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On product of generalized Appell polynomials

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Abstract. In this paper we investigate the product of arbitrary number of generalized Appell polynomials in terms of similar polynomials by studying the generating function of the corresponding coefficients. Some interesting special cases are also mentioned.

1. Erdélyi [3] studied the problem of expressing the product of Laguerre ([1]) polynomials as a series of Laguerre polynomials, i.e.

$$L_{m_1}^{(a_1)}(c_1x)L_{m_2}^{(a_2)}(c_2x)\ldots L_{m_k}^{(a_k)}(c_kx) = \sum_{s=0}^{m_1+\ldots+m_k} B_s^{(m_1,m_2,\ldots,m_k)} L_s^{(\beta)}(x), \tag{1}$$

by expressing the coefficients $B_s^{(m_1,m_2,\ldots,m_k)}$ in terms of Lauricella's hypergeometric function F_A of $(k+1)$ variables. Recently Carlitz [1] obtained the generating function of the coefficients $B_s^{(m_1,m_2,\ldots,m_k)}$ as

$$\sum_{m_1,\ldots,m_k=0}^{\infty} B_s^{(m_1,m_2,\ldots,m_k)} u_1^{m_1}u_2^{m_2}\ldots u_k^{m_k} = \prod_{j=1}^{k}[1-u_j]^{-1-a_j}\left(\sum_{j=1}^{k}\frac{c_ju_j}{1-u_j}\right)^s\Big/\left[1+\sum_{j=1}^{k}\frac{c_ju_j}{1-u_j}\right]^{\beta+s+1}. \tag{2}$$

Carlitz [1] also studied the product of an arbitrary number of Hermite polynomials in a series of Hermite polynomials by obtaining the generating function of the corresponding coefficients.

In this paper we investigate the product of generalized Appell polynomials in terms of similar polynomials by studying the generating function of the corresponding coefficients. In this connection we recall that a polynomial set $\{f_n(x)\}$ is referred here as a generalized Appell polynomial set if it has a generating function of the form:

$$\sum_{n=0}^{\infty}\frac{f_n(x)}{n!}t^n = f(t)\psi[xF(t)], \tag{3}$$

([1]) For notations and definitions see Rainville [4].

where

$$\psi(u) = \sum_{n=0}^{\infty} \gamma_n \frac{u^n}{n!}, \quad \gamma_n \neq 0: n = 0, 1, 2, \ldots,$$

(4)

$$f(t) = \sum_{n=0}^{\infty} f_n t^n, \quad f_0 \neq 0$$

and

$$F(t) = \sum_{n=0}^{\infty} h_n t^{n+1}, \quad h_0 \neq 0.$$

The condition that $\gamma_n \neq 0$ for $n = 0, 1, 2, \ldots$ ensures that the polynomial set $\{f_n(x)\}$ is a simple polynomial set [(4); Theorem 49]. But since $\{x^n\}$ is also a simple polynomial set, we can find constants $\{f'_{p,j}\}$ independent of x (but depending on p) such that [(4); Theorem 53]

$$x^p = \sum_{j=0}^{p} \binom{p}{j} f'_{p,j} f_j(x), \quad p = 0, 1, 2, \ldots \tag{5}$$

Now, let $\{f_{n_j}(x)\}$ $(j = 1, 2, \ldots, k)$ be generalized Appell polynomial sets having generating functions

$$\sum_{n=0}^{\infty} f_{n_j}(x) \frac{t^n}{n!} = f_j(t)\, \psi_j\big(xF_j(t)\big), \quad j = 1, 2, \ldots, k, \tag{6}$$

where for each $j = 1, 2, \ldots, k$, we have

$$\psi_j(u) = \sum_{n=0}^{\infty} \gamma_n^{(j)} \frac{u^n}{n!}, \quad \gamma_n^{(j)} \neq 0 \quad \text{for } n = 0, 1, 2, \ldots,$$

(7)

$$f_j(t) = \sum_{n=0}^{\infty} a_n^{(j)} t^n, \quad a_0^{(j)} \neq 0,$$

$$F_j(t) = \sum_{n=0}^{\infty} b_n^{(j)} t^{n+1}, \quad b_0^{(j)} \neq 0.$$

Further, let us also assume that

$$f_{n_j}(x) = \sum_{r=0}^{n_j} \binom{n_j}{r} f_{n_j,r} x^r \quad (j = 1, 2, \ldots, k;\ n_j = 0, 1, 2, \ldots),$$

$$f_{m_1}(a_1 x) f_{m_2}(a_2 x) \ldots f_{m_k}(a_k x) = \sum_{j=0}^{m_1+m_2+\ldots+m_k} C_j^{(m_1, m_2, \ldots, m_k)} f_j(x).$$

We are justified in these assumptions because the conditions $\gamma_n^{(j)} \neq 0$, $n = 0, 1, 2, \ldots$ and $a_0^{(j)} \neq 0$, $b_0^{(j)} \neq 0$ ensure that $\{f_{n_j}(x)\}$ $(j = 1, 2, \ldots, k)$ are simple polynomial sets.

Therefore, on making use of (1.5) and (1.7), we have

$$f_{m_1}(a_1x)f_{m_2}(a_2x)\dots f_{m_k}(a_kx)$$
$$=\sum_{r_1=0}^{m_1}\sum_{r_2=0}^{m_2}\cdots\sum_{r_k=0}^{m_k}\sum_{j=0}^{r_1+r_2+\dots+r_k}\binom{m_1}{r_1}\binom{m_2}{r_2}\cdots\binom{m_k}{r_k}\binom{r_1+r_2+\dots+r_k}{j}\times$$
$$\times f_{m_1,r_1}f_{m_2,r_2}\cdots f_{m_k,r_k}f'_{r_1+r_2+\dots+r_k,j}\,a_1^{r_1}a_2^{r_2}\dots a_k^{r_k}f_j(x),$$

or

(8) $$C_j^{(m_1,m_2,\dots,m_k)}$$
$$=\sum_{r_1=0}^{m_1}\sum_{r_2=0}^{m_2}\cdots\sum_{r_k=0}^{m_k}\binom{m_1}{r_1}\binom{m_2}{r_2}\cdots\binom{m_k}{r_k}\binom{r_1+r_2+\dots+r_k}{j}f_{m_1,r_1}f_{m_2,r_2}\cdots f_{m_k,r_k}\times$$
$$\times f'_{r_1+r_2+\dots+r_k,j}\,a_1^{r_1}a_2^{r_2}\dots a_k^{r_k}.$$

Hence,

(9) $$\sum_{m_1,m_2,\dots,m_k=0}^{\infty}C_j^{(m_1,m_2,\dots,m_k)}\frac{u_1^{m_1}u_2^{m_2}\dots u_k^{m_k}}{m_1!\,m_2!\dots m_k!}$$
$$=\sum_{r_1,r_2,\dots,r_k=0}^{\infty}\binom{r_1+r_2+\dots+r_k}{j}\frac{(a_1u_1)^{r_1}(a_2u_2)^{r_2}\dots(a_ku_k)^{r_k}}{r_1!\,r_2!\dots r_k!}f'_{r_1+r_2+\dots+r_k,j}\times$$
$$\times\sum_{m_1,m_2,\dots,m_k=0}^{\infty}f_{m_1+r_1,r_1}f_{m_2+r_2,r_2}\cdots f_{m_k+r_k,r_k}\frac{u_1^{m_1}u_2^{m_2}\dots u_k^{m_k}}{m_1!\,m_2!\dots m_k!}.$$

Now since

$$\sum_{m_j=0}^{\infty}f_{m_j}(x)\frac{t^{m_j}}{m_j!}=\sum_{m_j=0}^{\infty}\frac{t^{m_j}}{m_j!}\sum_{r=0}^{m_j}\binom{m_j}{r}f_{m_j,r}x^r=\sum_{r=0}^{\infty}\frac{(xt)^r}{r!}\sum_{m_j=0}^{\infty}f_{m_j+r,r}\frac{t^{m_j}}{m_j!}$$
$$(j=1,2,\dots,k),$$

on the other hand

$$\sum_{m_j=0}^{\infty}f_{m_j}(x)\frac{t^{m_j}}{m_j!}=f_j(t)\psi_j\big(xF_j(t)\big)=f_j(t)\sum_{r=0}^{\infty}\gamma_r^{(j)}x^r\frac{\{F_j(t)\}^r}{r!}$$
$$(j=1,2,\dots,k).$$

Therefore equating the coefficients of x^r in the above expressions, we have

(10) $$t^r\sum_{m_j=0}^{\infty}f_{m_j+r,r}\frac{t^{m_j}}{m_j!}=f_j(t)\gamma_r^{(j)}\{F_j(t)\}^r\qquad(j=1,2,\dots,k).$$

Substituting (1.10) in (1.9)

(11) $$\sum_{m_1,m_2,\dots,m_k=0}^{\infty} C_j^{(m_1,m_2,\dots,m_k)} \frac{u_1^{m_1} u_2^{m_2} \dots u_k^{m_k}}{m_1! m_2! \dots m_k!}$$

$$= f_1(u_1) f_2(u_2) \dots f_k(u_k) \sum_{r_1,r_2,\dots,r_k=0}^{\infty} \binom{r_1+r_2+\dots+r_k}{j} \frac{\{a_1 F_1(u_1)\}^{r_1} \dots \{a_k F_k(u_k)\}^{r_k}}{r_1! r_2! \dots r_k!} \times$$

$$\times \gamma_{r_1}^{(1)} \gamma_{r_2}^{(2)} \dots \gamma_{r_k}^{(k)} f'_{r_1+\dots+r_k,j}$$

$$= f_1(u_1) f_2(u_2) \dots f_k(u_k) \sum_{p=j}^{\infty} \binom{p}{j} f'_{p,j} \sum_{r_1+r_2+\dots+r_k=p} \gamma_{r_1}^{(1)} \gamma_{r_2}^{(2)} \dots \gamma_{r_k}^{(k)} \times$$

$$\times \frac{\{a_1 F_1(u_1)\}^{r_1} \{a_2 F_2(u_2)\}^{r_2} \dots \{a_k F_k(u_k)\}^{r_k}}{r_1! r_2! \dots r_k!}.$$

Now let us assume that

(12)

$$\left[\frac{d^n}{dx^n} [\psi_1(a_1 x) \psi_2(a_2 x) \dots \psi_k(a_k x)]\right]_{x=0} = \left[\frac{d^n}{dx^n} \psi[(a_1 + a_2 + \dots + a_k) x]\right]_{x=0},$$

which is equivalent to assuming that all the coefficients in the Maclaurin series expansion of the function $[\psi_1(a_1 x) \psi_2(a_2 x) \dots \psi_k(a_k x)]$ are equal to the corresponding coefficients in the Maclaurin series expansion of the function $\psi[(a_1 + a_2 + \dots + a_k) x]$, i.e.

(12)′ $$\sum_{r_1+r_2+\dots+r_k=p} \gamma_{r_1}^{(1)} \gamma_{r_2}^{(2)} \dots \gamma_{r_k}^{(k)} \frac{a_1^{r_1} a_2^{r_2} \dots a_k^{r_k}}{r_1! r_2! \dots r_k!} = (a_1 + a_2 + \dots + a_k)^p \frac{\gamma_p}{p!}.$$

Making use of (1.12)′ in (1.11) we have

(13) $$\sum_{m_1,m_2,\dots,m_k=0}^{\infty} C_j^{(m_1,m_2,\dots,m_k)} \frac{u_1^{m_1} u_2^{m_2} \dots u_k^{m_k}}{m_1! m_2! \dots m_k!}$$

$$= f_1(u_1) f_2(u_2) \dots f_k(u_k) \sum_{p=j}^{\infty} \binom{p}{j} f'_{p,j} \gamma_p \frac{[a_1 F_1(u_1) + \dots + a_k F_k(u_k)]^p}{p!}$$

$$= \frac{f_1(u_1) f_2(u_2) \dots f_k(u_k)}{j!} \sum_{l=0}^{\infty} \frac{1}{l!} f'_{l+j,j} \gamma_{l+j} [a_1 F_1(u_1) +$$

$$+ a_2 F_2(u_2) + \dots + a_k F_k(u_k)]^{l+j}.$$

For evaluating the sum on the right-hand side we observe that

$$\psi[xF(t)] = \sum_{p=0}^{\infty} \gamma_p x^p \frac{\{F(t)\}^p}{p!}$$
$$= \sum_{p=0}^{\infty} \gamma_p \frac{\{F(t)\}^p}{p!} \sum_{j=0}^{p} \binom{p}{j} f'_{p,j} f_j(x)$$
$$= \sum_{j=0}^{\infty} \frac{f_j(x)}{j!} \{F(t)\}^j \sum_{n=0}^{\infty} \frac{1}{n!} \gamma_{j+n} f'_{j+n,j} \{F(t)\}^n .$$

But since

$$\psi[xF(t)] = \frac{1}{f(t)} \sum_{j=0}^{\infty} f_j(x) \frac{t^j}{j!},$$

equating the coefficients of $f_j(x)$ in the above expressions, we get

$$\frac{t^j}{f(t)} = \sum_{n=0}^{\infty} \frac{1}{n!} \gamma_{j+n} f'_{j+n,j} \{F(t)\}^{j+n} .$$

Now since $h_0 \neq 0$, the inverse of $F(t)$ exists. Let it be $J(t)$ (i.e. $F(J(t)) = J(F(t)) = t$). Therefore, on replacing t by $J(t)$ in the above expression, we have

$$\text{(14)} \qquad \frac{\{J(t)\}^j}{f(J(t))} = \sum_{n=0}^{\infty} \frac{1}{n!} \gamma_{j+n} f'_{j+n,j} t^{j+n} .$$

Using (1.14) in (1.13) we obtain the following generating function of $C_j^{(m_1,m_2,\ldots,m_k)}$ (subject to (1.12) or (1.12)'):

$$\sum_{m_1,m_2,\ldots,m_k=0}^{\infty} C_j^{(m_1,m_2,\ldots,m_k)} \frac{u_1^{m_1} u_2^{m_2} \ldots u_k^{m_k}}{m_1! m_2! \ldots m_k!}$$
$$= \frac{f_1(u_1) f_2(u_2) \ldots f_k(u_k)}{j!} \frac{\{J(a_1 F_1(u_1) + a_2 F_2(u_2) + \ldots + a_k F_k(u_k)\}^j}{f[J(a_1 F_1(u_1) + a_2 F_2(u_2) + \ldots + a_k F_k(u_k))]} .$$

2. A large number of interesting results can be deduced as special cases of the result proved herein. As an illustration we mention a few of them:

(i) Product of polynomials of the Sheffer A-type zero (for definition and properties of these polynomials see Rainville [4]): If $\{f_n(x)\}$ and $\{f_{n_j}(x)\}$: $j = 1, 2, \ldots, k$, are of the Sheffer A-type zero, it is necessary as well as sufficient that ([4]; Theorem 72)

$$\psi(u) = \exp(u) \quad \text{and} \quad \psi_j(u) = \exp(u): \quad j = 1, 2, \ldots, k.$$

In this special case condition (1.12) is automatically satisfied and the generating function of the coefficients $C_j^{(m_1,m_2,\ldots,m_k)}$ is given by (1.15) (of course this time without the restriction (1.12)).

As a further special case let $f(t) = (1-t)^{-1-\beta}$, $F(t) = -t/(1-t)$; $f_j(t) = (1-t)^{-1-a_j}$ and $F_j(t) = -t/(1-t)$ $(j = 1, 2, \ldots, k)$. In this case $J(t) = -t/(1-t)$, and on some simplification we have a result that agrees with (1.2). In fact we could have obtained the following more general result, viz.,

$$\sum_{m_1,m_2,\ldots,m_k=0}^{\infty} E_j^{(m_1,m_2,\ldots,m_k)} u_1^{m_1} u_2^{m_2} \ldots u_k^{m_k}$$
$$= \frac{(1+u_1)^{1+a_1}(1+u_1)^{1+a_2}\ldots(1+u_k)^{a_k+1}}{(1-b_1u_1)(1-b_2u_2)\ldots(1-b_ku_k)} \frac{(a_1v_1+a_2v_2+\ldots+a_kv_k)^j}{(1+a_1v_1+a_2v_2+\ldots+a_kv_k)^{j+\beta+1}},$$

where

$$L_{m_1}^{(a_1+b_1m_1)}(a_1x) L_{m_2}^{(a_2+b_2m_2)}(a_2x) \ldots L_{m_k}^{(a_k+b_km_k)}(a_kx) = \sum_{j=0}^{m_1+m_2+\ldots+m_k} E_j^{(m_1,m_2,\ldots,m_k)} L_j^{(\beta)}(x),$$

and

$$v_j = u_j(1+v_j)^{b_j+1}, \qquad v_j(0) = 0.$$

For deriving this special case use is made of the following generating function for the index dependent on Laguerre polynomials due to Carlitz [2]:

$$\sum_{n=0}^{\infty} L_n^{(a+bn)}(x) t^n = \frac{(1+v)^{a+1}}{1-bv} \exp(-xv),$$

where b is some constant and v is a function of t defined as

$$v = t(1+v)^{b+1} \quad \text{and} \quad v(0) = 0.$$

Yet another interesting special case of this nature is obtained by choosing $f(t) = e^{-t^2}$, $F(t) = 2t$, $f_j(t) = e^{-t^2}$, $F_j(t) = 2t$ $(j = 1, 2, \ldots, k)$. In fact we find that if $H_n(x)$ are Hermite polynomials and

$$H_{m_1}(a_1x) H_{m_2}(a_2x) \ldots H_{m_k}(a_kx) = \sum_{j=0}^{m_1+m_2+\ldots+m_k} C_j^{(m_1,m_2,\ldots,m_k)} H_j(x),$$

then

$$\sum_{m_1,m_2,\ldots,m_k=0}^{\infty} C_j^{(m_1,m_2,\ldots,m_k)} \frac{u_1^{m_1}u_2^{m_2}\ldots u_k^{m_k}}{m_1!m_2!\ldots m_k!}$$
$$= \frac{(a_1u_1+a_2u_2+\ldots+a_ku_k)}{j!} \frac{e^{-(u_1^2+u_2^2+\ldots+u_k^2)}}{e^{-(a_1u_1+a_2u_2+\ldots+a_ku_k)^2}}.$$

It may be remarked that this result is different in nature to the one proved earlier by Carlitz [1] and it does not seem possible to derive the aforesaid result of Carlitz as an immediate special case of the result proved herein.

(ii) Products of Appell polynomials: If $\{f_n(x)\}$ and $\{f_{n_j}(x)\}$ are Appell type polynomials, then we necessarily have

$$\psi(u) = \exp u, \quad F(t) = t; \quad \psi_j(u) = \exp u, \quad F_j(t) = t \qquad \text{for each } j = 1, 2, \ldots, k,$$

whence the generating function of $C_j^{(m_1, m_2, \ldots, m_k)}$ is given by

$$\sum_{m_1, m_2, \ldots, m_k = 0}^{\infty} C_j^{(m_1, m_2, \ldots, m_k)} \frac{u_1^{m_1} u_2^{m_2} \ldots u_k^{m_k}}{m_1! m_2! \ldots m_k!} = \frac{f_1(u_1) f_2(u_2) \ldots f_k(u_k)}{j!} \frac{[a_1 u_1 + a_2 u_2 + \ldots + a_k u_k]^j}{f(a_1 u_1 + a_2 u_2 + \ldots + a_k u_k)}.$$

(iii) Setting $f(t) = (1+t^2)^{-\nu_1-\nu_2-\ldots-\nu_k}$, $F(t) = 2t/(1+t^2)$; $\psi(au) = (1-au)^{-\nu_1-\nu_2-\ldots-\nu_k}$, $f_j(t) = (1+t^2)^{-\nu_j}$, $F_j(t) = 2t/(1+t^2)$, $\psi_j(a_j u) = (1-au)^{-\nu_j}$ for each $j = 1, 2, \ldots, k$ it is easy to see that condition (1.12) is satisfied and we have the following interesting result:

If $\{C_n^\nu(x)\}$ are Gegenbaure polynomials and

$$C_{m_1}^{\nu_1}(ax) C_{m_2}^{\nu_2}(ax) \ldots C_{m_k}^{\nu_k}(ax) = \sum_{j=0}^{m_1+m_2+\ldots+m_k} E_j^{(m_1, m_2, \ldots, m_k)} C_j^{(\nu_1+\nu_2+\ldots+\nu_k)}(ax),$$

then

$$\sum_{m_1, m_2, \ldots, m_k = 0}^{\infty} E_j^{(m_1, m_2, \ldots, m_k)} u_1^{m_1} u_2^{m_2} \ldots u_k^{m_k} = (1+u_1^2)^{-\nu_1} (1+u_2^2)^{-\nu_2} \ldots (1+u_k^2)^{-\nu_k} \times$$
$$\times \frac{\left\{ J\left[2a\left(\frac{u_1}{1+u_1^2} + \frac{u_2}{1+u_2^2} + \ldots + \frac{u_k}{1+u_k^2} \right) \right] \right\}^j}{f\left[J\left(\frac{2au_1}{1+u_1^2} + \frac{2au_2}{1+u_2^2} + \ldots + \frac{2au_k}{1+u_k^2} \right) \right]},$$

where $J(x) = (1+\sqrt{1-x^2})/x$ and $f(t) = (1+t^2)^{-\nu_1-\nu_2-\ldots-\nu_k}$.

Specializing the ν_j's, further corresponding results for Legendre and Tchebycheff polynomials can be written out.

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