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A note on a mean ergodic theorem

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Abstract. Let H be a real, separable Hilbert space, and p a probability measure in H, satisfying $\int_H \|h\|^2 p(dh) < +\infty$. If U is a linear operator in H with $\|U^n\| < k$, for some constant k, for all $n > 1$, then via a strong law of large numbers, we prove that

$$\prod_{k=1}^{n^*} \frac{1}{n} U_p^k \to \delta_{x_0} \quad \text{weakly},$$

where $U_p^k(A) = p\,(U^k \in A)$ for Borel sets A in H, * denotes the usual operation of convolution, and δ_{x_0} is the probability concentrating on $x_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} U^k M$. Here $(M, h) = \int_H (g, h) p(dg)$, $h \in H$.

1. Introduction. Let H be a real, separable Hilbert space, and p a probability distribution in H (i. e. a normed regular measure defined on the Borel sets B of H) such that

$$\int_H \|h\|^2 p(dh) < \infty. \tag{1}$$

As in [5] the mathematical expectation E_p and the dispersion operator D_p of p are defined by the formulae:

$$(E_{p'}, h) = \int_H (g, h) p(dg), \qquad h \in H,$$

$$(D_p g, h) = \int_H (u - E_p, g)(u - E_p, h) p(du), \qquad g, h \in H.$$

If A is a linear operator in H we define:

$$A_p(Z) = p(A^{-1}Z), \qquad Z \in B.$$

DEFINITION 0. A sequence of probability distributions $\{p_n\}$ is said to be *weakly convergent to* p if

$$\int_H f(h)p_n(dh) \to \int_H f(h)p(dh)$$

for every real continuous bounded function f defined on H.

Recently, R. Jajte [3] proved the following theorem:

THEOREM. *Let U be a unitary operator in H, p a probability distribution in H satisfying* (1), *let M denote the mathematical expectation of p.*

Then the sequence of distributions

$$p_n = \prod_{k=1}^{n*} \frac{1}{n} U_p^k \tag{2}$$

converges weakly to a one point distribution δ_{x_0}, where

$$x_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} U^k M \tag{3}$$

*and * denotes the usual operation of convolution.*

The limit (3) exists by the ergodic theorem of J. V. Neumann. Thus the result can be regarded as a generalization of the ergodic theorem of J. V. Neumann if one observes that (2) reduces to (3) when $p = \delta_M$.

We must remark that the result was obtained by utilizing the properties of D_p and using some of the results in [5] as to the relation between the convergence of characteristic functionals and the convergence of their corresponding probability distributions.

It is known ([4], p. 54) that (3) remains true when U is a uniformly bounded linear operator (i. e. $\|U^n\| \leqslant k$, $n \geqslant 1$), where k is some constant, and in particular when U is a contraction (i. e. with norm less than or equal to one).

In this note we will show that Jajte's result is true for uniformly bounded linear operators. Our technique is entirely different from that used by R. Jajte. We shall consider a sequence of H-valued random variables associated with the sequence

$$\left\{p_n = \prod_{k=1}^{n*} \frac{1}{n} U_p^k\right\}.$$

We will prove a strong law of large numbers, from which follows the desired results.

2. Preliminaries. If X and Y are H-valued random variables, we write $X \sim Y$ to mean that X and Y have the same probability distribution.

On H, let X be the identity mapping of H to itself, with a probability distribution equal to p.

Let $X_0, X_1, X_2, \ldots$, be a sequence of independent H-valued random variables defined on some probability space

$$(\Omega, A, P) \text{ satisfying,}$$

$$X_j \sim U^j X, \quad j = 0, 1, 2, \ldots,$$

where $U^0 = I$ the identity operator.

Now we introduce some definitions.

DEFINITION 1. If X is an H-valued random variable, then $E X$, called the *expectation of* X, is defined by

$$(EX, h) = \int_{\Omega} (X, h) dP, \quad h \in H. \tag{4}$$

DEFINITION 2. If X is an H-valued random variable, then VX, called its *variance*, is defined by:

$$VX = \int_{\Omega} \|X - EX\|^2 dP. \tag{5}$$

3. Results. In what follows $X_0, X_1, \ldots$, are the same as in section 2. Also U will denote a uniformly bounded linear operator in H.

PROPOSITION 1. $EX_j = U^j EX_0$.

Proof. The existence follows from (1) (see [2], p. 77–80).

Now $EX_j = EU^j X_0$, since $X_0 \sim X$. On the other hand, $EU^j X_0 = U^j EX_0$ (see [2], p. 78), which completes the proof.

PROPOSITION 2. $VX_j \leqslant k^2 VX_0$, $j = 1, 2, \ldots$

Proof. Again the existence follows from (1).

Now $VX_j = VU^j X_0$, since $X_0 \sim X$

$$VX_j = \int_{\Omega} \|U^j X_0 - U^j EX_0\|^2 dP$$

by definition 2, and proposition 1.

According to the hypothesis $\|U^j\| \leqslant k, j \geqslant 1$; therefore

$$VX_j \leqslant k^2 VX_0, \quad j = 1, 2, \ldots$$

THEOREM 1.

$$\frac{X_1 + \ldots + X_n}{n} \to \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} U^k EX_0$$

with probability one.

Proof. Consider the sequence $X_1-EX_1, X_2-EX_2, \ldots$
By proposition 2

$$\sum_{j=1}^{+\infty} \frac{V(X_j-EX_j)}{j^2} < \infty,$$

so that (see [1], p. 47–48)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (X_j-EX_j) = 0 \quad \text{with probability one.}$$

Thus, by the completeness of H, the ergodic theorem for uniformly bounded operators ([4], p. 54), and proposition 1, we have:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} EX_j \quad \text{with probability one.}$$

The following theorem generalizes Jajte's result to the uniformly bounded case.

THEOREM 2.

$$\prod_{k=1}^{n^*} \frac{1}{n} U_p^k, \qquad n \geqslant 1$$

converges weakly to a one-point distribution δ_{x_0}, *where*

$$x_0 = \lim_{n\to 0} \frac{1}{n} \sum_{k=1}^{n} U^k EX_0.$$

Proof. As in the real valued case,

$$\prod_{k=1}^{n^*} \frac{1}{n} U_p^k$$

is the probability distribution of

$$\frac{X_1+\ldots+X_n}{n}.$$

So, by invoking the Lebesgue dominated convergence theorem, we have

$$\prod_{k=1}^{n^*} \frac{1}{n} U_p^k \text{ converges to } \delta_{x_0}, \text{ weakly (definition 0).}$$

where

$$x_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} U^k E X_0.$$

References

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