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Comparison theorems for purely non-linear parabolic differential operators

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Dedicated to the memory of Jacek Szarski

This paper deals with a system of non-linear parabolic operators of form (1) below. We prove, in Section 2, comparison theorems, of differential inequalities type, and uniqueness criterions in arbitrary domains under general assumptions resembling those made by M. Picone [4], [5] for a linear equation. Our theorems do not require any assumptions concerning the boundedness of the domain and, moreover, they include a considerable wider class of non-linear operators and give a better relation between the non-linearity and the growth of the solutions at infinity. In Section 3 we insert several consequences containing specific assumptions on the situation of the domain, on given functions f^k occurring in (1) and on the growth of the solutions. For example, as one of the corollaries we obtain a differential inequalities result, in an unbounded domain, under the assumption that functions $f^k(t, x, z, q, r)$ are Hölder continuous in r, q and Lipschitz continuous in z, the solutions in question belonging to a suitably chosen class of fast increasing functions, whereas theorems of this kind known in the literature (see e.g. [1]–[3], [9], [11]) contain the Lipschitz conditions with respect to all the variables z, q, r.

1. Notation and definitions. Let D be an arbitrary domain of the $(n+1)$-dimensional space $\boldsymbol{R}^{n+1}$ of points $(t, x) = (t, x_1, \ldots, x_n)$, contained in the strip $T_0 < t < T$. We allow $T = +\infty$, $T_0 = -\infty$. Let Γ be that part of the boundary ∂D of D which does not contain points lying on the plane $t = T$ if $T < +\infty$, $\Gamma = \partial D$ if $T = +\infty$, and $\Gamma = \emptyset$ if $D = \boldsymbol{R}^{n+1}$.

We shall treat the system of operators

$$(1)\qquad F^k(u) = f^k(t, x, u, u_x^k, u_{xx}^k) - u_t^k \quad (k = 1, \ldots, N),$$

where $u = (u^1, \ldots, u^N)$, $u_x^k = (u_{x_1}^k, \ldots, u_{x_n}^k)$ and u_{xx}^k is the matrix $(u_{x_i x_j}^k)_{i,j=1}^n$.

A function $u(t, x)$ will be said to be *regular* in an open set Ω if it is continuous in the closure $\bar{\Omega}$ and has continuous derivatives u_t, u_x, u_{xx} in Ω.

Throughout the paper we assume (without repetition) that the functions $f^k(t, x, z, q, r)$ are defined for $(t, x) \in D$, arbitrary $z = (z^1, \ldots, z^N)$, $q = (q_1, \ldots, q_n)$, $r = (r_{ij})$ and each f^k is non-decreasing in z^j, $j \neq k$.

An operator F^k of form (1) will be called *parabolic* in a set Ω with respect to a vector-function $u = (u^1, \ldots, u^N)$ with the regular component u^k in Ω, if for any symmetric matrix r and $(t, x) \in \Omega$ we have (cf. [9])

$$f^k\big(t, x, u(t, x), u_x^k(t, x), u_{xx}^k(t, x) + r\big) - \\ - f^k\big(t, x, u(t, x), u_x^k(t, x), u_{xx}^k(t, x)\big) \begin{cases} \geqslant 0 & \text{if } r \geqslant 0, \\ \leqslant 0 & \text{if } r \leqslant 0, \end{cases}$$

where inequality $r \geqslant 0$ $(r \leqslant 0)$ means that the quadratic form with the matrix r is semi-definite positive (negative).

Let $Q_\varrho = (\|x\| < \varrho) \cap (-\varrho < t < T)$, where $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$, $\varrho > 0$. We define C'_ϱ to be the closure of the union of the side surface and the base of the cylinder Q_ϱ, and we set $C_\varrho = \bar{D} \cap C'_\varrho$.

2. Comparison theorems.

THEOREM 1. *Let $u = (u^1, \ldots, u^N)$, $v = (v^1, \ldots, v^N)$ be continuous vector-functions in $\bar{D}$, $u \leqslant v$ (1) on Γ. Assume that there exists a vector-function*

$$h(t, x; \varepsilon) = \big(h^1(t, x; \varepsilon), \ldots, h^N(t, x; \varepsilon)\big) \colon \bar{D} \times (0, \varepsilon_0) \to \mathbf{R}^N, \quad \varepsilon_0 > 0,$$

with continuous and positive components in this set, having continuous derivatives $h_t, h_x, h_{xx}, h_\varepsilon$ in $D \times (0, \varepsilon_0)$, $h_\varepsilon > 0$, such that $h \to 0$ as $\varepsilon \to 0$, uniformly with respect to (t, x) in bounded subsets of D and that for any fixed $\varepsilon \in (0, \varepsilon_0)$ we have

$$(2) \qquad \liminf_{\varrho \to \infty} \left\{ \sup_{C_\varrho} \frac{u^k - v^k}{h^k} \right\} < 1 \ (^2) \qquad (k = 1, \ldots, N).$$

Let

$$\Omega^l = \{(t, x) \in D \colon u^l(t, x) > v^l(t, x)\}.$$

We assume that for every fixed k and $\varepsilon \in (0, \varepsilon_0)$, in the set Ω^k the components u^k, v^k are regular and either

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(1) Inequality between vectors is meant as the simultaneous inequalities between the components.

(2) If for some ϱ the set C_ϱ is void, then we set $\sup_{C_\varrho} f = 0$. Thus if D is bounded, (2) is automatically satisfied.

(a) $F^k(u) \geqslant F^k(v+h)$ *and* F^k *is parabolic with respect to* u *or* $v+h$,

or

(b) $F^k(u-h) \geqslant F^k(v)$ *and* F^k *is parabolic with respect to* $u-h$ *or* v.

Under these assumptions we have $u \leqslant v$ *in* D.

Proof. We confine ourselves to the proof in case (a) with F^k parabolic with respect to u. Let $(\bar{t}, \bar{x})$ be fixed and $\varepsilon \in (0, \varepsilon_0/4)$. By (2) there is $\varrho_0 > \|\bar{x}\| + |\bar{t}|$ such that

$$(3) \qquad u(t, x) - v(t, x) - h(t, x; \varepsilon) < 0 \quad \text{for } (t, x) \in C_{\varrho_0}.$$

We define $\theta(t) = 3 + (2/\pi)\arctan t$ and

$$(4) \qquad w^k(t, x) = u^k(t, x) - v^k(t, x) - h^k(t, x; \varepsilon\theta(t)).$$

Since $h^k(t, x; \varepsilon\theta(t)) \geqslant h^k(t, x; \varepsilon)$, we have $w^k < 0$ on C_{ϱ_0}. Consequently $w^k < 0$ $(k = 1, \ldots, N)$ on the whole part of the boundary of $D_{\varrho_0} := D \cap Q_{\varrho_0}$, where $t < T$. We shall show that these inequalities hold in D_{ϱ_0}. Suppose the contrary. Then there would exist an index k_0 and a point $(t^0, x^0) \in \Omega^{k_0} \cap Q_{\varrho_0}$ such that

$$(5) \qquad w^{k_0}(t^0, x^0) = 0, \quad w^k(t^0, x^0) \leqslant 0, \quad w^k(t, x) < 0$$
$$\text{for } (t, x) \in D_{\varrho_0} \cap [-\varrho_0, t^0)\ (k = 1, \ldots, N).$$

This implies

$$(6) \qquad w_x^{k_0}(t^0, x^0) = 0, \quad w_{xx}^{k_0}(t^0, x^0) \leqslant 0, \quad w_t^{k_0}(t^0, x^0) \geqslant 0.$$

Since

$$\frac{\partial}{\partial t} h^k(t, x; \varepsilon\theta(t)) > h_t^k(t, x; \varepsilon\theta(t)),$$

it follows that

$$(7) \qquad u_t^{k_0}(t^0, x^0) - v_t^{k_0}(t^0, x^0) - h_t^{k_0}(t^0, x^0; \tilde{\varepsilon}) > 0,$$

where $\tilde{\varepsilon} = \varepsilon\theta(t^0) \in (0, \varepsilon_0)$. On the other hand, by inequality (a) with ε replaced by $\tilde{\varepsilon}$, the left-hand side of (7) is less than or equal to

$$(8) \qquad \{f^{k_0}(t^0, x^0, u, u_x^{k_0}, u_{xx}^{k_0}) - f^{k_0}(t^0, x^0, u, u_x^{k_0}, u_{xx}^{k_0} - w_{xx}^{k_0})\} +$$
$$+ \{f^{k_0}(t^0, x^0, u, u_x^{k_0}, u_{xx}^{k_0} - w_{xx}^{k_0}) - f^{k_0}(t^0, x^0, v+h, v_x^{k_0} + h_x^{k_0}, v_{xx}^{k_0} + h_{xx}^{k_0})\}.$$

Now, the difference in the first bracket in (8) is, by (6) and the parabolicity, non-positive. By (4)–(6) and the monotonicity of f^{k_0}, the difference in the second bracket is non-positive also. This contradiction shows that $w^k < 0$ in D_{ϱ_0} and, in particular, at $(\bar{t}, \bar{x})$ $(k = 1, \ldots, N)$. Notice that, by (2), for $\varepsilon' < \varepsilon$ one can always find $\varrho_0' \geqslant \varrho_0$ such that (3) holds for $(t, x) \in C_{\varrho_0'}$. Repeating the above argument we deduce that w^k defined by (4) with ε substituted by ε' is also negative at $(\bar{t}, \bar{x})$. Thus letting $\varepsilon \to 0$ we obtain $u^k \leqslant v^k$ at $(\bar{t}, \bar{x})$ and the proof is complete.

For a vector $a = (a_1, \ldots, a_n)$ we shall write $|a| := (|a_1|, \ldots, |a_n|)$. Similar notation will be used for matrices. Let

$$G^k(h) := g^k(t, x, h, |h_x^k|, |h_{xx}^k|) - h_t^k \quad (k = 1, \ldots, N), \tag{9}$$

where $g^k(t, x, z, q, r)$ are functions defined for $(t, x) \in D$, vectors $z = (z^1, \ldots, z^N) \geqslant 0$, $q = (q_1, \ldots, q_n) \geqslant 0$ and matrices $r = (r_{ij})_{i,j=1}^n$ with non-negative elements.

The following theorem is a consequence of Theorem 1.

THEOREM 2. *Let u, v be regular vector-functions such that $u \leqslant v$ on Γ and $F^k(u) \geqslant F^k(v)$ in D $(k = 1, \ldots, N)$. Each F^k is assumed to be parabolic with respect to u or v in D. Suppose that f^k satisfy the inequalities*

$$\begin{aligned}[f^k(t, x, z, q, r) - f^k(t, x, \tilde{z}, \tilde{q}, \tilde{r})]\operatorname{sgn}(z^k - \tilde{z}^k) \\ \leqslant g^k(t, x, |z - \tilde{z}|, |q - \tilde{q}|, |r - \tilde{r}|) \quad (k = 1, \ldots, N).\end{aligned} \tag{10}$$

Moreover, we assume that there is a vector-function $h(t, x; \varepsilon)$ of the same property as in Theorem 1 such that for any fixed $\varepsilon \in (0, \varepsilon_0)$

$$G^k(h) \leqslant 0 \quad \text{in } D \ (k = 1, \ldots, N) \tag{11}$$

and (2) *is satisfied. Then $u \leqslant v$ in D.*

Proof. It is easily seen that

$$F^k(u) - F^k(v+h) \geqslant F^k(v) - F^k(v+h) \geqslant -G^k(h) \geqslant 0.$$

Thus if F^k is parabolic with respect to u, all the assumptions of Theorem 1 in case (a) are satisfied. If F^k is parabolic with respect to v the proof is similarly reduced to case (b) of Theorem 1.

The following two theorems deal with the uniqueness of solutions of the problem:

$$F^k(u) = 0 \quad \text{in } D \ (k = 1, \ldots, N), \qquad u(t, x) = \psi(t, x) \quad \text{on } \Gamma, \tag{12}$$

ψ being a given continuous vector-function.

THEOREM 3. *Let u be a regular solution of* (12). *Assume that there exist two vector-functions $h(t, x; \varepsilon)$, $\bar{h}(t, x; \varepsilon)$, having the same properties as function h in Theorem 1, such that for any fixed $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\lim_{\varrho \to \infty}\left\{\sup_{C_\varrho} \frac{|u^k|}{\min(h^k, \bar{h}^k)}\right\} = 0 \quad (k = 1, \ldots, N) \tag{13}$$

as well as

$$f^k(t, x, u+h, u_x^k + h_x^k, u_{xx}^k + h_{xx}^k) - f^k(t, x, u, u_x^k, u_{xx}^k) \leqslant h_t^k \tag{14}$$

and

$$f^k(t, x, u, u_x^k, u_{xx}^k) - f^k(t, x, u - \bar{h}, u_x^k - \bar{h}_x^k, u_{xx}^k - \bar{h}_{xx}^k) \leqslant \bar{h}_t^k \tag{15}$$

$(k = 1, \ldots, N)$, $(t, x) \in D$. *If, moreover, operators F^k are parabolic in D*

with respect to each solution of (12), *then* $u(t, x)$ *is the only solution in the class of all these functions* u *that satisfy condition* (13).

Proof. Let v be another solution of (12), satisfying (13). By (12), system (14) is equivalent to $F^k(u+h) \leqslant 0$ $(k = 1, \ldots, N)$. Since $F^k(v) = 0$, case (a) of Theorem 1 (with u, v interchanged) implies $v \leqslant u$ in D. Similarly, applying case (b) of Theorem 1 we get $u \leqslant v$. ■

THEOREM 4. *Let* u, v *be regular solutions of* (12) *and let* F^k *be parabolic with respect to* u *or* v *in* D. *Suppose functions* $f^k(t, x, z, q, r)$ *satisfy inequalities* (10). *Now we assume that each* f^k *or each* g^k *is non-decreasing in* z^j, $j \neq k$. *Further we assume that there exists a vector-function* $h(t, x;\ \varepsilon)$, *with the same properties as in Theorem* 1, *such that for any fixed* $\varepsilon \in (0, \varepsilon_0)$,

$$\liminf_{\varrho\to\infty}\left\{\sup_{C_\varrho}\frac{|u^k - v^k|}{h^k}\right\} < 1$$

and (11) *is satisfied. Then* $u = v$ *in* D.

This theorem, in the case where the functions f^k satisfy the monotonicity condition, follows immediatelly from Theorem 2. In the case where the functions g^k satisfy this condition the argument is similar to that used in the proof of Theorem 1. The reasoning concerning w^k defined by (4) is now carried over in respect to the functions

$$w^k(t, x) \overset{\mathrm{df}}{=} |u^k(t, x) - v^k(t, x)| - h^k\big(t, x;\ \varepsilon\theta(t)\big).$$

We omit the details.

Remark 1. If in assumptions (a) and (b) of Theorem 1 the weak inequality sign "$\geqslant$" is replaced by strong one "$>$", then the requirement that $h^k_\varepsilon > 0$ and that the derivative exists is superfluous. This can be shown by taking $\theta(t) \equiv 1$ and repeating the argument (now simplified) given in the proof of Theorem 1. Similar observations concern Theorems 2–4. In particular, replacing in (11) the sign "$\leqslant$" by "$<$", one can omit in Theorems 2 and 4 the assumption that $h^k_\varepsilon > 0$ and even that the derivative exists.

Remark 2. If the function h occurring in our theorems does not exist in the whole domain D, however, the interval (T_0, T) can be divided into a countable set of intervals $(T_0, t_1), \ldots, (t_m, t_{m+1}), \ldots$ such that for any set $D \cap (t_m, t_{m+1})$ there is a function h_m possessing required properties, then, evidently, the assertions of the theorems remain valid for the whole domain D.

3. Consequences. Now we discuss some consequences of the previous theorems. For the time being we assume $T_0 > -\infty$ but we do not assume that D is spatially bounded.

We say that $u = (u^1, \ldots, u^N)$ belongs to the class E if for any $\delta > 0$,

$$\liminf_{\|x\|\to\infty}\{u^k(t,x)\exp(-\delta\|x\|^2)\} \leqslant 0 \qquad (k = 1, \ldots, N) \tag{16}$$

uniformly in t, while we say $u \in K$ if there is integer $p > 0$ such that

$$u^k \leqslant e_p(\|x\|) \quad \text{ in } D \ (k = 1, \ldots, N),$$

where $e_p(s)$ is the function defined as follows

$$e_1(s) = e^s, \quad e_\nu(s) = e_1\big(e_{\nu-1}(s)\big) \quad \text{ for } \nu \geqslant 2,\ s \geqslant 0.$$

Notice that the class K does not include all the functions defined in D. For instance the function $u\colon D \to \mathbf{R}^1$ such that $u(t,x)|_{\|x\|=p} = e_p(p)$ for $p = 1, 2, \ldots$ does not belong to K.

PROPOSITION 1. *Let u, v be vector-functions regular in D ($T_0 > -\infty$), satisfying: $u \leqslant v$ on Γ, $F^k(u) \geqslant F^k(v)$ in D ($k = 1, \ldots, N$) and $u - v \in E$. Let operators F^k be parabolic with respect to u or v. Moreover, we assume that f^k satisfy* (10) *with*

$$g^k(t,x,z,q,r) \equiv A\sum_{i,j=1}^{n}\max(|r_{ij}|^a, |r_{ij}|) + B\sum_{i=1}^{n}\max(|q_i|^a, |q_i|) + \sum_{l=1}^{N} C_l^k|z^k|, \tag{17}$$

for some constants A, B, C_l^k, $0 < a < 1$. Then $u \leqslant v$ in D.

Proof. Let us write $h^k = \varepsilon\tilde{h}$, where

$$\tilde{h} = \exp\{(1+t)(\varepsilon^\eta\|x\|^2+1)+(C+1)t\}, \quad 0 < \varepsilon < 1/2,$$

$C = \max\limits_k \sum\limits_{l=1}^{n} C_l^k$ and η is a positive constant to be determined later on. Since $u - v \in E$, (2) holds true. It suffices to show (11) with g^k defined by (17) and to apply Theorem 2. We have

$$|h_{x_i}^k| = \varepsilon^{1+\eta}\tilde{h}(1+t)2|x_i| \leqslant \varepsilon^{1+\eta/2}\tilde{h}(1+T)(\varepsilon^\eta\|x\|^2+1).$$

Similarly

$$|h_{x_ix_j}^k| \leqslant 4\varepsilon^{1+\eta}\tilde{h}(1+T)^2(\varepsilon^\eta\|x\|^2+1).$$

Hence one easily derives

$$G^k(h) \leqslant h^k\{(\varepsilon^\eta\|x\|^2+1)[n^2A(1+T)^2 2^{3-(1+\eta)a} + nB(1+T)2^{1-(1+\eta/2)a} - 1] - 1\}.$$

Now it is seen that taking η large enough implies (11).

The result stated in Proposition 1 is known in the literature in the case where the functions f^k satisfy the Lipschitz condition (i.e. $a = 1$). In this case the class E can be extended by requiring (16) to hold only for large δ [1], [9], [11]. Choosing suitable h, one can easily show that the latter case is also contained in Theorem 2.

is non-empty. Let $\tilde{h} = \varepsilon h$, $\varepsilon > 0$. In Ω^k we obtain $L^k(|u|) \geqslant F^k(u)\operatorname{sgn} u^k = 0 \geqslant L^k(\tilde{h})$. Applying Theorem 1 with F^k, u, v, h substituted by L^k, $|u|, 0, \tilde{h}$ respectively, we deduce $|u| \leqslant 0$ in D.

In the case $N = 1$, $T_0 > -\infty$ Proposition 4 was established by M. Picone [4], [5]. Functions $h(t, x)$ satisfying inequalities of type $L^k(h) \leqslant 0$ depend on the growth of the coefficients as $\|x\| \to \infty$ and have been constructed for various purposes in the literature (see e.g. [2], [4], [8]).

As a final example illustrating the applicability of Theorem 1 we prove a maximum principle for a special strongly non-linear equation of the form

$$F(u) \equiv \sum_{i,j=1}^{n} a_{ij} |u_{x_i x_j}|^{\alpha_{ij}} \operatorname{sgn} u_{x_i x_j} + \sum_{i=1}^{n} b_i |u_{x_i}|^{\beta_i} + cu - u_t = 0. \tag{20}$$

Let D be an unbounded domain contained in a strip $0 < t < T$ ($\leqslant \infty$). We assume that $\alpha_{ij} \geqslant 1$, $\beta_i > 0$ are constants and each α_{ij}, $i \neq j$, is the harmonic mean of α_{ii} and α_{jj}, i.e.

$$\alpha_{ij} = 2\alpha_{ii}\alpha_{jj}/(\alpha_{ii} + \alpha_{jj}). \tag{21}$$

(This is automatically satisfied if, e.g., $a_{ij} = 0$ for $i \neq j$.) Moreover, we assume that functions a_{ij}, b_i are bounded and $c \leqslant 0$ in D as well as

$$2^{1-\alpha_{ii}} a_{ii} a_{ii} \geqslant \sum_{j=1, j\neq i}^{n} |a_{ij}| a_{ij} \quad (i = 1, \ldots, n). \tag{22}$$

Let us denote by Q the class of all functions w such that

$$\liminf_{\varrho \to \infty} \left\{ \sup_{C_\varrho} \frac{w(t, x)}{\sum_{i=1}^{n} (x_i^2 + 1)^{\lambda_i}} \right\} \leqslant 0, \tag{23}$$

where $\lambda_j = \min[\alpha_{jj}/(\alpha_{jj} - 1),\ \beta_j/2(\beta_j - 1)]$ for these j for which $\alpha_{jj} > 1$ and $\beta_j > 1$; $\lambda_k = \alpha_{kk}/(\alpha_{kk} - 1)$ for these k for which $\alpha_{kk} > 1$ and $\beta_k \leqslant 1$; $\lambda_l = \beta_l/2(\beta_l - 1)$ for such l that $\alpha_{ll} = 1$, $\beta_l > 1$; while λ_m is any number, say $> 1/2$, for m such that $\alpha_{mm} = 1$, $\beta_m \leqslant 1$. Thus in each case $\lambda_i > 1/2$ and depends on the non-linearity.

Let u be a regular solution to (20), such that $u \in Q$ and $u \leqslant M = \text{const} \geqslant 0$ on Γ. Then $u \leqslant M$ in D.

To prove this fact we use Theorem 1 with $N = 1$, $v = M$. We choose

$$h(t, x;\ \varepsilon) = \varepsilon(1 - \gamma t)^{-1} \sum_{i=1}^{n} (\varepsilon^{\eta} x_i^2 + 1)^{\lambda_i}, \qquad \varepsilon \in (0, 1),$$

where

$$\eta = \max_i\{\max[2(1 - \beta_i)/\beta_i,\ 0]\}, \qquad \gamma > A \max_i[(8\lambda_i^2)^{\alpha_{ii}}] + B \max_i[(4\lambda_i)^{\beta_i}],$$

$A = \sum_{i=1}^{n} a_{ii}$, $B = \sum_{i=1}^{n} |b_i|$. We first show that $F(M+h) < 0$ in $D_1 = D \cap \big(0 < t < (2\gamma)^{-1}\big)$. Indeed, we have

$$|h_{x_i}|^{\beta_i} \leqslant \varepsilon^{(1+\eta/2)\beta_i}(4\lambda_i)^{\beta_i}(\varepsilon^\eta x_i^2+1)^{(\lambda_i-1/2)\beta_i} \leqslant \varepsilon(4\lambda_i)^{\beta_i}(\varepsilon^\eta x_i^2+1)^{\lambda_i}.$$

Similarly

$$|h_{x_ix_i}|^{a_{ii}} \leqslant [\varepsilon^{1+\eta} 8\lambda_i^2(\varepsilon^\eta x_i^2+1)^{\lambda_i-1}]^{a_{ii}} \leqslant \varepsilon(8\lambda_i^2)^{a_{ii}}(\varepsilon^\eta x_i^2+1)^{\lambda_i}.$$

Further, $h_{x_ix_j} = 0$ for $i \neq j$. Hence one obtains $F(M+h) < 0$. It is easily seen that, by (23), condition (2) with $u^k = u$, $v^k = M$, $h^k = h$ is satisfied too.

In order to prove the parabolicity of F with respect to $M+h$ let us note first that if a matrix $r = (r_{ij})$ is semi-definite positive or negative, then $r_{ij}^2 \leqslant r_{ii}r_{jj}$. Hence, by (21) and Young inequality,

$$|r_{ij}|^{a_{ij}} \leqslant |r_{ii}|^{a_{ii}} a_{jj}/(a_{ii}+a_{jj}) + |r_{jj}|^{a_{jj}} a_{ii}/(a_{ii}+a_{jj}). \tag{24}$$

Therefore, if $r \geqslant 0$,

$$\varphi(r) := \sum_{i,j=1}^{n} a_{ij}|r_{ij}|^{a_{ij}} \operatorname{sgn} r_{ij} \geqslant \sum_{i=1}^{n}\left[a_{ii} - 2\sum_{j=1, j\neq i}^{n} |a_{ij}| \frac{a_{jj}}{a_{ii}+a_{jj}}\right]|r_{ii}|^{a_{ii}}, \tag{25}$$

and, by (22), $\varphi(r) \geqslant 0$. Similarly $r \leqslant 0 \Rightarrow \varphi(r) \leqslant 0$. Now let

$$\Phi(r) := \sum_{i=1}^{n} a_{ii}\big(|r_{ii}+h_{x_ix_i}|^{a_{ii}}\operatorname{sgn}(r_{ii}+h_{x_ix_i}) - |h_{x_ix_i}|^{a_{ii}}\operatorname{sgn} h_{x_ix_i}\big) + \\ + \sum_{i,j=1, i\neq j}^{n} a_{ij}|r_{ij}|^{a_{ij}}\operatorname{sgn} r_{ij}.$$

We have to show that $r \geqslant 0 \Rightarrow \Phi(r) \geqslant 0$ and $r \leqslant 0 \Rightarrow \Phi(r) \leqslant 0$. Note that $h_{x_ix_i} > 0$. In case $r \geqslant 0$ we apply the inequality $(a+b)^p \geqslant a^p + b^p$ $(a \geqslant 0,\ b \geqslant 0,\ p \geqslant 1)$ which implies $\Phi(r) \geqslant \varphi(r) \geqslant 0$. If $r \leqslant 0$ we consider two cases: $-h_{x_ix_i} \leqslant r_{ii} < 0$ and $r_{ii} < -h_{x_ix_i}$. In the first case we make use of the inequality $(a-b)^p \leqslant a^p - b^p$ $(a \geqslant b \geqslant 0,\ p \geqslant 1)$, whence

$$|r_{ii}+h_{x_ix_i}|^{a_{ii}} \leqslant -|r_{ii}|^{a_{ii}} + h_{x_ix_i}^{a_{ii}}.$$

Thus we obtain $\Phi(r) \leqslant \varphi(r) \leqslant 0$. In the second case, we use the inequality $(a-b)^p \geqslant 2^{1-p}a^p - b^p$ $(a \geqslant b \geqslant 0,\ p \geqslant 1)$ and similarly derive $\Phi(r) \leqslant 0$. Thus the proof for domain D_1 is complete. For domain D it follows from Remark 2.

Similarly one can prove the minimum principle.

In cases $a_{ij} \leqslant 1$, $\beta_i \leqslant 1$ one can use Propositions 1 or 2 and prove the maximum–minimum principle and the uniqueness for solutions of (20) belonging to the corresponding classes E or K. Now the factor

$2^{1-a_{ii}}$ in (22) should be omitted. If every $a_{ij} = a = \text{const} > 0$, then, instead of (22), the following less restrictive condition can be assumed:

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqslant (1-2^{1-\bar{a}})\sum_{i=1}^{n} a_{ii}\xi_i^2, \qquad \xi \in \boldsymbol{R}^n, \; \bar{a} = \max(a, 1).$$

The maximum principle itself, in unbounded domains, for strongly non-linear differential and differential-functional inequalities of parabolic type has been investigated extensively by R. Redheffer and W. Walter in papers [6], [7]. However, their results are concerned with bounded or slowly growing solutions and do not contain the examples given above.

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