgn}(A^K, v) & \text{if } H = K = Z_m, \\ -\text{sgn}(A^K, v) & \text{if } H = Z_\mu, \\ 0 & \text{if } H \neq Z_m, Z_\mu. \end{cases}$$

Applying again 1.2(e) we have $\deg_H(f, \Omega) = \deg_H(F, \Omega_0 \times U_1)$ for all $H \in \mathcal{H}$ and the proof of Theorem 4.1 is complete.

To sketch the proof of Lemma 4.3 we have to collect some basic facts concerning representations of the group $K = Z_m$. In what follows we identify Z_m with a subgroup of S^1. Let γ denote the generator of Z_m corresponding to $\exp(2\pi i/m)$. Suppose W is a representation of Z_m. According to the conventions we have adopted in Section 1, $\mathcal{W} = (W, \varrho)$, where W is a finite-dimensional linear space over R and $\varrho: Z_m \to \text{GL}(W)$ is a group homomorphism. W is called *irreducible* if there is no decomposition $W = W_1 \oplus W_2$ such that both W_1 and W_2 are positive-dimensional subrepresentations of W.

Let $L_K(W)$ denote the linear space of all linear K-equivariant maps of W into itself and $\text{Aut}_K(W)$ the group of all linear K-equivariant automorphisms of W: $I \in \text{Aut}_K(W)$ will denote the identity. For $j \in N$ define $\varrho_j: Z_m \to \text{GL}(2, R)$ by

$$\varrho_j(e^{i\theta}) = \begin{bmatrix} \cos j\theta & -\sin j\theta \\ \sin j\theta & \cos j\theta \end{bmatrix}.$$

Let $Q_j = (R^2, \varrho_j)$. We will need the following fact which is an easy consequence of the introduced definitions.

4.4. LEMMA. *For given $j \in N$ let $A_1 = \varrho_j(\gamma)$. If $A_1 \neq I, -I$ then $L_K(Q_j)$ and C are isomorphic as linear spaces over R.*

As a direct consequence of 4.4 we obtain the following simple observation.

4.5. COROLLARY. *Suppose $L_K(Q_j) \approx C$ and W is the direct sum of k copies of Q_j. Then $\text{Aut}_K(W)$ is isomorphic to $\text{GL}(k, C)$ ($=$ the group of all C-linear automorphisms of C^k).*

In the case where m is even and $m = 2\mu$ we let $Q_0 = (R, \varrho_0)$ where $\varrho_0: Z_m \to \text{GL}(1, R)$ is defined by $\varrho_0(\gamma) = -I$. In this case every linear map of Q_0 into itself is K-equivariant. Therefore we have the following:

4.6. COROLLARY. *Suppose W is the direct sum of k copies of Q_0. Then $\text{Aut}_K(W)$ is isomorphic to $\text{GL}(k, R)$.*

4.7. PROPOSITION (Serre [20]). *Suppose W is a nontrivial irreducible representation of Z_m.*

(*) *If m is odd and $\mu = (m-1)/2$ then W is 2-dimensional and there exists j, $1 \leq j \leq \mu$, such that W is equivalent to Q_j. Moreover, if $1 \leq \alpha < \beta \leq \mu$ then Q_α is not equivalent to Q_β.*

(**) *If m is even and $\mu = m/2$ then either W is 2-dimensional and there exists j, $1 \leqslant j < \mu$, such that W is equivalent to Q_j, or W is 1-dimensional and equivalent to Q_0. Moreover, if $0 \leqslant \alpha < \beta \leqslant \mu$ then Q_α is not equivalent to Q_β.*

4.8. Proposition. *Suppose that W is a representation of $K = Z_m$ such that $W^K = \{0\}$. Then admitting $W_j = 0$ as an empty sum, we have:*

(*) *If m is odd then $W = W_1 \oplus \ldots \oplus W_\mu$, where $\mu = (m-1)/2$ and each W_j is equivalent to direct sum of Q_j.*

(**) *If m is even and $m = 2\mu$ then $W = W_0 \oplus W_1 \oplus \ldots \oplus W_{\mu-1}$, where each W_j is equivalent to a direct sum of Q_j.*

Proof of Lemma 4.3. The Schur Lemma implies that with respect to the direct sum decomposition $W = W_0 \oplus W_1 \oplus \ldots \oplus W_\mu$ the linear map $A \in \mathrm{Aut}_K W$ is of the diagonal form

$$
A = \begin{bmatrix} A_0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_\mu \end{bmatrix}, \quad \text{where } A_j \colon W_j \to W_j.
$$

Thus $\mathrm{Aut}_K(W) = \mathrm{Aut}_K(W_0) \times \mathrm{Aut}_K(W_1) \times \quad \times \mathrm{Aut}_K(W_\mu)$. Since $\mathrm{GL}(k, C)$ is connected and $\mathrm{GL}(k, R)$ has two connected components the result follows.

Case 2. $K = G = S^1$.

Recall that in Case 2 we assume that the restriction of f to Ω is of class C^1, 0 is a regular value of this restriction, $M = f^{-1}(0)$ is connected (therefore diffeomorphic to S^1) and $G_x = S^1$ for all $x \in M$. Recall that there is a unique decomposition $V = V^G \oplus V^1 \oplus \ldots \oplus V^k$, where $V^i = R[k_i, m_i]$. For $x \in M$ let $A(x) = Df(x)$, thus the assignment $x \mapsto A(x)$ defines a continuous mapping from M into $L(V \oplus R, V)$. Moreover, for each $x \in M$, $A(x)$ is equivariant and epimorphic. The Schur Lemma implies that, with respect to the direct sum decompositions $V = V^G \oplus V^1 \oplus \ldots \oplus V^k$, $V \oplus R = (V^G \oplus R) \oplus V^1 \oplus \ldots \oplus V^k$, $A(x)$ is of the diagonal form

$$
A(x) = \begin{bmatrix} A^G(x) & & & 0 \\ & A_1(x) & & \\ & & \ddots & \\ 0 & & & A_k(x) \end{bmatrix}
$$

where $A^G(x) \colon V^G \oplus R \to V^G$ is a linear epimorphism and $A_i(x) \colon V^i \to V^i$ are linear isomorphisms.

The S^1 action defines a complex structure on V^j by $i*x = gx$, where $g \in S^1$, $g = e^{(i\pi/2m_j)}$. Moreover, an R-linear map of V^j is equivariant iff it is C-linear with respect to this structure. So, for $i = 1, \ldots, k$, V^i are complex linear spaces and $A_i(x)$ are complex automorphisms (cf. 4.5).

Recall that $\pi_1(\mathrm{GL}(m, C))$ (the fundamental group of $\mathrm{GL}(m, C)$) is isomorphic to Z and the isomorphism is induced by det. Let $d \colon S^1 \to M$ be a fixed

diffeomorphism such that $\text{sgn}(A^G(x), v) = -1$ where $v = d'(z)(1)$, $z = d^{-1}(x)$. We set $\gamma_l = A_l \circ d\colon S^1 \to \text{GL}(V^l, C)$. Let $p_l \in Z \approx \pi_1(\text{GL}(V^l, C))$ denote the class of the loop γ_l.

4.9. Theorem. *Suppose* $f \in C_G(\bar{\Omega}, \partial\Omega)$ *satisfies all the assumptions of Case 2. Then*

$$\deg_H(f, \Omega) = \begin{cases} p_0 & \text{if } H = S^1 = G, \\ p_l & \text{if } H = Z_{m_l}, \\ 0 & \text{if } H \neq Z_{m_l} \text{ and } H \neq G, \end{cases}$$

where $p_0 \in Z_2$ *is defined in Section 2.*

The idea of the proof is to find a normal G-map of simplest form in the homotopy class of f. We begin with the following statement whose proof is left to the reader.

4.10. Lemma. *Suppose that V is a complex linear space of dimension k and $e_1, \ldots, e_k$ is a basic of V over C. Then for every map $\gamma\colon S^1 \to \text{GL}(V, C)$ there exists a homotopy* $\Lambda\colon S^1 \times I \to \text{GL}(V, C)$ *such that $\Lambda(z, 0) = \gamma(z)$ and*

$$\Lambda(z, 1) = \begin{bmatrix} z^p & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \quad \text{in the basis } e_1, \ldots, e_k,$$

where p is the class of γ in $\pi_1(\text{GL}(V, C))$.

Denote by U_0 a tubular neighbourhood of M in the open set $\Omega^G \subset V^G \oplus R$. Identifying U_0 with a closed neighbourhood of the zero section in the normal bundle, we define a retraction $r\colon U_0 \to M$ by

$$r(x) = x_0 \quad \text{if} \quad x \in x_0 + N_{x_0}.$$

Next, let $D_1, \ldots, D_k$ be discs in $V^1, \ldots, V^k$ respectively such that $\Omega_1 = U_0 \times D_1 \times \cdots \times D_k$ is contained in Ω. For $\gamma_j = A_j \circ d\colon S^1 \to \text{GL}(V^j, C)$ we denote by A'_j the map $\Lambda_j(d^{-1}(\cdot), 1)\colon M \to \text{GL}(V^j, C)$, where Λ_j is given by Lemma 4.10 for $\gamma = \gamma_j$.

4.11. Proposition. *Let f satisfy the hypothesis of Theorem 4.9. Then for sufficiently small $U_0, D_1, \ldots, D_k$ the G-map $f_1\colon (\bar{\Omega}_1, \partial\Omega_1) \to (V, V\setminus\{0\})$ given by*

$$f_1(x, y_1, \ldots, y_k) = A^G(r(x))(x - r(x)) + \sum A'_j(r(x))y_j$$

is G-homotopic to f in $C_G(\bar{\Omega}_1, \partial\Omega_1)$.

Proof. Since $A'_j(x)\colon V^j \to V^j$, $1 \leqslant j \leqslant k$, are G-equivariant, f_1 is G-equivariant. It is evident that the G-map $\mathscr{H}\colon (\bar{\Omega}_1 \times I, \partial\Omega_1 \times I) \to (V, V\setminus\{0\})$ given by

$$\mathscr{H}(x, y_1, \ldots, y_k, t) = A^G(r(x))(x - r(x)) + \sum \Lambda_j(d^{-1}r(x), t)y_j$$

yields a G-homotopy between f_1 and the map

$$A^G(r(x))(x - r(x)) + \sum \Lambda_j(r(x))y_j.$$

This last map is G-homotopic to f in $C_G(\bar{\Omega}_1, \partial\Omega_1)$ by a Rouché principle argument. This proves the proposition.

In the remainder of the proof of Theorem 4.9 we assume $\Omega_1 = \Omega = U_0 \times D_1 \times \quad \times D_k$ to be as in Proposition 4.11. Starting from the map f_1 we next construct a normal G-map f_2 G-homotopic to f_1. For this purpose we introduce maps $h_j\colon M \times D_j \to V^j$, $1 \leqslant j \leqslant k$, setting

$$h_j(x, y) = \begin{cases} y & \text{if } \|y\| \leqslant \tfrac{1}{2}\varepsilon, \\ \alpha A'_j(x)(\varepsilon/\|y\|)y + (1-\alpha)(\varepsilon/2\|y\|)y & \text{if } \tfrac{1}{2}\varepsilon \leqslant \|y\| \leqslant \varepsilon, \end{cases}$$

where ε is the radius of D_j and $\alpha = 2\|y\|/\varepsilon - 1$.

4.12. Lemma. *Let h_j be as defined above.*

(1) h_j *is G-equivariant as a map of y.*

(2) *If $\|y\| = \varepsilon$ then $h_j(x, y) = A'_j(x)y$.*

(3) *If $h_j(x, y) = 0$ for $y \neq 0$ then*

$$A'_j(x) = \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

(The last means that $(d^{-1}(x))^{p_j} = -1$, or equivalently, x lies in the image of the set $C_j = \{z \in S^1 : z^{p_j} = -1\}$.)

(4) *For every $x \in C_j$ there exists only one orbit $Gy^0 \subset D_j$ such that $h_j(x, y) = 0$ for any $y \in Gy^0$. In fact, $y^0 = (y_1, 0, \ldots, 0)$ with $\|y^0\| = 2\varepsilon/3$, in the basis $e_1, \ldots, e_{k_j}$ of V_j over C (cf. 4.10).*

(5) *Let $y_1 = (0, 2\varepsilon/3)$ in the basis e_1, ie_1 of the plane $C \times \{0\} \times \quad \times \{0\} \subset V^j$. If $w = -e_1$ and $n = ie_1$ are respectively the tangent and normal vector to $Gy_1 \subset C$ then the matrix $(\partial h_j/\partial y)(x^0, y^0)$, $x^0 \in d(C_j)$, from the basis $\{n, w, e_2, ie_2, \ldots, e_{k_j}, ie_{k_j}\}$ to $\{e_1, ie_1, e_2, ie_2, \ldots, e_{k_j}, ie_{k_j}\}$ has the form*

$$\begin{bmatrix} 0 & 0 & & & 0 \\ -3 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$

(6) *In the R-basis $\{e_1, ie_1, e_2, ie_2, \ldots, e_{k_j}, ie_{k_j}\}$ we have*

$$(\partial h_j/\partial v)(x^0, y^0) = \begin{bmatrix} p_j\varepsilon/3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $v = d'(z)(1)$, $z = d^{-1}(x^0)$.

Proof. An easy computation shows (1)–(4). To prove (5), observe that

$$h_j(x^0, y^0 + tie_1) = h_j(x^0, (2\varepsilon/3 + t)ie_1) = [2\varepsilon - 3(2\varepsilon/3 + t)]ie_1,$$

which shows that $(\partial h_j/\partial n)(x^0, y^0) = -3ie_1$. The second column of the matrix in question is the zero vector, because w is a tangent vector to Gy^0 and h_j vanishes on this orbit. Finally, observe that h_j has the form $h_j(x, y) = (h_j^{\smallfrown}(x, y), y_2, \ldots, y_k)$, which gives (5). Also a direct computation shows (6). This completes the proof.

4.13. PROPOSITION. Let $h_j(x, y)$, $1 \leqslant j \leqslant k$, be the G-equivariant map of Lemma 4.12. Then the map $f_2: (\bar{\Omega}, \partial\Omega) \to (V, V \backslash \{0\})$ given by

$$f_2(x, y_1, \ldots, y_k) = A^G(r(x))(x - r(x)) + \sum h_j(r(x), y_j)$$

is G-equivariant, normal and G-homotopic to f_1 in $C_G(\bar{\Omega}, \partial\Omega)$.

Proof. Since the h_j are G-equivariant, f_2 is G-equivariant too. We will show that f_2 is normal for the constant $\varepsilon/2$. For any subgroup $H \subset S^1$ with $V_H \neq \emptyset$ we have

$$(V^H \oplus R)^\perp = V^{j_1} \oplus \ldots \oplus V^{j_r} \quad \text{for some } 1 \leqslant j_1 < \ldots < j_r \leqslant k.$$

It follows that $P^H(x, y_1, \ldots, y_k) = (y_{j_1}, \ldots, y_{j_r})$ for every $(x, y_1, \ldots, y_k) \in \bar{\Omega}$. If $\|P^H(x, y_1, \ldots, y_k)\| \leqslant \varepsilon/2$ then $\|y_{j_i}\| \leqslant \varepsilon/2$, and from the definition of h_j, we have $h_j(r(x), y_j) = y_j$ for $j = j_1, \ldots, j_r$. It follows that

$$P^H f_2(x, y_1, \ldots, y_k) = h_{j_1}(r(x), y_{j_1}) + \quad + h_{j_r}(r(x), y_{j_r}) = y_{j_1} + \quad + y_{j_r}$$

as desired. Let us join the maps f_1 and f_2 by the linear homotopy $H((x, y), t)$ $= tf_2(x, y) + (1-t)f_1(x, y)$. It is sufficient to show that H has no zero on $\partial\Omega \times I$. If $(x, y_1, \ldots, y_k) \in \partial\Omega$ then $x \in \partial U_0$ or $y_j \in \partial D_j$ for some $1 \leqslant j \leqslant k$. By symmetry, we can assume that $y_1 \in \partial D_1$. Denoting by P_1 the projection onto V^1, we have $P_1 H(x, y_1, \ldots, y_k, t) = th_1(r(x), y_1) + (1-t)$ $\times A_1'(r(x))y_1 = A_1'(r(x))y_1 \neq 0$. Hence $H(x, y_1, \ldots, y_k, t) \neq 0$. The proof is complete.

Observe that for a given $(y_1, \ldots, y_k)$, if $f_2(x, y_1, \ldots, y_k) = 0$ then $x \in \bigcap C_j$ where the j are such that $y_j \neq 0$. This means that there exists $(x, y_1, \ldots, y_k) \in f_2^{-1}(0)$ with at least two nonzero coordinates y_s, y_r if $C_s \cap C_r \neq \emptyset$. We shall avoid this possibility in the next step of our procedure. For fixed $C_1, \ldots, C_k$ we can find $z_1, \ldots, z_k \in S^1$ such that the sets $z_1 C_1, \ldots, z_k C_k$ are mutually disjoint. Define $\theta_j: M \to M$ by $\theta_j(x) = d(z_j^{-1}(d^{-1}(x)))$.

4.14. PROPOSITION. Let θ_j, $1 \leqslant j \leqslant k$, be as defined above. Then the map $f_3: (\bar{\Omega}, \partial\Omega) \to (V, V \backslash \{0\})$ given by

$$f_3(x, y_1, \ldots, y_k) = A^G(r(x))(x - r(x)) + \sum h_j(\theta_j r(x), y_j)$$

has the following properties:

(i) f_3 is a G-equivariant, normal and G-homotopic to f_2.

(ii) If $f_3(x, y_1, \ldots, y_k) = 0$ then at most one y_i is different from zero. Moreover, $\Omega_H \cap f_3^{-1}(0) = \varnothing$ if $H \neq Z_{m_j}$, $1 \leqslant j \leqslant k$, $H \neq S^1$, and $\Omega_H \cap f_3^{-1}(0)$ consists of p_j distinct orbits if $H = Z_{m_j}$.

Proof. Since the h_j are G-equivariant in the second variable, f_3 is G-equivariant. Observe next that homotopies joining the maps θ_j to the identity on M induce a G-homotopy between f_2 and f_3. By the same argument as in Proposition 4.13, f_3 is normal. If $f_3(x, y_1, \ldots, y_k) = 0$ and $y_j \neq 0$ then $\theta_j(x) \in d(C_j)$, or equivalently $d^{-1}(x) \in z_j \cdot C_j$. Since $z_1 C_1, \ldots, z_k C_k$ are mutually disjoint, it follows that only one y_i is different from zero. Suppose that $(x, y_1, \ldots, y_k) \in \Omega_H \cap f_3^{-1}(0)$ where $H = Z_{m_i}$. On account of the above, $(x, y_1, \ldots, y_k) = (x, 0, \ldots, 0, y_i, 0, \ldots, 0)$. This shows that $\Omega_H \cap f_3^{-1}(0) = \{d(z_i C_i) + G y_i\}$, which means that the last set consists of p_i distinct orbits. Furthermore, if $H \neq Z_{m_j}$, $j = 1, \ldots, k$, $H \neq S^1$ and $\Omega_H \neq \varnothing$ then any point of Ω_H has at least two coordinates y_i different from zero, which implies $\Omega_H \cap f_3^{-1}(0) = \varnothing$. This ends the proof.

Proof of Theorem 4.9. By G-homotopy invariance, $\deg_H(f, \Omega) = \deg_H(f_3, \Omega)$ for every H. Suppose that $H \neq S^1$ and $H \neq Z_{m_j}$, $j = 1, \ldots, k$. Since $\Omega_H \cap f_3^{-1}(0) = \varnothing$ and f_3 is normal, $\deg_H(f, \Omega) = 0$. Finally, let $H = Z_{m_j}$, and $p = |p_j|$. From what has already been proved, we have

$$\Omega_H \cap f_3^{-1}(0) = \{x_1 + Gy, \ldots, x_p + Gy\},$$

where $\{x_1, \ldots, x_p\} = d(z_j C_j) \subset M$ and $y \in D_j$, $\frac{1}{2}\varepsilon < \|y\| < \varepsilon$. Fix $x \in \{x_1, \ldots, x_p\}$. The derivative $Df_3(x + y)$ has the form

$$
\begin{array}{ccccc}
N_x & T_x & V^1 & V^j & V^k \\
\end{array}
$$

$$
\begin{bmatrix}
A_N & 0 & & & \\
0 & 0 & 1 & & \\
& & & 1 & \\
& C & & D & \\
& & & & 1 \\
& & & & 1 \\
\end{bmatrix}
\begin{array}{c}
V^G \\
V^1 \\
\\
V^j \\
\\
V^k \\
\end{array}
$$

with zeros in the remaining places, where T_x, $N_x \subset V^G \oplus R$ are respectively the tangent and normal spaces to M at x, $A_N = A^G/N_x$, $C = (\partial h_j/\partial x)(x, y)$ and $D = (\partial h_j/\partial y)(x, y)$. Since $G(x + y) \subset \Omega_H$ is an isolated orbit in the zero set of a normal map f_3, we can compute $\deg_H(f_3, \Omega)$ by definition. Let w be the unit tangent vector to $G(x + y)$ at $(x + y)$ (as in 4.12(5)). We observe that

$$\text{sgn}(Df_3(x + y), w) = \text{sgn}(A^G(x), v) \cdot \text{sgn}(p_j) \cdot (-1).$$

Since $\text{sgn}(A^G(x), v) = -1$, we have $\deg_H(f_3, \Omega) = p_j$. The proof of Theorem 4.9 is complete.

5. S^1-degree in Banach spaces. In this section we will extend the results of the preceding sections to the case of G-equivariant maps between subsets of Banach G-spaces $(G = S^1)$. Our approach imitates closely the Leray–Schauder extension of the classical degree theory.

5.1. Definition. Let E be a Banach space and let $GL(E)$ denote the group of all its linear automorphisms. E is called a *Banach G-space* or a *representation* of G if there is given a homomorphism $\varrho\colon G \to GL(E)$ such that the map $\mu\colon G \times E \to E$ defined by $\mu(g, x) = \varrho(g)x$ is continuous. Note that, by the Banach–Steinhaus theorem, assuming the continuity of ϱ as a map from G into $GL(E)$ equipped with the strong topology, one obtains an equivalent definition of a Banach G-space.

Note also that since $G = S^1$ is compact, we may assume that ϱ satisfies the following condition:

(*) $\varrho(g)\colon E \to E$ is an isometry for every $g \in G$.

Indeed, if $\|\cdot\|$ is a norm in E then the formula

$$\|x\|_* = \int_G \|\varrho(g)x\| \, d\mu,$$

where μ denotes the normalized Haar measure on G, defines an equivalent norm such that $\|\varrho(g)x\|_* = \|x\|_*$ for all $g \in G$.

In what follows we assume that E is a Banach G-space with a norm $\|\cdot\|$ satisfying (*). We denote by $E \oplus R$ the direct product of E and R, i.e. $E \oplus R$ is a Banach G-space by $g(x, t) = (gx, t)$. We also let $Q\colon E \oplus R \to E$ denote the projection.

Recall that if X is a topological space then a continuous map $f\colon X \to E$ is called *compact* if the closure of $f(X)$ is a compact subset of E.

5.2. Definition. Let E be a Banach G-space and let X be a topological G-space. A continuous map $f\colon X \to E$ is called a *compact G-map* if it is G-equivariant and compact. We say that a continuous map $h\colon X \times [0, 1] \to E$ is a *compact G-homotopy* if it is compact and $h(gx, t) = gh(x, t)$ for all $x \in X$, $g \in G$ and $t \in [0, 1]$.

Our aim is to define a degree for S^1-equivariant maps of the form $Q + f\colon \bar{\Omega} \to E$, where Ω is an open bounded invariant subset of $E \oplus R$ and f is a compact G-map such that $(Q + f)(\partial\Omega) \subset E \setminus \{0\}$.

5.3. Theorem. *Let Ω run through the family of all bounded invariant subsets of $E \oplus R$ and $f\colon X \to E$ through compact G-maps such that $X \subset E \oplus R$ is invariant, $\bar{\Omega} \subset X$ and $(Q + f)(\partial\Omega) \subset E \setminus \{0\}$. Then there exists an $\mathscr{A}$-valued function $\mathrm{Deg}(Q + f, \Omega)$, called the S^1-degree, satisfying the following conditions:*

(a) *If $\deg_H(Q + f, \Omega) \neq 0$ then $(Q + f)^{-1}(0) \cap \Omega^H \neq \varnothing$.*

(b) *If $\Omega_0 \subset \Omega$ is open, invariant and $(Q + f)^{-1}(0) \cap \Omega \subset \Omega_0$ then $\mathrm{Deg}(Q + f, \Omega) = \mathrm{Deg}(Q + f, \Omega_0)$.*

(c) *If Ω_1, Ω_2 are open invariant subsets of Ω such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $(Q+f)^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$ then $\mathrm{Deg}(Q+f, \Omega) = \mathrm{Deg}(Q+f, \Omega_1) + \mathrm{Deg}(Q+f, \Omega_2)$.*

(d) *If h: $\bar\Omega \times [0, 1] \to E$ is a compact G-homotopy such that $x + h(x, t) \neq 0$ for all $(x, t) \in \partial\Omega \times [0, 1]$ then $\mathrm{Deg}(Q + h_0, \Omega) = \mathrm{Deg}(Q + h_1, \Omega)$.*

(e) *Suppose $E = E_1 \oplus E_2$ and $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 (resp. Ω_2) is an open bounded invariant subset of E_1 (resp. $E_2 \oplus R$). Let Q_2: $E_2 \oplus R \to E_2$ denote the projection. Suppose further that $0 \in \Omega_1$ and f_2: $\bar\Omega_2 \to E_2$ is a compact map such that $Q_2(y) + f_2(y) \neq 0$ for all $y \in \partial\Omega_2$. Define f: $\bar\Omega \to E$ by $f(x, y) = f_2(y)$, where $(x, y) \in \bar\Omega = \bar\Omega_1 \times \bar\Omega_2 \subset E_1 \oplus (E_2 \oplus R)$. Then $\mathrm{Deg}(Q+f, \Omega) = \mathrm{Deg}(Q_2 + f_2, \Omega_2)$.*

Recall that a G-map f: $X \to E$, where X is a G-space, is called *finite-dimensional* if there exists a finite-dimensional subspace $L \subset E$ such that $f(X) \subset L$.

The proof of Theorem 5.3 is based on the following approximation lemma.

5.4. Lemma. *Let X be a bounded closed invariant subset of a Banach G-space F. A G-map f: $X \to E$ is compact if and only if it is the limit of a uniformly convergent sequence of finite-dimensional G-maps.*

Proof. The condition is evidently sufficient. Our proof of necessity follows the classical lines (cf. Dugundji–Granas [9]). We will prove that for every $\varepsilon > 0$ there exists a G-equivariant finite-dimensional ε-approximation of f. For a given $\varepsilon > 0$ choose an ε-net $\{y_1, \ldots, y_k\}$ for $f(X)$. Since the set of elements of E whose orbits lie in a finite-dimensional subspace of E is dense in E (Mostow [18]), we may assume that $gy_i \subset V_i$, $i = 1, \ldots, k$, where V_i is a finite-dimensional linear G-subspace of E. Thus there exists a continuous map φ_ε: $X \to V = V_1 + \ldots + V_k$ such that $\|f(x) - \varphi_\varepsilon(x)\| < \varepsilon$ for all $x \in X$. Set

$$f_\varepsilon(x) = \int_G g\varphi_\varepsilon(g^{-1}x)\,d\mu.$$

Clearly f_ε is a G-map, $f_\varepsilon(X) \subset V$ and $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ for all $x \in X$. The proof is complete.

Suppose now that Ω is an open bounded invariant subset of $E \oplus R$, and f: $\bar\Omega \to E$ is a compact G-map such that $(Q + f)(\partial\Omega) \subset E \setminus \{0\}$. Since $(Q + f)(\partial\Omega)$ is closed in E, we have $\eta = \inf\{\|Q(x) + f(x)\|: x \in \partial\Omega\} > 0$. By 5.4 we can find a G-invariant finite-dimensional linear subspace $V \subset E$ and a G-map f_ε: $\bar\Omega \to E$ such that $f_\varepsilon(\bar\Omega) \subset V$ and $\|f(x) - f_\varepsilon(x)\| < \eta$ for all $x \in \bar\Omega$. Thus $Q + f_\varepsilon$: $(\bar\Omega, \partial\Omega) \to (E, E \setminus \{0\})$ and $(Q + f_\varepsilon)(\bar\Omega \cap V \oplus R) \subset V$. Therefore, by Section 3, $\mathrm{Deg}(Q + f_\varepsilon, \Omega \cap V \oplus R)$ is defined.

5.5. Definition. $\mathrm{Deg}(Q+f, \Omega) = \mathrm{Deg}(Q+f_\varepsilon, \Omega \cap V \oplus R)$, where V and f_ε are as above.

Using Lemma 5.4, the homotopy invariance and the contraction property of the S^1-degree in the finite-dimensional case (1.2(d) and (e)), one can check

that the definition of $\text{Deg}(Q+f, \Omega)$ does not depend on the choice of f_ε and V. The argument is the same as in the case of the Leray–Schauder degree.

Proof of Theorem 5.3. To prove (a) take a sequence ε_n such that $\lim \varepsilon_n = 0$ and $\varepsilon_n < \inf\{\|Q(x)+f(x)\|: x \in \partial\Omega\}$. By 5.4 there exists a sequence $\{f_n\}$, where $f_n: \bar\Omega \to V_n$ is a G-map, V_n is a finite-dimensional linear G-subspace of E and f_n is an ε_n-approximation of f. By 5.5, $\text{Deg}(Q+f_n, \Omega \cap V_n \oplus R)$ $= \text{Deg}(Q+f, \Omega) \neq 0$. Therefore, by 1.2(a), for every n there exists $x_n = (y_n, t) \in \Omega^H$ such that $y_n + f_n(y_n, t_n) = 0$. Since $f(\bar\Omega)$ is compact we may assume that $\{f(x_n)\}$ is convergent. Then

$$\|y_n - y_m\| = \|f_n(x_n) - f_m(x_m)\| \leqslant \|f(x_n) - f(x_m)\| + \varepsilon_n + \varepsilon_m$$

and thus $\{y_n\}$ converges. We may assume that $(y_n, t_n) \to (y_0, t_0)$. Therefore $y_0 + f(y_0, t_0) = 0$. This completes the proof of (a).

(b) and (c) are easy consequences of 1.2(b) and (c), respectively. To prove (d) suppose that $h: \bar\Omega \times [0, 1] \to E$ is a compact homotopy such that $Q(x) + h(x, t) = 0$ for all $(x, t) \in \partial\Omega \times [0, 1]$. Given $\varepsilon > 0$, using again 5.4, we find a finite-dimensional ε-approximation $h_\varepsilon: \bar\Omega \times [0, 1] \to V$ such that $Q(x) + h_\varepsilon(x, t) \neq 0$ for all $(x, t) \in \partial\Omega \times [0, 1]$. Then, by 5.5 and 1.2(d),

$$\text{Deg}(Q+h_0, \Omega) = \text{Deg}(Q+(h_\varepsilon)_0, \Omega \cap V \oplus R) = \text{Deg}(Q+(h_\varepsilon)_1, \Omega \cap V \oplus R)$$
$$= \text{Deg}(Q+h_1, \Omega).$$

Finally, (e) follows easily from Definition 5.5 and 1.2(e), which completes the proof of Theorem 5.3.

With a view to applications discussed in the last sections it is convenient to extend the definition of the S^1-degree to a slighty larger class of maps. Consider two Banach G-spaces E and F together with a linear G-isomorphism $T: E \to F$. Let $\Omega \subset E \oplus R$ be an open G-invariant bounded set. We will consider G-maps of the form $TQ+f: \bar\Omega \to F$ such that f is compact, where $Q: E \oplus R \to E$ denotes the projection.

5.6. Definition. Suppose that Ω is an open G-invariant bounded subset of $E \oplus R$ and $f: \bar\Omega \to F$ is a compact G-map such that $(TQ+f)(\partial\Omega) \subset F\backslash\{0\}$. Define the S^1-degree of $TQ+f$ by $\text{Deg}(TQ+f, \Omega, T) = \text{Deg}(Q+T^{-1}f, \Omega)$.

As a direct consequence of Definition 5.6 we obtain the following reformulation of Theorem 5.3.

5.7. Theorem. *Let Ω run through the family of all open bounded invariant subsets of $E \oplus R$ and $f: X \to F$ through compact G-maps such that $X \subset E \oplus R$ is invariant, $\bar\Omega \subset X$ and $(TQ+f)(\partial\Omega) \subset F\backslash\{0\}$. Then there exists an $\mathscr{A}$-valued function $\text{Deg}(TQ+f, \Omega, T)$, called the S^1-degree, satisfying the following conditions:*

(a) *If $\deg_H(TQ+f, \Omega, T) \neq 0$ then $(TQ+f)^{-1}(0) \cap \Omega^H \neq \varnothing$.*

(b) *If $\Omega_0 \subset \Omega$ is open, invariant and $(TQ+f)^{-1}(0) \cap \Omega \subset \Omega_0$ then $\text{Deg}(TQ+f, \Omega, T) = \text{Deg}(TQ+f, \Omega_0, T)$.*

(c)   If Ω_1, Ω_2 are open invariant subsets of Ω such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $(TQ+f)^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$ then $\mathrm{Deg}(TQ+f, \Omega, T) = \mathrm{Deg}(TQ+f, \Omega_1, T)$ $+ \mathrm{Deg}(TQ+f, \Omega_2, T)$.

(d)   If $h: \bar{\Omega} \times [0, 1] \to F$ is   a   compact   G-homotopy   such   that $TQ(x) + h(x, t) \neq 0$ for   all $(x, t) \in \partial\Omega \times [0, 1]$ then $\mathrm{Deg}(TQ+h_0, \Omega, T)$ $= \mathrm{Deg}(TQ+h_1, \Omega, T)$.

(e)   Suppose $E = E_1 \oplus E_2$. For $i = 1, 2$ set $F_i = T(E_i)$ and let $T_i: E_i \to F_i$ denote the restriction. Suppose also that $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 (resp. Ω_2) is an open bounded invariant subset of E_1 (resp. $E_2 \oplus R$). Assume further that $0 \in \Omega_1$ and $f_2: \bar{\Omega}_2 \to F_2$ is a compact map such that $TQ(y) + f_2(y) \neq 0$ for all $y \in \partial\Omega_2$. Define $f: \bar{\Omega} \to F$ by $f(x, y) = f_2(y)$, where $(x, y) \in \Omega = \Omega_1 \times \Omega_2 \subset E_1 \oplus (E_2 \oplus R)$. Then $\mathrm{Deg}(TQ+f, \Omega, T) = \mathrm{Deg}(TQ_2+f_2, \Omega_2, T_2)$, where $Q_2: E_2 \oplus R \to E_2$ denotes the projection.

6. Computations in Banach spaces.

Let E denote a real infinite-dimensional Banach space. We will use the following notations:

$L(E)$ = the Banach algebra of all linear and continuous operators from E into itself,

$I \in L(E)$ — the identity operator,

$K(E)$ = the ideal of compact operators in $L(E)$,

$\mathrm{GL}(E)$ = the group of all invertible operators in $L(E)$,

$L_C(E)$ = $\{A \in L(E): A = I + B, B \in K(E)\}$,

$\mathrm{GL}_C(E)$ = $\mathrm{GL}(E) \cap L_C(E)$.

If E_1, E_2 are closed subspaces of E such that $E = E_1 \oplus E_2$ (i.e. E_1 and E_2 are complementary) and $A \in L(E)$ then the decomposition $E = E_1 \oplus E_2$ determines a matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{ij}: E_j \to E_i$, $i, j = 1, 2$.

6.1. Definition. Let $B \in K(E)$. A real number μ is called a *characteristic value* of B if $\mathrm{Ker}(I - \mu B) \neq \{0\}$.

Note that μ is a characteristic value of B if and only if $\mu \neq 0$ and μ^{-1} is an eigenvalue of B. Therefore by the Riesz–Schauder theorem, any closed interval contains only a finite number of characteristic values of B. For a characteristic value μ we let $L(\mu) = \bigcup \mathrm{Ker}(I - \mu B)^k$; $L(\mu)$ is called the *generalized kernel* of $I - \mu B$. It is known that $\dim L(\mu) < \infty$.

Suppose now that $A = I - B \in \mathrm{GL}_C(E)$. Denote by $\{\mu_1 < \ldots < \mu_p\}$ the set of all characteristic values of B contained in $[0, 1]$. Let $L = L(\mu_1) + \ldots + L(\mu_p)$ ($L = \{0\}$ if B has no characteristic value in $[0, 1]$) and set $\mathrm{sgn}\, A = (-1)^d$, where $d = \dim L$. The following proposition collects standard properties of $\mathrm{sgn}\, A$.

6.2. PROPOSITION (cf. Kato [16]).

(a) $GL_C(E)$ *has two connected components.* A_1, $A_2 \in GL_C(E)$ *are in the same connected component if and only if* $\operatorname{sgn} A_1 = \operatorname{sgn} A_2$.

(b) *If* $E = E_1 \oplus E_2$ *and* $A \in GL_C(E)$ *is of the form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

then $\operatorname{sgn} A = \operatorname{sgn} A_{11} \operatorname{sgn} A_{22}$.

Consider now the Banach space $E \oplus R$ and let Q: $E \oplus R \to E$ denote the projection. Let $L(E \oplus R, E)$ denote the Banach space of linear operators from $E \oplus R$ into E. Let

$K(E \oplus R, E) = $ the subspace of compact operators in $L(E \oplus R, E)$,

$SL_C(E) = \{A \in L(E \oplus R, E)$: $A(E \oplus R) = E$ and $A = Q + B$, $B \in K(E \oplus R, E)\}$.

Suppose now that $A \in SL_C(E)$. Since A is a Fredholm operator of index 1 and A is surjective, $\dim \operatorname{Ker} A = 1$. Choose $v \in \operatorname{Ker} A$ such that $v \neq 0$ (note that this determines an orientation of $\operatorname{Ker} A$). Let ξ: $E \oplus R \to R$ be a linear functional such that $\xi(v) = 1$. Define $A^\sim$: $E \oplus R \to E \oplus R$ by $A^\sim(x) = (A(x), \xi(x))$. Clearly $A^\sim \in GL_C(E \oplus R)$. Set

$$\operatorname{sgn}(A, v) = \operatorname{sgn} A^\sim$$

It is easy to check that the above definition is correct and independent of the choice of ξ.

From 6.2 we obtain at once

6.3. PROPOSITION. *If* $A \in SL_C(E)$, $E = E_1 \oplus E_2$, $v \in \operatorname{Ker} A \subset E_2$, $v \neq 0$ *and, with respect to the decompositions* $E \oplus R = E_1 \oplus E_2 \oplus R$, $E = E_1 \oplus E_2$, A *has the representation*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad \begin{matrix} E_1 & & E_1 \\ \oplus & \to & \oplus \\ E_2 \oplus R & & E_2 \end{matrix}$$

then $\operatorname{sgn}(A, v) = \operatorname{sgn} A_{11} \operatorname{sgn}(A_{22}, v)$.

Consider now two Banach spaces E, F together with an isomorphism T: $F \to E$. Let Q: $F \oplus R \to E$ be defined by $Q(y, t) = T(y)$. Let $K(F, E)$ (resp. $K(F \oplus R, E)$) denote the space of all compact linear maps from F (resp. $F \oplus R$) into E. Set

$GL_C(T) = \{A$: $A = T + A_0$: $F \to E$ is an isomorphism, $A_0 \in K(F, E)\}$,

$SL_C(T) = \{A$: $A = Q + A_0$: $F \oplus R \to E$, $A_0 \in K(F \oplus R, E)$, $A(F \oplus R) = E\}$.

For $A \in GL_C(T)$ we set

$$\mathrm{sgn}_T A = \mathrm{sgn}(T^{-1} A).$$

Take $B \in SL_C(T)$; obviously $T^{-1}B \in SL_C(F)$. As in the case $T = I$ we have $\dim \mathrm{Ker}\, B = 1$. For $v \in \mathrm{Ker}\, B$, $v \neq 0$, we set

$$\mathrm{sgn}_T(B, v) = \mathrm{sgn}(T^{-1}B, v).$$

As a direct consequence of the above definitions, we have

6.4. PROPOSITION. *Suppose* $A \in GL_C(T)$, $B \in SL_C(T)$, $v \in \mathrm{Ker}\, B$, $v \neq 0$.

(a) $T \in GL_C(A)$ *and* $\mathrm{sgn}_T A = \mathrm{sgn}_A T$.

(b) *If* η: $[0, 1] \to GL_C(T)$ *is continuous, then* $\mathrm{sgn}_T \eta(0) = \mathrm{sgn}_T \eta(1)$.

(c) $\mathrm{sgn}_T(B, v) = \mathrm{sgn}_T A \, \mathrm{sgn}_A(B, v)$.

Suppose now $F = F_1 \oplus F_2$ and let $E_i = T(F_i)$, $i = 1, 2$. Clearly $E = E_1 \oplus E_2$. Let T_i: $F_i \to E_i$, $i = 1, 2$, denote the corresponding restrictions. Suppose further that $B \in SL_C(T)$ and, with respect to the decompositions $F \oplus R = F_1 \oplus F_2 \oplus R$, $E = E_1 \oplus E_2$.

$$B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

and $v \in \mathrm{Ker}\, B \subset F_2 \oplus R$. Then we have

6.5. PROPOSITION. $\mathrm{sgn}_T(B, v) = \mathrm{sgn}_{T_1}(B_{11}) \cdot \mathrm{sgn}_{T_2}(B_{22}, v)$.

Suppose now that we have isomorphisms of Banach spaces:

$$T: F \to E, \quad R: F_1 \to E_1, \quad \Phi: F \oplus R \to F_1 \oplus R, \quad \Psi: E \to E_1.$$

Suppose further that $RQ_1 \Phi = \Psi T Q$, where Q, Q_1 denote the corresponding projections.

6.6. PROPOSITION. *Assume that* $A \in SL_C(T)$ *and* $x \in \mathrm{Ker}\, A$, $x \neq 0$. *Then* $\Psi A \Phi^{-1} \in SL_C(R)$ *and* $\mathrm{sgn}_R(\Psi A \Phi^{-1}, \Phi(x)) = \mathrm{sgn}_T(A, x)$.

Now we are ready to formulate an infinite-dimensional generalization of Theorem 4.1. As above, we consider two Banach G-spaces E, F together with a G-equivariant isomorphism T: $F \to E$. Suppose that $\Omega \subset F \oplus R$ is bounded, open, invariant and f: $\bar{\Omega} \to E$ is a C^1-map of the form $f(x, p) = Tx + \Phi(x, p)$, where Φ is a compact G-map. Suppose further that there exists $a \in \Omega$ such that $G_a = Z_k \neq S^1$, $f^{-1}(0) = Ga \subset \Omega$ and $Df(a)$: $F \oplus R \to E$ is surjective. Let $K = Z_k$ and let f^K: $\bar{\Omega}^K \to E^K$ denote the restriction of f. Clearly $Df^K(a)$: $F^K \oplus R \to E^K$ is also surjective. Let v denote the tangent vector to the orbit Ga at a. Notice that $v \in \mathrm{Ker}\, Df^K(a)$. Finally, let $R = T^K$: $F^K \to E^K$ denote the restriction of T.

6.7. THEOREM. *With the above assumptions we have:*

(a) *If* $\mathrm{sgn}_T(Df(a), v) = \mathrm{sgn}_R(Df^K(a), v)$ *then*

$$\deg_H(f, \Omega, T) = \begin{cases} \mathrm{sgn}_R(Df^K(a), v) & \text{if } H = K = Z_k, \\ 0 & \text{if } H \neq K. \end{cases}$$

(b) *If* $\mathrm{sgn}_T(Df(a), v) = -\mathrm{sgn}_R(Df^K(a), v)$ *then* k *is even and, letting* $k = 2m$, *we have*

$$\deg_H(f, \Omega, T) = \begin{cases} \mathrm{sgn}_R(Df^K(a), v) & \text{if } H = K = Z_k, \\ -\mathrm{sgn}_R(Df^K(a), v) & \text{if } H = Z_m, \\ 0 & \text{if } H \neq Z_k \text{ and } H \neq Z_m. \end{cases}$$

We shall use a linear slice in the proof of 6.7.

6.8. THEOREM. *Let E be a Banach representation of $G = S^1$, x_0 a point of E, and K the isotropy group of x_0. Then there exist a K-equivariant linear subspace $N \subset E$ (of codimension equal to the dimension of G/K), a ball $B_\varepsilon(N)$ in N with center at x_0 and an open neighbourhood U of Gx_0 such that the map $\eta: G \times_K B_\varepsilon(N) \to U$ given by $\eta([g, v]) = gv$ is a G-homeomorphism.*

(See Jodel–Marzantowicz [15] for the proof of this theorem for any compact Lie group.)

Proof of Theorem 6.7. First we observe that Ω can be replaced by the open tube $U \approx G \times_K B_\varepsilon(N)$ given by 6.8. For a given G-map Ψ from U into some Y we denote by Ψ_N its restriction to $U_N = \{e\} \times_K B_\varepsilon(N) \subset N$. Obviously Ψ_N is K-equivariant. As in 4.2 the assignment $\Psi \mapsto \Psi_N$ gives a one-to-one correspondence between G-maps from $(\bar{U}, \partial U)$ into $(Y, Y \backslash \{0\})$ and K-maps from $(\bar{U}_N, \partial U_N)$ into $(Y, Y \backslash \{0\})$ for any G-representation Y. The inverse assignment is $\Psi_N \mapsto G \times_K \Psi_N$, where $G \times_K \Psi_N([g, v]) = g\Psi_N(v)$. The same assignment gives a one-to-one correspondence between equivariant homotopy classes too.

In order to derive $\deg_{S^1}(f, U, T)$ we have to compute $\deg_{S^1}(T^{-1} \circ f, U)$. Note that $T^{-1} \circ f = Q + T^{-1} \circ \Phi$, where $Q: F \oplus R \to F$ is the projection. Let $\Psi = T^{-1} \circ f$; then $D\Psi(x_0) = Q + T^{-1} \circ D\Phi(x_0)$ is K-equivariant, so the formula $\Psi^\wedge(x) = D\Psi(x_0)(x - x_0)$ defines a K-equivariant map. Note also that since $Gx_0 = M = f^{-1}(0) \cap U$, M is a smooth manifold of dimension 1 and there exists a unique tangent vector x_0'. Moreover, N is a normal space at x_0', i.e. $N \oplus \mathrm{span}(x_0') = F \oplus R$. From a Rouché principle argument applied to the isomorphism $\Psi_N^\wedge$ it follows that there exists an open K-invariant subset $U_\varepsilon \subset U_N$ such that Ψ_N and $\Psi_N^\wedge$ are K-homotopic as maps of pairs $(U_\varepsilon, \partial U_\varepsilon) \to (F, F \backslash \{0\})$. As a consequence, Ψ is G-homotopic to $G \times_K \Psi_N^\wedge$ on GU_ε.

Next, we G-homotopically replace $G \times_K \Psi_N^\wedge$ by a G-map which is a finite-dimensional perturbation of Q.

6.9. Lemma. *There exists a K-homotopy* η: $[0, 1] \times N \to F$ *such that*:

(a) *for every* t *the map* $\eta(t, \cdot)$ *is K-equivariant, compact and linear*,

(b) *for every* t *the map* $Q + \eta(t, \cdot)$ *is an isomorphism*,

(c) $\eta(0, \cdot) = T^{-1} Df_N(x_0)$,

(d) $\eta(1, \cdot)(N) \subset V$, *where* V *is a G-invariant finite-dimensional subspace of* F.

Proof. The lemma can be proved by a modification of nonequivariant arguments of the proof of the fact that a compact linear perturbation of identity can be deformed through isomorphisms into a finite-dimensional perturbation of identity.

After this preparation we are able to complete the proof of Theorem 6.7. Denote GU_ε by Ω_1. Observe first that we can assume $M = \Psi^{-1}(0) \cap \Omega_1$ to be a subset of a finite-dimensional subrepresentation $V_1 \subset F \oplus R$, where $V_1 = W \oplus R$. (Since G is abelian, $(F \oplus R)^K$ is a subrepresentation of $F \oplus R$.) Let $y \in (F \oplus R)^K$ be a vector having a finite-dimensional orbit which is close enough to x_0. Since U_N is a slice there exists $g \in G$ such that $y_0 = gy \in U_N$. It is easy to check that the map $\Psi''(x) = G \times_K \Psi_N(x + x_0 - y_0)$ is G-homotopic to Ψ, and $\Psi''^{-1}(0) \cap \Omega_1 = Gy_0$. Let $V_2 \subset F$ be the subrepresentation given by 6.9, and $V = V_1 + V_2$. Let $\Psi^\sim = Q|_N + \eta(1, \cdot)$ where η is given by 6.9. From 6.9 and 6.4(b) it follows that

$$\mathrm{sgn}\big(D\Psi(x_0), x_0'\big) = \mathrm{sgn}\big(D(G \times_K \Psi^\sim)(x_0), x_0'\big),$$

$$\mathrm{sgn}\big(D\Psi^K(x_0), x_0'\big) = \mathrm{sgn}\big(D(G \times_K \Psi^\sim)^K(x_0), x_0'\big),$$

since $x_0' \in \mathrm{Ker} D\Psi(x_0) = \mathrm{Ker} D(G \times_K \Psi^\sim)(x_0) \subset (F \oplus R)^K$. By the properties of the S^1-degree,

$$\mathrm{Deg}(\Psi, \Omega_1) = \mathrm{Deg}(G \times_K \Psi^\sim, \Omega_1).$$

Since $G \times_K \Psi^\sim = Q + \Phi^\sim$ with $\Phi^\sim(\Omega_1) \subset V$, from Definition 5.5 it follows that

$$\mathrm{Deg}(\varphi|_V, \Omega_V) = \mathrm{Deg}(G \times_K \Psi^\sim, \Omega_1),$$

where $\Omega_V = \Omega_1 \cap V$ and φ is the restriction of $G \times_K \Psi^\sim$ to Ω_V. Let next F_1 be a subspace of F such that $F_1 \oplus V = F$. With respect to this decomposition $D(G \times_K \Psi^\sim)(x_0)$ has the form

$$D(G \times_K \Psi^\sim)(x_0) = \begin{bmatrix} D\varphi(x_0) & S \\ 0 & \mathrm{id} \end{bmatrix}.$$

From 6.3 it follows that

$$\mathrm{sgn}\big(\varphi'(x_0), x_0'\big) = \mathrm{sgn}\big(D(G \times_K \Psi^\sim)(x_0), x_0'\big),$$

$$\mathrm{sgn}\big(\varphi'(x_0)^K, x_0'\big) = \mathrm{sgn}\big(D(G \times_K \Psi^\sim)^K(x_0), x_0'\big).$$

Now Theorem 4.1 applied to $(\varphi|_V, \Omega_V)$ gives the conclusion of Theorem 6.7.

7. S^1-degree and periodic solutions of ODE. Throughout this section we shall assume that $\varphi\colon R^n \to R^n$ is a C^∞ map. Consider the autonomous differential equation

$$(*) \qquad\qquad y'(t) = \varphi\big(y(t)\big).$$

For each $v \in R^n$ there exists a unique solution of $(*)$, $y(t) = \eta(v, t)$, satisfying $y(0) = \eta(v, 0) = v$ and defined in an open t-interval I_v containing 0. We will denote by $\Omega \subset R^n \times R$ the union of the sets $\{v\} \times I_v$, where $v \in R^n$. Clearly Ω is open and $\eta\colon \Omega \to R^n$ is a C^∞ mapping which satisfies, wherever defined, the group property

$$\eta\big(\eta(v, s), t\big) = \eta(v, s+t).$$

Set

$$\Omega_+ = \{(v, t) \in \Omega\colon t > 0\}, \qquad \Sigma = \{(v, p) \in \Omega_+\colon \eta(v, p) = v\}.$$

Recall that $v \in R^n$ is called a *stationary point* of $(*)$ if $\varphi(v) = 0$. We let

$$\text{Sing}(\varphi) = \{v \in R^n\colon \varphi(v) = 0\};$$

note that $\text{Sing}(\varphi) \times (0, \infty) \subset \Sigma$.

Define the *nonlinear Fuller map* associated with the equation $(*)$, $\pi\colon \Omega \to R^n$, by $\pi(w, \lambda) = w - \eta(w, \lambda)$. We let $P\colon R^n \oplus R \to R^n$ denote the linear map $D\pi(v, p)$ and call it the *linear Fuller map* associated with $(*)$ at (v, p).

We say that $v \in R^n$ is a *p-periodic point with respect to* $(*)$ if $(v, p) \in \Sigma$. If v is a nonstationary p-periodic point then there is the least $q > 0$ such that $\eta(v, q) = v$ and $p = mq$. We call p the *period*, q the *least period*, and $m = q^{-1}p$ the *multiplicity* of (v, p).

Suppose now that $v \in R^n$ is nonstationary and p-periodic. Let P denote the linear Fuller map of $(*)$ at (v, p) and let $A\colon R^n \to R^n$ denote the linear map $D\eta_p(v)$, where $\eta_p(w) = \eta(w, p)$. Since η_p is a local diffeomorphism, A is a linear automorphism of R^n. Moreover, $A\big(\varphi(v)\big) = \varphi(v)$ and thus 1 belongs to the spectrum $\sigma(A)$ of A. We say that v is *elementary* if the linear subspace of R^n corresponding to the eigenvalue 1 is one-dimensional and A has no eigenvalue of absolute value 1 different from 1 (note that v is elementary if the periodic orbit which starts from v is hyperbolic). Let L denote the one-dimensional linear subspace spanned by $\varphi(v)$ and N the linear subspace corresponding to $\sigma(A)\backslash\{1\}$. Clearly $R^n = N \oplus L$. Note also that $P(N) = N$ and $P(0, 1) = -\varphi(v)$. The above shows that if v is elementary then $P(R^n \oplus R) = R^n$.

Together with $(*)$ we will consider the equation

$$(**) \qquad\qquad x'(t) = p\varphi\big(x(t)\big).$$

As a direct consequence of the introduced definitions we have:

7.1. Remark. Let $(v, p) \in R^n \times (0, \infty)$. Then $(v, p) \in \Omega_+$ if and only if equation $(**)$ has a solution $x\colon [0, 1] \to R^n$ satisfying the initial condition $x(0) = v$, and the solution is given by $x(t) = \eta(v, pt)$.

We will use the following notations:

$C^1 = C^1([0, 1], R^n) =$ the Banach space of C^1-functions $x: [0, 1] \to R^n$ with the standard norm,

$C = C([0, 1], R^n) =$ the Banach space of continuous functions $x: [0, 1] \to R^n$ with the standard norm,

$E^1 = \{x \in C^1: x(0) = x(1), x'(0) = x'(1)\}$,

$E = \{x \in C: x(0) = x(1)\}$,

$S: C^1 \to C \oplus R^n, \quad S(x) = (x' + x, x(0) - x(1))$,

$T: E^1 \to E$, the restriction of S.

Clearly, both S and T are isomorphisms of Banach spaces.

For $x: [0, 1] \to R^n$ and $\varrho = e^{2\pi i \theta} \in G = S^1 = \{z \in C: |z| = 1\}$ define

$$(\varrho x)(t) = \begin{cases} x(t+\theta) & \text{for } t+\theta \leq 1, \\ x(t+\theta-1) & \text{for } t+\theta > 1. \end{cases}$$

With the above-defined S^1 action E^1 and E are S^1-Banach spaces and T is an equivariant isomorphism. Define $f: E^1 \oplus R \to E$ by $f(x, \lambda) = x' - \lambda \varphi(x(\cdot))$. The following is evident:

7.2. Remark. f is equivariant and $f - T$ is completely continuous.

The following theorem states the main result of this section:

7.3. Theorem. *Suppose that* $v \in R^n$ *is an elementary p-periodic point and* (v, p) *is of multiplicity* m *with respect to equation* (*) *and define* $\xi \in E^1$ *by* $\xi(t) = \eta(v, pt)$. *Then there exists an open bounded subset* $\mathscr{U} \subset E^1 \times (0, \infty)$ *such that* $f^{-1}(0) \cap \mathscr{U} = G(\xi, p)$ *and* $f(x, \lambda) \neq 0$ *for all* $(x, \lambda) \in \partial\mathscr{U}$ *and we have*

(a) *if* $\mathrm{sgn}(P, \varphi(v)) = \mathrm{sgn}(P_m, \varphi(v))$ *then*

$$\deg_H(f, \mathscr{U}, T) = \begin{cases} \mathrm{sgn}(P, \varphi(v)) & \text{if } H = Z_m, \\ 0 & \text{if } H \neq Z_m, \end{cases}$$

(b) *if* $\mathrm{sgn}(P, \varphi(v)) = -\mathrm{sgn}(P_m, \varphi(v))$ *then* m *is even and letting* $m = 2\mu$ *we have*

$$\deg_H(f, \mathscr{U}, T) = \begin{cases} -\mathrm{sgn}(P, \varphi(v)) & \text{if } H = Z_m, \\ \mathrm{sgn}(P, \varphi(v)) & \text{if } H = Z_\mu, \\ 0 & \text{if } H \neq Z_m, H \neq Z_\mu, \end{cases}$$

where P_m *denotes the linear Fuller map of* (*) *at* (v, q) *and* P *denotes the linear Fuller map of* (*) *at* (v, p).

We postpone the proof of Theorem 7.3 to the end of this section. First we will prove the following result which plays a crucial role in our argument.

7.4. PROPOSITION. *Suppose that $v \in R^n$ is an elementary p-periodic point with respect to equation (*). Define $\xi \in E^1$ by $\xi(t) = \eta(v, pt)$. Then $Df(\xi, p)$ is surjective, $\xi' \in \operatorname{Ker} Df(\xi, p)$ and*

$$\operatorname{sgn}_T(Df(\xi, p), \xi') = \operatorname{sgn}(P, \varphi(v)),$$

where P denotes the linear Fuller map of (*) at (v, p).

Before proving 7.4 we have to prove auxiliary results. We use the following notation:

$$f^\wedge \colon C^1 \oplus R \to C, \quad f^\wedge(x, \lambda) = x' - \lambda \varphi(x(\cdot)),$$

$$f^\sim \colon C^1 \oplus R \to C \oplus R^n, \quad f^\sim(x, \lambda) = (f^\wedge(x, \lambda), x(0) - x(1)),$$

$$\Phi = Df(\xi, p), \quad \Phi^\wedge = Df^\wedge(\xi, p), \quad \Phi^\sim = Df^\sim(\xi, p),$$

$$\psi \colon \Omega \to C^1 \oplus R, \quad \psi(w, \lambda) = (x, \lambda), \text{ where } x(t) = \eta(w, t\lambda),$$

$$\Lambda^\sim \colon C^1 \oplus R \to R^n, \quad \Lambda^\sim(x, p) = x(0) - x(1).$$

It is easy to check that $\Phi^\wedge(x, \lambda) = x'(\cdot) - pD\varphi(\xi(\cdot))x(\cdot) - \lambda\varphi(\xi(\cdot))$. Note that $f^\wedge(\psi(w, \lambda)) = 0$ for all $(w, \lambda) \in \Omega$ and thus $D\psi(v, p)(R^n \oplus R) \subset \operatorname{Ker} \Phi^\wedge$. Let

$$\Lambda \colon \operatorname{Ker} \Phi^\wedge \to R^n$$

denote the restriction of $\Lambda^\sim$ and

$$\gamma \colon C^1 \cap \operatorname{Ker} \Phi^\wedge \to R^n, \quad \gamma(x) = x(0).$$

Clearly γ is an isomorphism.

7.5. LEMMA. *With the above notations $\Lambda(\operatorname{Ker} \Phi^\wedge) = R^n$ and $\operatorname{sgn}(P, \varphi(v)) = \operatorname{sgn}_\gamma(\Lambda, \xi')$.*

Proof. Clearly $\pi = \Lambda^\sim \psi$ and thus $P = \Lambda D\psi(v, p)$. Since P is surjective, Λ is also surjective. Let $B \colon R^n \to \operatorname{Ker} \Phi^\wedge$ denote the restriction of $D\psi(v, p)$. Since $\gamma(\psi(w, p)) = w$, B is injective. Moreover, $B(R^n) = C^1 \cap \operatorname{Ker} \Phi^\wedge$. Thus, since $D\psi(v, p)(0, 1) \notin C^1 \cap \operatorname{Ker} \Phi^\wedge$ and $\dim \operatorname{Ker} \Phi^\wedge = n + 1$, $D\psi(v, p)$ maps $R^n \oplus R$ isomorphically onto $\operatorname{Ker} \Phi^\wedge$. Note that $P = \Lambda D\psi(v, p)$ and γB is the identity in R^n and $B(\varphi(v)) = p^{-1} \xi'$. Therefore 7.5 follows from 6.6.

7.6. LEMMA. *With the above notations we have:*

(a) $\Phi^\sim$ *is surjective and $\operatorname{Ker} \Phi^\sim$ is the one-dimensional subspace of $C^1 \oplus R$ spanned by ξ'.*

(b) $\operatorname{sgn}_S(\Phi^\sim, \xi') = \operatorname{sgn}(P, \varphi(v))$.

Proof. First observe that $\Phi^\sim(x, \lambda) = (\Phi^\wedge(x, \lambda), \Lambda^\sim(x, \lambda))$. Since $\Lambda^\sim$ maps $\operatorname{Ker} \Phi^\wedge$ onto R^n and $\Phi^\wedge$ is surjective, it follows that $\Phi^\sim$ is surjective and the proof of (a) is complete.

To prove (b) note first that $\operatorname{Ker}\Phi^\sim = \operatorname{Ker}\Phi^\wedge \cap \operatorname{Ker}A^\sim$ Define $\Psi\colon C^1 \to C\oplus R^n$ by $\Psi(x) = (\Phi^\wedge(x, 0), x(0))$. Obviously Ψ is an isomorphism. Let $F = \Psi^{-1}(C)$, $F_0 = \Psi^{-1}(R^n)$. We have the direct sum decomposition

$$\Psi = \begin{bmatrix} \Gamma & 0 \\ 0 & \gamma \end{bmatrix} \quad \begin{matrix} F \\ \oplus \\ F_0 \end{matrix} \to \begin{matrix} C \\ \oplus, \\ R^n \end{matrix}$$

where $\Gamma\colon F \to C$ denotes the restriction of Ψ and $\gamma(x) = x(0)$.

Recall that $A\colon \operatorname{Ker}\Phi^\wedge \to R^n$ denotes the restriction of $A^\sim$ Since $F_0 \subset \operatorname{Ker}\Phi^\wedge$, $C^1 \oplus R = F \oplus \operatorname{Ker}\Phi^\wedge$ We have

$$\Phi^\sim = \begin{bmatrix} \Gamma & 0 \\ -\gamma_1 & A \end{bmatrix} \quad \begin{matrix} F \\ \oplus \\ \operatorname{Ker}\Phi^\wedge \end{matrix} \to \begin{matrix} C \\ \oplus, \\ R^n \end{matrix}$$

where $\gamma_1(x) = x(1)$.

Using 6.5 we have $\operatorname{sgn}_\Psi(\Phi^\sim, \xi') = \operatorname{sgn}_\Gamma(\Gamma)\cdot\operatorname{sgn}_\gamma(A, \xi')$. If we let $S_\tau(x) = (x'+x, x(0)-\tau x(1))$, then for each $\tau \in [0, 1]$, $S_\tau \in \operatorname{GL}_C(S)$. Thus, by 6.4(b), $\operatorname{sgn}_S S_0 = 1$. Now define $\eta\colon [0, 1] \to \operatorname{GL}_C(S)$ by

$$\eta(\tau)(x) = ((1-\tau)\Phi^\wedge(x, 0) + \tau(x'+x), x(0)).$$

Since $\eta(0) = \Psi$ and $\eta(1) = S_0$ we have, using again 6.4(b), $\operatorname{sgn}_S \Psi = 1$. Therefore, by 6.4(c),

$$\operatorname{sgn}_\Psi(\Phi^\sim, \xi') = \operatorname{sgn}_S(\Phi^\sim, \xi')$$

and the proof is finished.

7.7. LEMMA. $\operatorname{sgn}_S(\Phi^\sim, \xi') = \operatorname{sgn}_T(\Phi, \xi')$.

Proof. Let $R\colon S^{-1}(C) \to C$ be the restriction of S, and $\Phi_1\colon S^{-1}(C) \oplus R \to C$ the restriction of $\Phi^\sim$ Consider the direct sum decompositions

$$\Phi^\sim = \begin{bmatrix} \Phi_1 & \Phi_0 \\ 0 & S_0 \end{bmatrix} \quad \begin{matrix} S^{-1}(C) \oplus R \\ \oplus \\ S^{-1}(R^n) \end{matrix} \to \begin{matrix} C \\ \oplus, \\ R^n \end{matrix}$$

where S_0 denotes the restriction of S, and

$$\Phi_1 = \begin{bmatrix} \Phi & \Phi_2 \\ 0 & R_0 \end{bmatrix} \quad \begin{matrix} R^{-1}(E) \oplus R \\ \oplus \\ R^{-1}(E_0) \end{matrix} \to \begin{matrix} E \\ \oplus, \\ E_0 \end{matrix}$$

where $E_0 = \{x \in C\colon x(t) = tw, w \in R^n\}$ and R_0 denotes the restriction of R. We have $C = E \oplus E_0$. Applying 6.5 we obtain the desired result.

Proof of Proposition 7.4. The equality $\text{sgn}_T(Df(\xi, p), \xi')$ $= \text{sgn}(P, \varphi(v))$ follows at once from 7.6 and 7.7.

The last step in our proof of 7.3 is the following generalization of 7.4.

7.8. PROPOSITION. *Suppose that $v \in R^n$ is an elementary p-periodic point of multiplicity m with respect to equation (∗). Assume $m = kv$ and let $K = Z_k$. Let $f^K \colon (E^1)^K \oplus R \to E^K$ denote the restriction of f. Define $\xi \in E^1$ by $\xi(t) = \eta(v, pt)$. Then $Df^K(\xi, p)$ is surjective, $\xi \in \text{Ker} Df^K(\xi, p)$ and*

$$\text{sgn}_T(Df^K(\xi, p), \xi') = \text{sgn}(P_k, \varphi(v)),$$

where P_k denotes the linear Fuller map of (∗) at $(v, k^{-1}p)$.

Proof. Consider the following isomorphisms of Banach spaces:

$\Gamma \colon (E^1)^K \oplus R \to E^1 \oplus R$, $\Gamma(x, \lambda) = (y, \lambda)$, where $y(t) = kx(k^{-1}t)$,

$\Psi \colon E^K \to E$, $\Psi(x) = y$, where $y(t) = x(k^{-1}t)$,

$R \colon E^1 \to E$, $R(x) = x' + k^{-1}x$.

Together with (∗) we consider the equation

(α_*) $y'(t) = \psi(y(t))$,

where $\psi \colon R^n \to R^n$ is defined by $\psi(w) = \varphi(k^{-1}w)$.

Recall that we have assumed that, with respect to equation (∗), $v \in R^n$ is a nonstationary elementary p-periodic point and (v, p) is of multiplicity m. Furthermore, we have denoted by $\xi \in C^1$ the solution of the equation $x'(t) = p\varphi(x(t))$ satisfying the initial condition $x(0) = v$. Define

$$F \colon E^1 \oplus R \to E, \quad F(y, \lambda) = y' - \lambda\psi(y(\cdot)),$$

and let $(\zeta, p) = \Gamma(\xi, p)$. Since $\Psi f^K = F\Gamma$, we have $\Psi Df^K(\xi, p) = DF(\zeta, p)\Gamma$

Furthermore, $\Gamma(\xi') = k\zeta'$. Therefore, applying 6.6, we have

$$\text{sgn}_T(Df^K(\xi, p), \xi') = \text{sgn}_R(Df(\zeta, p), \zeta').$$

Set $\omega(\tau) = (1 - \tau)T + \tau R$; clearly $\omega(\tau)$ is an isomorphism for all $\tau \in [0, 1]$; hence, by 6.4, $\text{sgn}_T R = 1$ and thus $\text{sgn}_R(DF(\zeta, p), \zeta') = \text{sgn}_T(DF(\zeta, p), \zeta')$. Thus

(i) $\text{sgn}_T(Df^K(\xi, p), \xi') = \text{sgn}_T(DF(\zeta, p), \zeta')$.

Note that $\pi^\sim$, the nonlinear Fuller map for (α_*), is given by

(ii) $\pi^\sim(w, \lambda) = w - k\eta(k^{-1}w, k^{-1}\lambda)$.

Since v is an elementary p-periodic point for (∗), kv is one for (α_*). Let $P^\sim$ denote the linear Fuller map for (α_*) at (kv, p). From (ii) it follows at once that $\text{sgn}(P^\sim, \psi(kv)) = \text{sgn}(P_k, \varphi(v))$. Therefore using (i) and 7.4 we obtain $\text{sgn}_T(Df^K(\xi, p), \xi') = \text{sgn}(P_k, \varphi(v))$.

Proof of Theorem 7.3. The theorem follows at once from Proposition 7.8 and Theorem 6.7.

8. S^1-degree and the Fuller index. As in Section 7 we consider the autonomous differential equation

$$(*) \qquad\qquad y'(t) = \varphi\big(y(t)\big),$$

where $\varphi\colon R^n \to R^n$ is a C^∞ map. In what follows we keep the notations introduced at the beginning of the preceding section.

Suppose U is an open subset of $R^n \oplus R$ such that $\bar{U} \subset \Omega_+$. We say that U is *admissible for* φ if $\partial U \cap \Sigma = \varnothing$ (i.e. there is no periodic point on the boundary of U). Recall that if U is admissible for φ then there is defined a rational number $i_F(\varphi, U)$, the Fuller index of φ with respect to U (cf. Fuller [11], Chow-Mallet-Paret-Yorke [5]). Assuming that U is admissible for φ let $C = U \cap \Sigma$; clearly C is compact. Define $\Phi\colon C \to E^1 \oplus R$ by

$$\Phi(v, p) = (x, p), \quad \text{where } x(t) = \eta(v, tp).$$

Since Φ is continuous, $\Phi(C)$ is compact. Moreover, $\Phi(C) \subset f^{-1}(0)$ and $f^{-1}(0)\backslash\Phi(C)$ is closed in $E^1 \oplus R$. Therefore there exists an open bounded subset $\mathscr{U} \subset E^1 \oplus R$ such that $f^{-1}(0) \cap \mathscr{U} = \Phi(C)$ and $f^{-1}(0) \cap \partial\mathscr{U} = \varnothing$. Thus $\mathrm{Deg}(f, \mathscr{U}, T)$ is defined. For $m \in N$ set

$$d_m(\varphi, U) = \deg_H(f, \mathscr{U}, T), \quad \text{where } H = Z_m.$$

Now we are able to state the principal result of this section.

8.1. Theorem. *If $U \subset \Omega_+$ is admissible for φ then*

$$i_F(\varphi, U) = \sum_{m \in N} m^{-1} d_m(\varphi, U).$$

Proof. First, consider a special case: there is an elementary periodic point $(v, p) \in U$ such that

$$C = \{\eta(v, t); \; 0 \leqslant t \leqslant p\}.$$

In other words, we assume that C is one periodic orbit. Let m be the multiplicity of (v, p) and $m \cdot q = p$. Let (as in 7.3) P_m denote the linear Fuller map of $(*)$ at (v, q) and if m is even, $m = 2\mu$, let P denote the linear Fuller map of $(*)$ at (v, p). It follows from 7.3 that there are two possibilities:

(a) $\mathrm{sgn}\big(P, \varphi(v)\big) = \mathrm{sgn}\big(P_m, \varphi(v)\big)$;

in this case $d_k(\varphi, U) = 0$ for $k \neq m$, $d_k(\varphi, U) = \mathrm{sgn}\big(P, \varphi(v)\big)$ for $k = m$ and because from the definition of the Fuller index we have $i_F(\varphi, U) = m^{-1}\mathrm{sgn}\big(P, \varphi(v)\big)$, the conclusion follows at once;

(b) m is even and $\mathrm{sgn}\big(P, \varphi(v)\big) = -\mathrm{sgn}\big(P_m, \varphi(v)\big)$;

in this case

$$i_F(\varphi, U) = m^{-1}\operatorname{sgn}(P, \varphi(v)) = (\mu^{-1} - m^{-1})\operatorname{sgn}(P, \varphi(v))$$

$$= \mu^{-1}d_\mu(\varphi, U) + m^{-1}d_m(\varphi, U)$$

and the conclusion follows.

Thus we have proved our theorem in the case where φ has one elementary periodic orbit in U. Therefore, using the additivity of the S^1-degree and the definition of the Fuller index completes the proof.

Note that from Theorem 8.1 we see at once that the Fuller index may be defined in terms of the S^1-degree. The crucial step made by Fuller in [11] was to prove the homotopy invariance of $i_F(\varphi, U)$; using the S^1-degree one can obtain an independent proof of this property.

References

[1] J. F. Adams, *Lectures on Lie Groups*, Benjamin, New York 1969.

[2] J. Amann and S. Weiss, *On the uniqueness of the topological degree*, Math. Z. 130 (1973), 39–54.

[3] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York 1972.

[4] N.-S. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Springer, 1982.

[5] N.-S. Chow, J. Mallet-Paret and J. A. Yorke, *Global Hopf bifurcation from a multiple eigenvalue*, Nonlinear Anal. 2 (1978), 753–763.

[6] E. N. Dancer, *A new degree for S^1-invariant gradient mappings and applications*, preprint, 1984.

[7] T. tom Dieck, *Transformation Groups and Representation Theory*, Lecture Notes in Math. 766, Springer, 1979.

[8] A. Dold, *Fixed point index and fixed point theorem for euclidean neighbourhood retracts*, Topology (1965), 1–8.

[9] J. Dugundji and A. Granas, *Fixed Point Theory*, Monografie Mat., Polish Scientific Publishers, Warszawa 1982.

[10] G. Dylawerski, *A relation between the Dancer S^1-degree and the generalized S^1-degree*, to appear.

[11] F. B. Fuller, *An index of fixed point type for periodic orbits*, Amer. J. Math. 89 (1967), 133–148.

[12] K. Gęba, I. Massabó and A. Vignoli, *Generalized topological degree and bifurcation*, in: *Nonlinear Functional Analysis and its Aplications*, S. P. Singh (ed.), Reidel, 1986, 55–73.

[13] A. Granas, *The Leray–Schauder index and the fixed point theory for arbitrary ANR'S*, Bull. Soc. Math. France 100 (8) (1972), 209–228.

[14] J. Ize, I. Massabó and A. Vignoli, *Global results on continuation and bifurcation for equivariant maps*, in: *Nonlinear Functional Analysis and its Aplications*, S. P. Singh (ed.), Reidel, 1986, 75–111.

[15] J. Jodel and W. Marzantowicz, *The existence of the linear slice for an orbit of a representation in a Banach space*, notes.

[16] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1966.

[17] M. A. Krasnosel'skiĭ and P. P. Zabreĭko, *Geometric Methods of Nonlinear Analysis* (in Russian), Nauka, Moscow 1975.

[18] G. D. Mostow, *Cohomology of topological groups and solv-manifolds*, Ann. of Math. 73 (1961), 20–48.

[19] R. L. Rubinsztein, *On the equivariant homotopy of spheres*, Dissertationes Math. 134 (1976).

[20] J.-P. Serre, *Représentations linéaires des groupes finis*, Hermann, Paris 1967.

[21] S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa 17 (1963), 97–116.

INSTYTUT MATEMATYKI, UNIWERSYTET GDAŃSKI
INSTITUTE OF MATHEMATICS, UNIVERSITY OF GDAŃSK
Wita Stwosza 57, 80-952 GDAŃSK, POLAND