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Coefficient problem for a class of Mocanu-Bazilevič functions

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Abstract. Let $M(\alpha, \sigma, A)$, $\alpha > 0$, $0 \leqslant \sigma < 1$, $A > \frac{1}{2}$ denote the class of Mocanu–Bazilevič univalent functions f in the open unit disk $E(|z| < 1)$ for which $\left|\left(J(\alpha, f(z)) - \sigma\right)/(1-\sigma) - A\right| < A$, $z \in E$, where

$$J(\alpha, f(z)) \equiv (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right].$$

The coefficient problem for the class $M(\alpha, \sigma, A)$ is completely solved; the results obtained are sharp.

1. Let S denote the class of functions f which are regular and univalent in the open unit disk $E(|z| < 1)$ and normalized by the conditions

$$f(0) = f'(0) - 1 = 0. \tag{1}$$

Let C_r denote the image of the circle $|z| = r$, $0 < r < 1$, under the mapping $f \in S$, $z = re^{i\theta}$. Then the angle $\varphi = \arg\{f(re^{i\theta})\}$ represents the argument of the radius vector from the origin to the point $f(re^{i\theta})$, while $\psi = \arg\{i\, re^{i\theta} f'(re^{i\theta})\}$ is the argument of the tangent vector to C_r at $f(re^{i\theta})$. Many special subclasses of S are obtained by imposing restrictions on the behavior of these arguments. Thus, if S_σ^* and K_σ denote the subclasses of S of starlike and convex functions of order σ, $0 \leqslant \sigma < 1$, respectively, then $f \in S_\sigma^*(K_\sigma)$ if and only if $\partial\varphi/\partial\theta \geqslant \sigma$ $(\partial\psi/\partial\theta \geqslant \sigma)$ for each r, $0 < r < 1$ [11]. The concept of α-convexity of order σ is obtained by combining these conditions on the arguments. Thus, if $\alpha \geqslant 0$ is a real number, a function f which is regular in E, satisfies (1) and is such that $f(z)\, f'(z)/z \neq 0$ for $z \in E$, is said to be an *α-convex function* of order σ, $0 \leqslant \sigma < 1$, if the Mocanu angle $\mu = (1-\alpha)\varphi + \alpha\psi$ is an increasing function of θ for fixed values of r [9] and satisfies $\partial\mu/\partial\theta \geqslant \sigma$ for a given α [5]. We denote the class of all functions f that are α-convex of order σ by $M(\alpha, \sigma)$. If we write

$$J\left(\alpha, f(z)\right) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right],$$

then $f \in M(\alpha, \sigma)$ if and only if $\operatorname{Re} J(\alpha, f(z)) \geqslant \sigma$ holds for all $z \in E$. Obviously, $M(0, \sigma) = S_\sigma^*$ and $M(1, \sigma) = K_\sigma$. In [5] it is proved that $f \in M(\alpha, \sigma)$ for $\alpha > 0$ if and only if f is a Bazilevič function [1] of type $1/\alpha$ and order σ of the form

$$f(z) = \left[\frac{1}{\alpha}\int_0^z [g(\zeta)]^{1/\alpha}\zeta^{-1}d\zeta\right]^\alpha, \quad z \in E, \tag{2}$$

where $g \in S_\sigma^*$, and the powers appearing in (2) as well as in the sequel are meant as principal values. The class $M(\alpha, \sigma)$ therefore consists of Mocanu–Bazilevič type of univalent functions of order σ.

2. The subclass of Mocanu–Bazilevič functions to be investigated in this paper is defined as follows: Let f be regular in E and satisfy (1); then $f \in M(\alpha, \sigma, A)$ in E, if there exist real numbers α, σ and A, $\alpha > 0$, $0 \leqslant \sigma < 1$ and $A > \frac{1}{2}$, such that for given α, σ and A

$$\left|\frac{J(\alpha, f(z)) - \sigma}{1-\sigma} - A\right| < A, \quad z \in E. \tag{3}$$

Obviously, $M(\alpha, \sigma, A_1) \subset M(\alpha, \sigma, A_2)$ for given α and σ whenever $A_1 < A_2$, and $M(\alpha, \sigma, \infty) \equiv M(\alpha, \sigma)$. We also note that for given α and σ the class $M(\alpha, \sigma, \frac{1}{2})$ is empty, since $J(\alpha, z) = 1$; in this case the identity function $f(z) = z$ yields equality in (3). Other different classes of this general kind have been studied, e.g., in [8] and [10], where members of the respective classes turn out to be bounded in E. The members of $M(\alpha, \sigma, A)$ are, however, not bounded in E.

Let $P(A)$ denote the class of functions p which are regular in E and satisfy the conditions that $p(0) = 1$, $|p(z) - A| < A$ for $z \in E$ and a fixed $A > \frac{1}{2}$. Let Ω denote the class of all functions ω regular in E and satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < |z|$ for $z \in E$. It is known [4] that if $p \in P(A)$, then

$$p(z) = \frac{1+\omega(z)}{1-a\omega(z)}, \quad a = 1 - 1/A, \tag{4}$$

where $\omega \in \Omega$.

Members of the class $M(\alpha, \sigma, A)$ can be represented in terms of the members of $P(A)$. Thus, in view of (3), $f \in M(\alpha, \sigma, A)$ if an only if

$$J(\alpha, f(z)) - \sigma = (1-\sigma)p(z), \quad p \in P(A). \tag{5}$$

THEOREM 1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M(\alpha, \sigma, A)$, *and let* $s_1 = 0$, $s_m = (1-\alpha)\ (\beta_m - a_m) + \alpha\gamma_{m-1}$,

$$t_1 = (1+a)(1-\sigma), \tag{6}$$

$$t_m = (1-\sigma+a\alpha-a\sigma)a_m + a(1-\alpha)\beta_m + a\alpha\gamma_{m-1}, \quad m = 2, 3, \ldots,$$

where a_m, β_m, γ_m *are defined by (with* $a_1 = 1$)

$$a_m = \sum_{k=1}^{m} (m-k+1) a_k a_{m-k+1},$$

(7)
$$\beta_m = \sum_{k=1}^{m} k(m-k+1) a_k a_{m-k+1},$$

$$\gamma_m = \sum_{k=1}^{m} k(k+1) a_{k+1} a_{m-k+1},$$

and $a = 1-1/A$. *Then the coefficients* a_n *satisfy the following quadratic inequality:*

(8)
$$|s_n|^2 \leqslant \sum_{m=1}^{n-1} \{|t_m|^2 - |s_m|^2\}, \quad n = 2, 3, \ldots$$

Equality in (8) *holds for the function*

(9)
$$f_*(z) = \left[\frac{1}{a} \int_0^z \zeta^{\frac{1}{a}-1} (1-\varepsilon a\zeta)^{-(1+a)(1-\sigma)/aa} d\zeta\right]^a, \quad |\varepsilon| = 1.$$

Proof. Substituting (4) in (5) we get after some simplification

(10)
$$(1-a)\{z[f'(z)]^2 - f(z)f'(z)\} + azf(z)f''(z)$$
$$= \{a(1-a)z[f'(z)]^2 + (1-\sigma+aa-a\sigma)f(z)f'(z) + aazf(z)f''(z)\}\,\omega(z).$$

Given $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we note that

$$f(z)f'(z) = \sum_{m=1}^{\infty} a_m z^m, \quad [f'(z)]^2 = \sum_{m=1}^{\infty} \beta_m z^{m-1}, \quad f(z)f''(z) = \sum_{m=1}^{\infty} \gamma_m z^m,$$

where a_m, β_m, γ_m are defined by (7). Thus, relation (10) becomes

$$(1-a) \sum_{m=1}^{\infty} (\beta_m - a_m) z^m + a \sum_{m=1}^{\infty} \gamma_m z^{m+1}$$
$$= \left\{a(1-a) \sum_{m=1}^{\infty} \beta_m z^m + (1-\sigma+aa-a\sigma) \sum_{m=1}^{\infty} a_m z^m + aa \sum_{m=1}^{\infty} \gamma_m z^m\right\} \omega(z),$$

which, in view of the substitutions (6), simplifies to

$$\sum_{m=1}^{\infty} s_m z^m = \left\{\sum_{m=1}^{\infty} t_m z^m\right\} \omega(z),$$

or

(11)
$$\left|\sum_{m=1}^{n} s_m z^m + \sum_{m=n+1}^{\infty} h_m z^m\right| \leqslant \left|\sum_{m=1}^{\infty} t_m z^m\right|,$$

where the series $\sum_{m=n+1}^{\infty} h_m z^m$ is absolutely and uniformly convergent in compacta on E. Putting $z = re^{i\theta}$ and performing the indicated integrals we get

$$\int_0^{2\pi} \Big| \sum_{m=1}^{n} s_m r^m e^{im\theta} + \sum_{m=n+1}^{\infty} h_m r^m e^{im\theta} \Big|^2 d\theta \leqslant \int_0^{2\pi} \Big| \sum_{m=1}^{n-1} t_m r^m e^{im\theta} \Big|^2 d\theta$$

which by Parseval's identity gives as $r \to 1$

$$\sum_{m=1}^{n} |s_m|^2 \leqslant \sum_{m=1}^{n-1} |t_m|^2, \quad n = 2, 3, \ldots,$$

and inequality (8) is obtained on transposing terms.

The method used in this proof is due to Clunie [2], which has also been used in [7], [8] and [10].

Inequality (8) in the particular cases for $n = 2, 3$ gives

$$|a_2| \leqslant \frac{(1+a)(1-\sigma)}{1+a}, \tag{12}$$

$$|a_3| \leqslant \frac{(1+a)(1-\sigma)[1+3\alpha+2a+5a\alpha+\alpha a^2-\sigma(1-a)(1+3a)]}{2(1+\alpha)^2(1+2a)}. \tag{13}$$

The presence of the parameter α makes it very difficult to further simplify the quadratic inequality (8), or employ any induction scheme on it. However, some special cases can be obtained from (8) as follows:

If $f \in M(\alpha, \sigma)$, the bounds in (12) and (13) are

$$|a_2| \leqslant \frac{2(1-\sigma)}{1+\alpha}, \quad |a_3| \leqslant \frac{(1-\sigma)(3+8\alpha+\alpha^2)}{(1+\alpha)^2(1+2\alpha)},$$

which for $\sigma = 0$ reduce to the result given in [5].

If $f \in M(0, \sigma, A)$, inequality (8) becomes

$$\sum_{k=2}^{n} (k-1)^2 |a_k|^2 \leqslant (1+a)^2(1-\sigma)^2 + \sum_{k=2}^{n-1} \{1+ak-(1+a)\sigma\}^2 |a_k|^2,$$

which by induction gives

$$|a_n| \leqslant \prod_{k=0}^{n-2} \frac{|1+(k+1)a-(1+a)\sigma|}{k+1}, \quad n = 2, 3, \ldots \tag{14}$$

Equality in (14) holds for the function

$$f(z) = z(1-\varepsilon az)^{-(1+a)(1-\sigma)/a}, \quad |\varepsilon| = 1.$$

If $f \in M(0, \sigma, \infty) \equiv S^*_\sigma$, inequality (14) gives the known result [11]

$$|a_n| \leqslant \prod_{k=0}^{n-2} \frac{k+2(1-\sigma)}{k+1}, \quad n = 2, 3, \ldots,$$

which is sharp for $f(z) = z(1-\varepsilon z)^{-2(1-\sigma)}$, $|\varepsilon| = 1$ (see also [7]).

If $f \in M(1, \sigma, A)$, inequality (8) becomes

$$\sum_{k=2}^{n} k^2(k-1)^2 |a_k|^2 \leqslant (1+a)^2(1-\sigma)^2 + \sum_{k=2}^{n-1} \{(1+a)(1-\sigma)k + ak(k-1)\}^2 |a_k|^2,$$

which by induction leads to

$$|a_n| \leqslant \prod_{k=0}^{n-2} \frac{(1-a)(1-\sigma)+ka}{k+2}, \quad n = 2, 3, \ldots \tag{15}$$

Equality in (15) holds for the function

$$f(z) = \int_0^z (1-\varepsilon a\zeta)^{-(1+a)(1-\sigma)/a} d\zeta, \quad |\varepsilon| = 1.$$

If $f \in M(1, \sigma, \infty) \equiv K_\sigma$, inequality (15) gives the known result [11]

$$|a_n| \leqslant \prod_{k=0}^{n-2} \frac{k+2(1-\sigma)}{k+2}, \quad n = 2, 3, \ldots,$$

which is sharp for

$$f(z) = \int_0^z (1-\varepsilon\zeta)^{-2(1-\sigma)} d\zeta = \frac{1-(1-\varepsilon z)^{2\sigma-1}}{2\sigma-1},$$

$$|\varepsilon| = 1.$$

In view of these particular cases it appears that the function $f_*(z)$ defined by (9) is an extremal function for the class $M(a, \sigma, A)$.

3. The coefficient problem for the class $M(a, \sigma, A)$ can be completely solved as follows: Let us write the function $f_*(z)$, defined by (9), as

$$f_*(z) = zH(z), \tag{16}$$

where

$$H(z) = \left\{1 + \sum_{n=1}^{\infty} b_n z^n\right\}^a, \tag{17}$$

$$b_n = \frac{\varepsilon^n}{n!\, a^n(1+na)} \prod_{k=0}^{n-1} [(1+a)(1-\sigma)+kaa]. \tag{18}$$

Then, if $f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu \in M(\alpha, \sigma, A)$ it is easy to see that $|a_{n+1}| \leqslant \frac{H^{(n)}(0)}{n!}$, $n = 1, 2, \ldots$ In fact, the following result holds:

THEOREM 2. *Let* $f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu \in M(\alpha, \sigma, A)$. *Let* $S(n)$ *be the set of all n-tuples* $(x_1, x_2, \ldots, x_n)$ *of non-negative integers for which* $\sum_{i=1}^{n} i x_i = n$, *and for each such n-tuple define* q *by* $\sum_{i=1}^{n} x_i = q$. *If*

$\gamma(\alpha, q) = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-q)$ *with* $\gamma(\alpha, 0) = \alpha$, *and* $c_n = |b_n|$, *where* b_n *are given by* (18), *then for* $n = 1, 2, \ldots$

$$|a_{n+1}| \leqslant \sum \frac{\gamma(\alpha, q-1) c_1^{x_1} c_2^{x_2} \ldots c_n^{x_n}}{x_1! \, x_2! \ldots x_n!}, \tag{19}$$

where summation is taken over all n-tuples in $S(n)$.

Proof. In view of (16) and (17), since

$$H(z) = [h(z)]^\alpha = 1 + \sum_{n=1}^{\infty} a_{n+1} z^n, \tag{20}$$

where $h(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, we obtain on differentiating (20)

$$H'(z) = \alpha \frac{h'(z)}{h(z)}, \quad H(z) = \sum_{n=1}^{\infty} n a_{n+1} z^{n-1},$$

which on using the power series for h, h' and H gives

$$\Big(\sum_{n=1}^{\infty} n a_{n+1} z^{n-1}\Big)\Big(1 + \sum_{n=1}^{\infty} b_n z^n\Big) = \alpha \Big(\sum_{n=1}^{\infty} n b_n z^{n-1}\Big)\Big(1 + \sum_{n=1}^{\infty} a_{n+1} z^n\Big). \tag{21}$$

For fixed integer $n \geqslant 1$ we equate the coefficients of z^{n-1} in (21) and find that

$$\sum_{k=0}^{n} [k - \alpha(n-k)] b_{n-k} a_{k+1} = 0 \quad (c_0 = a_1 = 1). \tag{22}$$

Since $c_0 = 1$, we can solve (22) for a_{n+1} and get

$$|a_{n+1}| \leqslant -\frac{1}{n} \sum_{k=0}^{n-1} [k - \alpha(n-k)] c_{n-k} |a_{k+1}|. \tag{23}$$

Equation (23) is a recursion formula that allows us to compute $|a_{n+1}|$ from these with smaller index and as such determines a sequence of $|a_{n+1}|$ in a unique manner. Thus, in order to prove this theorem it would suffice to show that for each integer n the coefficients a_{n+1} defined by the equality in (19) do indeed satisfy (23). For this purpose, we proceed by induction, i.e., we assume for each $k = 1, 2, \ldots, n-1$

$$(24)\qquad |a_{n+1}| \leqslant \sum \frac{\gamma(a, j-1) c_1^{x_1} c_2^{x_2} \ldots c_k^{x_k}}{x_1!\, x_2! \ldots x_k!},$$

where $j = \sum_{i=1}^{k} x_i$ and the sum is taken over $S(k)$, the set of all non-negative k-tuples $(x_1, x_2, \ldots, x_k)$ for which $\sum_{i=1}^{k} i x_i = k$. Now if $k < n$, we can enlarge the k-tuple to an n-tuple by adjoining suitably many zeros.

Then any solution of

$$(25)\qquad \sum_{i=1}^{n} i x_i = k, \quad k < n,$$

in non-negative integers must give $x_i = 0$ for $i = k+1, k+2, \ldots, n$, and the inclusion of the factors $c_i^{x_i}/x_i!$ in (24) does not change the value because these factors are 1 for $i = k+1, k+2, \ldots, n$. Hence (24) can be replaced by

$$(26)\qquad |a_{n+1}| \leqslant \sum \frac{\gamma(a, j-1) c_1^{x_1} c_2^{x_2} \ldots c_n^{x_n}}{x_1!\, x_2! \ldots x_n!}, \quad k \leqslant n,$$

where $j = \sum_{i=1}^{n} x_i$, and the sum is taken over the set $S(k)$ of all non-negative nteger solutions of (25). We use (26) in the right-hand side R of (23). Then

$$(27)\qquad R \equiv -\frac{1}{n} \sum_{k=0}^{n-1} \sum_{S(k)} \frac{[k - a(n-k)]\gamma(a, j-1) c_{n-k} c_1^{x_1} c_2^{x_2} \ldots c_n^{x_n}}{x_1!\, x_2! \ldots x_n!}.$$

Now let $(y_1, y_2, \ldots, y_n)$ be any fixed n-tuple in $S(n)$, so that

$$(28)\qquad \sum_{i=1}^{n} i y_i = n, \quad \sum_{i=1}^{n} y_i = q.$$

We are to determine the coefficient C of $c_1^{y_1} c_2^{y_2} \ldots c_n^{y_n}$ in (27). This coefficient may arise from combining several terms from the sum and in fact such terms arise if and only if $c_{n-k} c_1^{x_1} c_2^{x_2} \ldots c_n^{x_n} = c_1^{y_1} c_2^{y_2} \ldots c_n^{y_n}$.

To be specific, let a be an index for which $y_a \geqslant 1$, and let $x_i = y_1$ if $i \neq a$, and let $x_a = y_a - 1$. For this fixed a, we have $j = \sum_{i=1}^{n} x_i = q-1$. In (27) we set $n-k = a$. If A is the set of a for which $y_a \neq 0$, then

$$C = -\frac{1}{n}\sum_{a\in A}\frac{(n-a-\alpha a)\gamma(\alpha, q-2)}{x_1!\,x_2!\,\ldots\,x_n!}. \tag{29}$$

Inserting the factor y_a in the numerator and denominator of (29) we have

$$C = -\sum_{a\in A}\frac{y_a(n-a-\alpha a)\gamma(\alpha, q-2)}{ny_1!\,y_2!\,\ldots\,y_n!}$$

$$= \frac{\gamma(\alpha, q-2)}{ny_1!\,y_2!\,\ldots\,y_n!}\sum_{a\in A} y_a(\alpha a + a - n).$$

If $y_a = 0$, then the corresponding term in the sum is zero. Hence, using (28)

$$C = \frac{\gamma(\alpha, q-2)}{ny_1!\,\ldots\,y_n!}\sum_{a=1}^{n}(\alpha a y_a + a y_a - n y_a)$$

$$= \frac{\gamma(\alpha, q-2)}{ny_1!\,y_2!\,\ldots\,y_n!}\, n[\alpha - (q-1)]$$

$$= \frac{\gamma(\alpha, q-1)}{y_1!\,\ldots\,y_n!},$$

which is precisely the coefficient of $c_1^{y_1}c_2^{y_2}\ldots c_n^{y_n}$ required on the right-hand side of (19). Since the argument holds for each fixed $(y_1, y_2, \ldots, y_n)$, the proof is complete.

The bounds in (19) are sharp and for $\alpha > 0$ attained by $f_*(z)$ defined in (9).

The technique used by Goodman in [3] has been employed to get the bounds in (19) in the compact form.

All particular cases discussed in Section 2 for $\alpha > 0$ follow from (19). For $\sigma = 0$, $\alpha = 1$ the above theorem reduces to the result announced in [6]. This theorem solves the coefficient problem for the class $M(\alpha, \sigma, A)$ completely.

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