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Remarks on algebraic concomitants of the Riemann–Christoffel curvature tensor in a three-dimensional space

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Introduction. In the paper [1] the problem of determination of all regular algebraic concomitants which are densities in the sense of Weyl, of the Riemann–Christoffel curvature tensor in a three-dimensional space has been solved. The authors apply a method of differential equations.

In the present note we show how one can solve this problem by a tensorial method which does not require any assumptions of regularity of the unknown function.

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Let $R_{ij,kl}$ be a Riemann–Christoffel curvature tensor of type (0,4) in a three-dimensional Riemannian (or pseudo-Riemannian) space V_3. It fulfils the identities

$$R_{ij,kl} = R_{kl,ij}, \tag{1}$$

$$R_{ij,kl} = -R_{ji,kl}, \tag{2}$$

$$i, j, k, l = 1, 2, 3$$

and the first Bianchi identity

$$R_{[ij,k]l} = 0,$$

which in our case of the three-dimensional space V_3, as is well known, is a simple consequence of (1) and (2). So (1) and (2) are the only symmetry properties of $R_{ij,kl}$.

Let ε^{ijk}, ε_{ijk} be the Ricci symbols in V_3 which are tensor densities of weights -1 and 1 respectively. We have

$$\varepsilon^{mjk}\varepsilon_{mpq} = \varepsilon^{jk}_{pq}, \tag{3}$$

where ε^{jk}_{pq} is the alternating operator equal to $\delta^{[j}_p\delta^{k]}_q$. Let us put

$$\tilde{R}^{ij} \overset{\text{df}}{=} R_{kl,pq}\varepsilon^{ikl}\varepsilon^{jpq}. \tag{4}$$

Conversely in view of (3) and (1), (2) we obtain

$$R_{ij,kl} = \tfrac{1}{4}\tilde{R}^{pq}\varepsilon_{pij}\varepsilon_{qkl}. \tag{5}$$

$\tilde{R}^{ij}$ is a tensor density of type $(2, 0)$ and of weight -2 submitted to the transformation rule

(6) $$\tilde{R}^{i'j'} = J^2 A^{i'}_i A^{j'}_j \tilde{R}^{ij}.$$

By (1) and (4) we have
$$\tilde{R}^{ij} = \tilde{R}^{ji}.$$

(5) shows that the correspondence (4) between $R_{ij,kl}$ and $\tilde{R}^{ij}$ is $1-1$ and invariant. Thus, these two geometric objects are equivalent in the sense of Wagner or Gołąb. Consequently, the problem of determination of algebraic concomitants of $R_{ij,kl}$ is equivalent to that for $\tilde{R}^{ij}$. For instance, if we are looking for all the concomitants of $R_{ij,kl}$ which are, densities, we may compute them from $\tilde{R}^{ij}$ and then, using relations (4), represent them as functions of $R_{ij,kl}$.

In order to find a general form for the density concomitants σ (of a fixed type) of $\tilde{R}^{ij}$ let us recall that if $\sigma(\tilde{R}^{ij}) \neq 0$ is one, then any other density concomitant χ (of this type) is of the form

(3) $$\chi = \varphi(\tilde{R}^{ij})\sigma,$$

wherein φ is a scalar concomitant of $\tilde{R}^{ij}$ (i.e. an absolute invariant of the quantities $\tilde{R}^{ij}$) ([1]).

By (6), any scalar concomitant of the symmetric tensor density $\tilde{R}^{ij}$ must fulfil the functional equation

(8) $$\varphi(J^2 A^{i'}_i A^{j'}_j \tilde{R}^{ij}) = \varphi(\tilde{R}^{ij})$$

for any matrix $[A^{i'}_i]$ such that $J = \det(A^{i'}_i) \neq 0$. In matrix notation (8) takes the form

(9) $$\varphi(J^2 A\tilde{R}A^T) = \varphi(\tilde{R}); \quad A = [A^{i'}_i], \tilde{R} = [\tilde{R}^{ij}],$$

A^T – the transposed matrix A.

Putting
$$B = JA^T; \quad \det B = J^4, \ A = (\det B)^{-3/4} B^T$$
we get

(10) $$\varphi(B^T \tilde{R} B) = \varphi(\tilde{R})$$

for any matrix B with a positive determinant ([2]). (10) may be treated as an functional equation for the invariants of the quadratic form

(11) $$\tilde{R}^{ij}\xi_i \xi_j$$

([1]) Because χ/σ is a scalar.

([2]) The positiveness of $\det B$ does not influence the solutions of (10) because $(-B)^T\tilde{R}(-B) = B^T\tilde{R}B$ and if $\det B < 0$, then $\det(-B) > 0$.

and the only solutions are functions of the rank r and the signature s of this form, which we refer to the quantities $\tilde{R}^{ij}$.

Thus we have shown that

$$\varphi(\tilde{R}^{ij}) = f(r, s), \tag{12}$$

where f is an arbitrary function. If $\det\tilde{R} \neq 0$, then $r = 3$ and r may be omitted in (12).

Since the tensor $R_{ij,kl}$ has an even number of indices, it cannot admit any other types of density concomitants except the Weyl densities [4]

$$\sigma' = |J|^{-w}\sigma; \quad w \text{ is the weight of } \sigma. \tag{13}$$

For such density concomitants $\sigma = \sigma(\tilde{R}^{ij})$ we get the following matrix functional equation:

$$\sigma(J^2 A\tilde{R}A^T) = |J|^{-w}\sigma(\tilde{R}).$$

As before, this equation is equivalent to the following:

$$\sigma(B^T\tilde{R}B) = |B|^{-w/4}\sigma(\tilde{R}). \tag{14}$$

It has been shown in [4] that equation of type (14) has no non-vanishing solution if $\det\tilde{R} = 0$ ([3]).

On the other hand, if $\det R \neq 0$, then by (6) $\sigma = \det(\tilde{R}^{ij})$ is a non-vanishing Weyl density of weight -4, i.e.

$$\det(\tilde{R}^{i'j'}) = |J|^4\det(\tilde{R}^{ij}).$$

Now we come to the conclusion that every density concomitants χ of a weight w of $\tilde{R}^{ij}$ is of the form

$$\chi = f(s)|\det(\tilde{R}^{ij})|^{-w/4}, \tag{15}$$

where s is the signature of the form (11) and f an arbitrary function.

One can compute $\det(\tilde{R}^{ij})$ in terms of $R_{ij,kl}$. We have

$$\det(\tilde{R}^{ij}) = \frac{1}{3\,!}\tilde{R}^{i_1j_1}\tilde{R}^{i_2j_2}\tilde{R}^{i_3j_3}\varepsilon_{i_1i_2i_3}\varepsilon_{j_1j_2j_3}$$

and in consequence, by using (3) and (4), we get

$$\det(\tilde{R}^{ij}) = \tfrac{2}{3}R_{i_1j_1,k_1l_1}R_{i_2j_2,k_2l_2}R_{i_3j_3,k_3l_3}\varepsilon^{i_1i_2i_3}\varepsilon^{j_1j_2j_3}\varepsilon^{k_1k_2k_3}\varepsilon^{l_1l_2l_3}.$$

Remark 1. Formula (15) provides the densities which are concomitants of the tensor $R_{ij,kl}$ itself. But we may ask for densities as combined concomitants of the pair

$$\{R_{ij,kl}, g_{ij}\}, \tag{16}$$

([3]) This is equivalent to the fact that a singular quadratic form (11) has no relative invariant.

g_{ij} being the metric tensor in V_3. Since each V_n is (locally) conformally an Euclidean one, we have

$$R_{ij,kl} = -\frac{1}{n-2}(R_{ik}g_{jl}+R_{jl}g_{ik}-R_{il}g_{jk}-R_{jk}g_{il}) - \\ -\frac{K}{(n-1)(n-2)}(g_{jk}g_{il}-g_{jl}g_{ik}),$$

where R_{ij} is the Ricci tensor and K the scalar curvature, and consequently the pair (16) is equivalent to the pair

$$\{R_{ij}, g_{ij}\}. \tag{17}$$

(17) consists of two covariant symmetric tensors of type $(0, 2)$ and the algebraic concomitants (including densities) of such a pair have been found in [4].

Remark 2. The quantities $\tilde{R}^{ij}$ and R_{ij} are connected with each other by the formula

$$R_{ij} = -\tfrac{1}{4}\varepsilon_{ikl}\varepsilon_{jpq}g^{kp}\tilde{R}^{lq}.$$

Remark 3. By replacing each pair (i, j) of skew-symmetric indices by one single index a such that (a, i, j) is an even permutation of $(1, 2, 3)$ we get from the essential components of $R_{ij,kl}$ a 3×3 matrix $[R_{ab}]$ by putting $R_{ab} = R_{ij,kl}$ (see [2], [3]). From (4) it follows that

$$[\tilde{R}^{ij}] = 4[R_{ab}],$$

which may be useful for computing $\det(\tilde{R}^{ij})$.

Remark 4. The factor $f(s)$ in (15) does not occur in the result obtained in [1], because s is not a differentiable function of components $R_{ij,kl}$.

Remark 5. The above method of determining algebraic concomitants of density type for $R_{ij,kl}$ in a three-dimensional space V_3 may be applied, after some modifications, to the problem of determining such concomitants of a tensor

$$R_{i_1\ldots i_{n-1}, j_1,\ldots,j_{n-1}}, \quad i_1, \ldots, i_{n-1}, j_1, \ldots, j_{n-1} = 1, \ldots, n$$

for n odd, arbitrary and $\geqslant 3$ [4], which has the following symmetry properties

$$R_{j_1\ldots j_{n-1}, i_1,\ldots,i_{n-1}} = R_{i_1,\ldots,i_{n-1}, j_1,\ldots,j_{n-1}},$$
$$R_{[i_1,\ldots,i_{n-1}], j_1,\ldots,j_{n-1}} = 0.$$

[4] In the case $n = 3$ no modifications are needed.

References

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