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Certain subclasses of the class of typically real functions

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Abstract. In the paper new functions classes T_a, $a \geqslant 0$, are introduced and examined. Each T_a, $a \geqslant 0$, is a subclass of the family of functions regular in the disc $K = \{z: |z| < 1\}$. T_0 is precisely the class of typically real functions and T_1 is the class of functions convex in the direction of the imaginary axis, with real coefficients.

For functions of class T_a, $a > 0$, a structural formula and an estimation of the coefficients are found. Also the region of values of the system of n-Taylor coefficients of class T_a is described.

Introduction. Let H denote the class of functions of the form:

$$f(z) = z + a_2 z^2 + \ldots \tag{1}$$

regular in the disc $K = \{z: |z| < 1\}$.

Denote by $C \subset H$ the class of functions univalent in K and convex in the direction of the imaginary axis meets the image by f of every circle $|z| = r$, $0 < r < 1$, at no more than two points.

Let $T \subset H$ denote the class of typically real functions, i.e., functions of form (1) taking on real values $f(z)$ if and only of z is real.

The class C has been introduced by M. S. Robertson [11] in 1936; the research work connected with the class T has been initiated in 1932 by W. Rogosiński [12].

In 1969 P. T. Mocanu [9] defined the classes M_a, $0 \leqslant a \leqslant 1$, of a-starlike functions which are a "combination" of the well-known classes of starlike and convex functions.

Z. J. Jakubowski, using the properties of the class T of typically real functions as well as the properties of the family $\bar{C}$ of functions convex in the direction of the imaginary axis, with real coefficients, suggested analogous research to be done in the classes T and $\bar{C}$.

In the present paper we introduce and examine a new family of classes T_a, $a \geqslant 0$, such that $T_0 = T$ and $T_1 = \bar{C}$. We find a structural formula, an estimation of coefficients and the region of values of the system of n-Taylor coefficients of a function of class T.

1. Let $\bar{H} \subset H$ be the class of functions with real coefficients, i.e., such that $f^{(n)}(0) \in R$, $n = 2, 3, \ldots$, where R is the set of real numbers.

Denote by $\bar{C} \subset C$ the class functions univalent in K and convex in the direction of the imaginary axis, with real coefficients.

The following facts are classical:

LEMMA A. *If $f \in \bar{C}$, then f is real only on the real axis.*

LEMMA B. *A function f is in $\bar{C}$ if and only if $f^{(n)}(0) \in R$, $n = 2, 3, \ldots$, and*

$$\mathrm{re}\{(1-z^2)f'(z)\} > 0, \quad z \in K. \tag{2}$$

THEOREM A [11]. *If $f \in \bar{C}$, then*

$$|a_n| \leqslant 1, \quad n = 2, 3, \ldots \tag{3}$$

Equality in (3) takes place if and only if

$$f^*(z) = \frac{z}{1-\varepsilon z}, \quad \varepsilon = \pm 1, \quad f^*(z) = \frac{z}{1-z^2}.$$

Moreover, it has been shown [8] that then

$$-1 \leqslant a_{2n} \leqslant 1, \quad -1 + \frac{2}{2n+1} \leqslant a_{2n+1} \leqslant 1,$$

the estimations being sharp.

Now consider the class T of typically real functions of form (1).

The definition of this class is often given in the following form:

Let $T \subset \bar{H}$ denote the class of functions satisfying the condition

$$\mathrm{im}\, z \cdot \mathrm{im} f(z) > 0, \quad z \neq \bar{z},\ z \in K.$$

The following lemmas are well known:

LEMMA C. *If $f \in T$, then $f^{(n)}(0) \in R$, $n = 2, 3, \ldots$*

LEMMA D. *If $f \in T$, then $f(z) = \overline{f(\bar{z})}$, $z \in K$.*

LEMMA E. *A function f is in T if and only if $f^{(n)}(0) \in R$, $n = 2, 3, \ldots$ and*

$$\mathrm{re}\left\{\frac{1-z^2}{z} f(z)\right\} > 0, \quad z \in K. \tag{4}$$

LEMMA F. *If $f \in \bar{C}$, then $zf'(z) \in T$.*

Denote by $\mathfrak{B}[a, b]$ the family of functions β defined in the interval $[a, b]$, non-decreasing and normalized by the condition $\int_a^b d\beta(t) = 1$.

Observe that the family $\mathfrak{B}[a, b]$ is the set of probability measures on the segment $[a, b]$, [13].

THEOREM B [5]. *A function f belongs to T if and only if there exists a $\beta \in \mathfrak{B}[-1,1]$ such that*

$$f(z) = \int_{-1}^{1} \frac{z}{1-2tz+z^2}\, d\beta(t), \qquad z \in K.$$

THEOREM C [12]. *If $f \in T$, then*

$$|a_n| \leqslant n, \qquad n = 2, 3, \ldots \tag{5}$$

Equality in (5) takes place if and only if

$$f^*(z) = \frac{z}{(1-\varepsilon z)^2}, \qquad \varepsilon = \pm 1. \tag{6}$$

Moreover, it has been shown [8] that the sharp estimations

$$-1 \leqslant a_{2n} \leqslant 1, \qquad 1-2n \leqslant a_{2n+1} \leqslant 2n+1, \qquad n = 1, 2, \ldots, \tag{7}$$

hold.

Observe also the following properties of the classes considered

$$\bar{C} \subset \bar{S} \subset T \subset \bar{H} \subset H,$$

where $\bar{S} \subset \bar{H}$ is a subclass of the class of univalent functions.

2. Basing on the definition and the properties of the classes $\bar{C}$ and T, we introduce the following function classes.

DEFINITION. Let $T_a \subset \bar{H}$, $a \geqslant 0$, denote the class of functions satisfying the condition

$$\operatorname{re}\left\{(1-a)\frac{1-z^2}{z} f(z) + a(1-z^2) f'(z)\right\} > 0, \qquad z \in K. \tag{8}$$

Condition (8) is equivalent to the condition

$$(1-a)\frac{f(z)}{z} + af'(z) = \frac{p(z)}{1-z^2}, \qquad z \in K,$$

where $p \in \wp$ ($\wp$ denoting the class of Carathéodory functions [3]) and $p^{(n)}(0) \in R$, $n = 1, 2, \ldots$, and also to the condition

$$azf'(z) + (1-a)f(z) = F(z), \qquad z \in K, \tag{9}$$

where $F \in T$.

Directly from the definitions of the class T_a and from (2), (4) we get

$$T_0 = T, \qquad T_1 = \bar{C}.$$

We shall prove

THEOREM 1. *If $f \in T_a$, $a > 0$, then*

$$f(z) = \frac{1}{a} z^{1-\frac{1}{a}} \int_0^z \zeta^{-2+\frac{1}{a}} F(\zeta)\, d\zeta, \quad z \in K, \tag{10}$$

$$f(z) = \frac{1}{a} z^{1-\frac{1}{a}} \int_0^z \zeta^{-1+\frac{1}{a}} \frac{p(\zeta)}{1-\zeta^2}\, d\zeta, \quad z \in K, \tag{11}$$

where $F \in T$, $p \in \wp$, $p^{(n)}(0) \in R$, $n = 1, 2, \ldots$ The converse is also true.

Proof. If $f \in T_a$, $a > 0$, then some function $F \in T$ satisfies condition (9). By theorems [6] asserting to the existence of solutions of a differential equation in the set of complex functions equation (9) has a solution. It is easily verified that the following function satisfies that equation

$$f(z) = \frac{1}{a} z^{1-\frac{1}{a}} \int_0^z \zeta^{-2+\frac{1}{a}} F(\zeta)\, d\zeta + c_1 z^{1-\frac{1}{a}},$$

for $z \in K \setminus (-1, 0\rangle$, C_1 — const. If follows from the norming conditions for functions $f \in T$ that $C_1 = 0$.

Since f admits extension on to whole disc K, it follows that every function f belonging to the family T_a is of form (10).

Conversely, observe that for every function $F \in T$ the function (10) is regular in K (we choose this particular branch of the power $\zeta^{-2+\frac{1}{a}}$ which for $0 < \zeta < 1$ attains main values) and it satisfies equation (9). Thus $f \in T_a$.

In an analogous way we prove relation (11).

Formulae (10) and (11) can be expressed in the form

$$f(z) = \frac{1}{a} \int_0^1 t^{-2+\frac{1}{a}} F(zt)\, dt, \quad z \in K, \tag{12}$$

$$f(z) = \frac{1}{a} \int_0^1 t^{-1+\frac{1}{a}} \frac{p(zt)}{1-z^2t^2}\, dt, \quad z \in K. \tag{12'}$$

From the equality $T_0 = T$, the integral representation (12) and the definition of the class T we get

THEOREM 2. *For an arbitrary $a \geqslant 0$*

$$T_a \subset T. \tag{13}$$

Moreover, the following theorem holds

THEOREM 3. *If* $0 \leqslant a_1 \leqslant a_2$, *then*

$$T_{a_2} \subset T_{a_1}.$$

Proof. In the cases: $a_1 = a_2$, $a_1 = 0$ and $a_2 > 0$ the theorem is true.

Suppose that there exist number a_1, a_2, $0 < a_1 < a_2$ and a function $f \in T_{a_2}$ such that $f \notin T_{a_1}$. It follows from (10) that for certain $\zeta \in K$

$$\operatorname{re}\left\{(1-a_1)\frac{1-\zeta^2}{\zeta}f(\zeta)+a_1(1-\zeta^2)f'(\zeta)\right\} \leqslant 0$$

holds.

By assumption

$$\operatorname{re}\left\{(1-a_2)\frac{1-\zeta^2}{\zeta}f(\zeta)+a_2(1-\zeta^2)f'(\zeta)\right\} > 0.$$

Thus

$$\operatorname{re}\left\{(a_1-a_2)\frac{1-\zeta^2}{\zeta}f(\zeta)\right\} > 0.$$

Since $a_1 - a_2 < 0$, we have $\operatorname{re}\left\{\frac{1-\zeta^2}{\zeta}f(\zeta)\right\} < 0$, which contradicts (13) and (4).

COROLLARY 1. *For every* $a \geqslant 1$, $T_a \subset \bar{C}$.

COROLLARY 2. *The functions* f *belonging to* T_a, $a \geqslant 1$, *are univalent in the disc* K.

Next we shall give some examples of functions illustrating relations between the classes examined. To this end let us denote

$$I(a, z, f) = (1-a)\frac{1-z^2}{z}f(z)+a(1-z^2)f'(z), \qquad z \in K.$$

EXAMPLE 1. Consider the function defined by the formula

$$f_1(z) = z, \qquad z \in K.$$

Since $I(a, z, f_1) = 1$ for every $z \in K$, we see that $f_1 \in T_a$ for every $a \geqslant 0$.

EXAMPLE 2. The function defined by the formula

$$f_2(z) = \frac{z}{(1+z)^2}, \qquad z \in K,$$

belongs to T. Let $a > 0$; then

$$I(a, z, f_2) = (1-a)\frac{1-z}{1+z}+a\left(\frac{1-z}{1+z}\right)^2.$$

Since $\lim_{z\to -i} \operatorname{re}\{I(a, z, f_2)\} = -a$, we see that $f_2 \notin T_a$, $a > 0$. Thus the family T is in fact wider than any class T_a, $a > 0$.

EXAMPLE 3. The function defined by the formula

$$f_3(z) = \frac{z}{1+z}, \quad z \in K,$$

belongs to the class $\bar{C}$. Let $a > 1$; then

$$I(a, z, f_3) = (1-a)(1-z) + a\frac{1-z}{1+z}.$$

Since $\lim_{z \to -i} \operatorname{re}\{z(a, z, f_3)\} = 1-a < 0$, we see that $f_3 \notin T_a$, $a > 1$. Thus the family $\bar{C}$ is in fact wider than any class T_a, $a > 1$.

3. Now we prove a structural formula for the class T_a.

THEOREM 4. *A function $f \in T_a$, $a > 0$ if and only if there exists a $\beta \in \mathfrak{B}[-1, 1]$ such that*

$$f(z) = \int_{-1}^{1} \left[\frac{1}{a}\int_{0}^{1} t^{-2+\frac{1}{a}} \frac{zt}{1-2\tau zt + z^2t^2}\, dt\right] d\beta(\tau), \quad z \in K. \tag{14}$$

Proof. Take any function $f \in T_a$, $a > 0$, and consider relation (12). Since the function $F(zt)$ satisfies the assumptions of Theorem B, then for every function $f \in T_a$, $a > 0$, there exists a function $\beta \in \mathfrak{B}[-1, 1]$ such that

$$f(z) = \frac{1}{a} t^{-2+\frac{1}{a}} \left[\int_{-1}^{1} \frac{zt}{1-2\tau zt + z^2t^2}\, dt\right] d\beta(\tau), \quad z \in K.$$

By the Fubini theorem on the change of succession of integrating in a double Stieltjes integral, we get the assertion of the theorem.

Also the converse is valid.

COROLLARY 3. *The class T_a is compact, connected an convex.*

This follows e.g. from the general theorems concerning classes of functions which can be represented by an integral formula ([5], [12]).

The following theorem follows directly from the definition of the class T_a, $a \geqslant 0$, equation (9) and estimation (5) in Theorem C

THEOREM 5. *If $f \in T_a$, $a \geqslant 0$, then*

$$|a_n| \leqslant \frac{n}{1+a(n-1)}, \quad n = 1, 2, \ldots \tag{15}$$

The estimation is sharp. Equality holds if and only if

$$f^*(z) = zG\left(\frac{1}{a}, 2, 1+\frac{1}{a}, \varepsilon z\right), \quad \varepsilon = \pm 1, a > 0, z \in K, \tag{16}$$

where

$$(17) \qquad G(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt$$

$$= \frac{\Gamma(c)}{\Gamma(c)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \cdot \frac{z^k}{k!};$$

$\Gamma(x)$ is the Euler gamma function, $\operatorname{re} a > 0, \operatorname{re}(c-a) > 0, \operatorname{re} b > 0$.

Proof. Let a function f of form (1) be an arbitrary functions of the family T_a, $a > 0$. From differential equation (9) we obtain

$$f(z) = z + \sum_{n=2}^{\infty} \frac{b_n}{1-a(n-1)} z^n, \quad z \in K,$$

where b_n, $n = 2, 3, \ldots$ are the coefficients of the expansion of the relevant function $F \in T$. Thus

$$(18) \qquad a_n = \frac{b_n}{1+a(n-1)}, \quad n = 2, 3, \ldots$$

Hence by (5), (15) follows.

Since in estimation (5) equality holds only for function (6), then, in view of (12), equality in (15) holds for the function

$$f^*(z) = \frac{1}{a} \int_0^1 t^{-2+\frac{1}{a}} zt(1-zt)^{-2}dt, \quad \varepsilon = \pm 1, \ z \in K.$$

According to notation (17), the sharpness of estimation is realized by function (16).

Thus Theorem 5 is a natural generalization of the classical results of W. Rogosiński [12] ($a = 0$) and M. S. Robertson [11] ($a = 1$).

On the other hand we hand obtained another proof of the fact that, for every function $F \in T$, the particular solution $f(z)$ ($f(0) = 0$, $f'(0) = 1$) of differential equation (9) is a regular function in the disc K.

It also follows from estimation (15) that the "limiting" family $T_{+\infty}$ is reduced to the identity function.

Moreover, from (7) and from relation (18) we get sharp estimations in the class T_a, $a \geqslant 0$,

$$-\frac{2n}{1+a(2n-1)} \leqslant a_{2n} \leqslant \frac{2n}{1+a(2n-1)},$$

$$\frac{1-2n}{1+2na} \leqslant a_{2n+1} \leqslant \frac{1+2n}{1+2na}, \quad n = 1, 2, \ldots$$

4. We shall determine next the region of values of the system of coefficients of functions of class T_α.

Consider the function

$$F(z,\tau) = z/(1-2\tau z+z^2), \qquad -1 \leqslant \tau \leqslant 1,\ z \in K.$$

Observe that

$$F(z,\tau) = \begin{cases} z/(1+z)^2 & \text{for } \tau = -1, \\ z/(1-2\tau z+z^2) & \text{for } -1 < \tau < 1, \\ z/(1-z)^2 & \text{for } \tau = 1, \end{cases} \tag{19}$$

is a regular function in K. Thus

$$F(z,\tau) = z+\sum_{n=2}^{\infty} b_n(\tau) z^n, \qquad z \in K,\ -1 \leqslant \tau \leqslant 1,$$

where

$$b_n(\tau) = \begin{cases} (-1)^{n+1} n & \text{for } \tau = -1, \\ \dfrac{\sin n\varphi}{\sin\varphi} & \text{for } -1 < \tau < 1,\ e^{i\varphi} = \tau + i\sqrt{1-\tau^2}, \\ n & \text{for } \tau = 1, \end{cases}$$

$n = 2, 3, \ldots$

Let $f(z,\tau)$ denote the function

$$f(z,\tau) = \frac{1}{a}\int_0^t t^{-2+\frac{1}{a}} F(zt,\tau)\,dt, \qquad a > 0,\ z \in K,$$

where $F(z,\tau)$ is the function given by formula (19).

Thus the function $f(z,\tau)$ possesses the following properties: Let $z \in K$, $a > 0$, $-1 \leqslant \tau \leqslant 1$; then

(a) $f(z,\tau)$ is regular in K;

(b) $f(z,\tau) = \overline{f(\bar{z},\tau)}$;

(c) If $\xi = \tau + i\sqrt{1-\tau^2}$, then, for $\tau \neq \pm 1$,

$$f(z,\tau) = \frac{z\xi}{\xi-\bar{\xi}} G\left(\frac{1}{a}, 1, 1+\frac{1}{a}, z\xi\right) - \frac{z\bar{\xi}}{\xi-\bar{\xi}} G\left(\frac{1}{a}, 1, 1+\frac{1}{a}, z\bar{\xi}\right);$$

(d) If $\xi = e^{i\varphi}$, $0 < \varphi < \pi$, $a \geqslant 0$, then

$$f(z,\tau) = \sum_{k=0}^{\infty} \frac{1}{1+ka} \cdot \frac{\sin(k+1)\varphi}{\sin\varphi} z^{k+1};$$

(e) $f^{(n)}(0,\tau) \in R$, $n = 0, 1, 2, \ldots$;

(f) $f(z,\tau) \in T$.

Thus from Theorem 4 we get

THEOREM 6. *A function* $f \in T_a$, $a \geqslant 0$, *if and only if there exists* $\beta \in \mathfrak{B}[-1,1]$, *such that*

$$f(z) = \int_{-1}^{1} f(z,\tau)\,d\beta(\tau), \qquad z \in K, \tag{20}$$

where for $-1 \leqslant \tau \leqslant 1$

$$f(z,\tau) = \begin{cases} \dfrac{1}{a}\displaystyle\int_0^1 t^{-2+\frac{1}{a}} \frac{zt}{1-2\tau zt+z^2t^2}\,dt & \text{for } a > 0, \\ \dfrac{z}{1-2\tau z+z^2} & \text{for } a = 0; \end{cases} \tag{I}$$

or for $a > 0$

(II) $f(z,\tau)$

$$= \begin{cases} zG\left(\dfrac{1}{a}, 2, 1+\dfrac{1}{a}; -z\right) & \text{for } \tau = -1, \\ \dfrac{z\xi}{\xi-\bar{\xi}}G\left(\dfrac{1}{a}, 1, 1+\dfrac{1}{a}; z\xi\right) - \dfrac{z\bar{\xi}}{\xi-\bar{\xi}}G\left(\dfrac{1}{a}, 1, 1+\dfrac{1}{a}; z\bar{\xi}\right), & -1 < \tau < 1, \\ zG\left(\dfrac{1}{a}, 2, 1+\dfrac{1}{a}; z\right) & \text{for } \tau = 1; \end{cases}$$

or

$$f(z,\tau) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k(k+1)}{1+ka} z^{k+1} & \text{for } \varphi = \pi, \\ \displaystyle\sum_{k=0}^{\infty} \frac{1}{1+ka}\,\frac{\sin(k+1)\varphi}{\sin\varphi} z^{k+1} & \text{for } 0 < \varphi < \pi, \\ \displaystyle\sum_{k=0}^{\infty} \frac{1+k}{1+ka} z^{k+1} & \text{for } \varphi = 0. \end{cases} \tag{III}$$

Put for $n = 2, 3, \ldots$ and $a \geqslant 0$

$$a_n(\tau) = \begin{cases} \dfrac{(-1)^n n}{1+a(n-1)} & \text{for } \tau = -1, \\ \dfrac{1}{1+a(n-1)}\,\dfrac{\sin n\varphi}{\sin\varphi} & \text{for } -1 < \tau < 1, \\ \dfrac{n}{1+a(n-1)} & \text{for } \tau = 1. \end{cases} \tag{21}$$

Then formula (20) implies the following

THEOREM 7. *If* $f \in T_a$, $a \geqslant 0$, *then for* $n = 2, 3, \ldots$

$$a_n = \int_{-1}^{1} a_n(\tau) d\beta(\tau),$$

where $\beta \in \mathfrak{B}[-1, 1]$ *and* $a_n(\tau)$ *are defined by formulae* (21).

Let $V_n(a)$, $n \geqslant 2$, $a \geqslant 0$, denote the region of values of the system $(a_2, a_3, \ldots, a_n)$ of coefficients of the Taylor expansion (1) of functions of class T_a.

By Carathéodory's theorem [7] and by Theorem 7 we have

THEOREM 8. *If* $f \in T_a$, $a \geqslant 0$, *the region of values* $V_n(a)$ *is the convex hull of the* $(n-1)$*-dimensional real space* $(\omega_1, \ldots, \omega_{n-1})$, *where for* $e^{i\varphi} = \tau + i\sqrt{1-\tau^2}$

$$\omega_{k-1} = \begin{cases} \dfrac{(-1)^k k}{1+a(k-1)} & \text{for} \quad \varphi = \pi, \\ \dfrac{1}{1+a(k-1)} \dfrac{\sin k\varphi}{\sin\varphi} & \text{for} \quad 0 < \varphi < \pi, \\ \dfrac{k}{1+a(k-1)} & \text{for} \quad \varphi = 0, \end{cases}$$

$k = 2, 3, \ldots, n$.

COROLLARY 4. *In the case* $n = 3$ *the region of values* $V_3(a)$ *is the convex hull of the curve whose equation is*

$$a_3 = \frac{(a+1)}{2a+1} a_2^2 - \frac{1}{2a+1}, \qquad -\frac{2}{a+1} \leqslant a_2 \leqslant \frac{2}{a+1}.$$

In the case $n = 4$, $V_4(a)$ *is the convex hull of a the curve with the equation*

$$a_4 = \frac{(a+1)(2a+1)}{3a+1} a_2 a_3 - \frac{a+1}{3a+1} a_2,$$

$$a_3 = \frac{(a+1)}{2a+1} a_2^2 - \frac{1}{2a+1}, \qquad -\frac{2}{a+1} \leqslant a_2 \leqslant \frac{2}{a+1}.$$

Remark. Theorem 8 is a generalization of an analogous theorems in the classes T and $\bar{C}$.

5. The results of [1] and Theorem 4 yield

THEOREM 9. *If* $f \in T_a^{\mathbf{R}}$, $a \geqslant 0$, *then the region of values of the evaluation functional*

(22) $$F(f) = f(z), \quad z \text{ is a fixed point of } K,$$

is a convex hull of the curve with the equation

$$\omega = f(z, \tau), \qquad -1 \leqslant \tau \leqslant 1,$$

where $f(z, \tau)$ *is of form* (I) [(II) or (III)].

Remark. Let us denote

$$C_0 = \frac{ir}{4\sin\psi}\left[G\left(\frac{1}{a}, 1, 1+\frac{1}{a}; r\right) + G\left(\frac{1}{a}, 1, 1+\frac{1}{a}; -r\right)\right],$$

$$\varrho = \frac{r}{2a\sin\psi}\int_0^1 t^{-1+\frac{1}{a}}\frac{1}{r^2t^2-1}\cdot\frac{\bar{z}+\frac{1}{\bar{z}}-2\tau}{z+\frac{1}{z}-2\tau}\cdot dt.$$

It appears that the region of values of the functional (22) is included in that section of the disc $K(C_0, \varrho)$ determined by the chord AB with $A = \left(zG\left(\frac{1}{a}, 2, 1+\frac{1}{a}; z\right)\right)$ and $B = \left(zG\left(\frac{1}{a}, 2, 1+\frac{1}{a}; -z\right)\right)$ which contains the centre of the disc.

In the case $a = 1$ we get the region of values of functional (22) in the class $\bar{C}$.

6. Finally we show that the following theorem is true

THEOREM 10. *If* $f \in T_a$, $a > 0$, *then*

1° $\overline{\mathrm{co}}\, T_a = T_a$,

where $\overline{\mathrm{co}}\, A$ *stands for the closure of the convex hull of the set* A;

2° *In the integral representation* (14) *or* (20) *of a function of this class the probability measure* $\beta \in \mathfrak{B}[-1, 1]$ *is uniquely determined*;

3° *The set of extreme points of this class is the set of functions of the form*

$$E_{T_a} = \{f(z, \tau):\ -1 \leqslant \tau \leqslant 1,\ z \in K\},$$

where $f(z, \tau)$ *is given by formula* (I).

Proof. 1° follows from the convexity of the class and from the definition of the convex hull.

The second property is a consequence of the uniqueness of the measure β in the integral representation in class T [13].

3° follows from Brickman's theorem [2] and from Theorem 6.

Theorem 10, especially 3°, gives a possibility of examining real functionals on the class T_a. It is a generalization of the analogous topics considered in the classes T and $\bar{C}$.

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