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On M. Kac's probabilistic formula for the solution of the telegraphist's equation

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Abstract. A new, more direct proof is given of M. Kac's probabilistic formula for the solution of the telegraphist's equation, connected with a random walk on R^1.

The solution of the Cauchy problem for the equation of a vibrating string

$$\frac{\partial^2 u(t,x)}{\partial t^2} = v^2 \frac{\partial^2 u(t,x)}{\partial x^2}, \qquad (t,x)\in R^2,$$

$$u(0,x) = u_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = u_1(x), \qquad x\in R^1,$$

is given by the d'Alembert formula

$$u(t,x) = \tfrac{1}{2}\big(u_0(x+vt)+u_0(x-vt)\big)+\tfrac{1}{2}\int_{-t}^{t} u_1(x+v\tau)\,d\tau .$$

An analogous formula for the solution of the Cauchy problem for the telegraphist's equation

$$\text{(1)}\qquad \begin{aligned} &\frac{\partial^2 u(t,x)}{\partial t^2} + 2a\frac{\partial u(t,x)}{\partial t} = v^2 \frac{\partial^2 u(t,x)}{\partial x^2}, \qquad (t,x)\in R^2,\\ &u(0,x) = u_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = u_1(x), \qquad x\in R^1, \end{aligned}$$

is

$$\text{(2)}\qquad u(t,x) = \frac{1}{2}\frac{\partial}{\partial t}\Big[e^{-at}\int_{-t}^{t} I_0(ai\sqrt{t^2-\tau^2})\,u_0(x+v\tau)\,d\tau + \\ + \tfrac{1}{2}e^{-at}\int_{-t}^{t} I_0(ai\sqrt{t^2-\tau^2})\,u_1(x+v\tau)\,d\tau\Big],$$

where i is the imaginary unit and I_0 is the Bessel function of the first kind and of the order zero (see Courant [1], Chapter VI, § 12, 6). M. Kac [3] exhibited a connection betwen the telegraphist's equation and a random walk on R^1 and proved that, for $t \geqslant 0$, the first member of the right-hand side of (2), which gives the solution of (1) for $u_1 \equiv 0$, is equal to

$$\tfrac{1}{2} E u_0\Big(x + v\int_0^t (-1)^{N_a(\tau)} d\tau\Big) + \tfrac{1}{2} E u_0\Big(x - v\int_0^t (-1)^{N_a(\tau)} d\tau\Big),$$

where E stands for the mean value of a random variable and $N_a(t)$, $t \geqslant 0$, is the homogeneous Poisson stochastic process with the mean value $EN_a(t) = at$.

In the original proof of his formula M. Kac uses the Laplace transformation and in an ingenious manner performs calculations on characterictic functions. Here we want to present a more direct proof of the Kac's formula, in a version extended to the full expression (2). Our method will employ the group-theoretical aspects of the formulas and therefore it will be natural to use the language of the theory of one-parameter groups and semigroups of linear operators.

1. A homogeneous Markov process with values in a non-commutative group. Let $N_a(t)$, $t \geqslant 0$, be the homogeneous Poisson stochastic process with the mean value $EN_a(t) = at$, $a = \text{const} > 0$. This process has independent increments and takes only the values $0, 1, \ldots$ with the probabilites

$$P\big(N_a(t) = n\big) = \frac{(at)^n}{n!} e^{-at}.$$

For any $t \geqslant 0$ consider the stochastic variables

$$(-1)^{N_a(t)} \quad \text{and} \quad \xi_a(t) = \int_0^t (-1)^{N_a(\tau)} d\tau.$$

They have the following interpretation. Suppose that a point of R^1 moves with velocity equal to 1 or -1, which changes at random in such a manner that the number of changes through the time interval from 0 to t is $N_a(t)$. If at the moment $t = 0$ the point is at the origin of R^1 and has the velocity $+1$, then $\xi_a(t_0)$ is its position at the moment $t = t_0$ and $(-1)^{N_a(t_0)}$ is its velocity at this moment. The natural phase space for such a motion is the non-commutative group $\mathscr{G}$ constituted by $R^1 \times \times \{1, -1\}$ under the multiplication rule of pairs

$$(\xi, k)(\eta, l) = (l\xi + \eta, kl), \qquad \xi, \eta \in R^1, \ k, l = \pm 1.$$

Let g_t, $t \geqslant 0$, be the stochastic process with values in $\mathscr{G}$ defined by the formula

$$g_t = \big(\xi_a(t), (-1)^{N_a(t)}\big).$$

For any non-negative t and h we have

$$g_{t+h}g_t^{-1} = \Big(\int_0^h (-1)^{N_a(t+\tau)-N_a(t)}d\tau, \, (-1)^{N_a(t+h)-N_a(t)}\Big),$$

from which we see that

1° the random variables $g_{t+h}g_t^{-1}$ and g_t are independent and that

2° the random variables $g_{t+h}g_t^{-1}$ and g_h have the same distribution.

This means that $g_t, t \geqslant 0$, is a homogeneous Markov process with values in the group $\mathscr{G}$.

For any $t \geqslant 0$ let μ_t be the probabilistic Borel measure on $\mathscr{G}$ defined by the formula

$$\mu_t(B) = P(g_t \in B),$$

where B is an arbitrary Borel subset of $\mathscr{G}$. The measures μ_t have compact supports, namely

$$\operatorname{supp}\mu_t \subset \{(\xi, k): \, |\xi| \leqslant t, \; k = \pm 1\}, \tag{3}$$

and we have

$$\mu_t = e^{-at}\delta_{(t,1)} + \nu_t, \tag{4}$$

where $\delta_{(t,1)}$ is the unit mass at the point $(t, 1) \in \mathscr{G}$ and ν_t is a non-negative Borel measure on $\mathscr{G}$ such that

$$\int_{\mathscr{G}} \nu_t(dg) = P\big(N_a(t) \geqslant 1\big) = 1 - e^{-at}. \tag{5}$$

Moreover, it follows from 1° and 2° that

$$\mu_{t+s} = \mu_t * \mu_s, \qquad t, s \geqslant 0, \tag{6}$$

where $*$ denotes the convolution of measures on $\mathscr{G}$ (see Grenander [2], § 2.2).

2. Perturbation of a one-parameter group of diagonal operators by means of the process g_t. Let $G(\xi)$, $-\infty < \xi < \infty$, be a one-parameter strongly continuous group of continuous linear automorphisms of a Banach space E. Let $v = \text{const} > 0$ and let U be the representation of the group $\mathscr{G}$ by continuous linear automorphisms of $E \times E$, defined by the formula

$$U(\xi, k)\cdot(\varphi, \psi) = \begin{cases} \big(G(v\xi)\varphi, G(-v\xi)\psi\big) & \text{if } k = 1, \\ \big(G(-v\xi)\psi, G(v\xi)\varphi\big) & \text{if } k = -1, \end{cases}$$

where (ξ, k) and $(\varphi, \psi) \in E \times E$. It follows from (3) that, for any $t \geqslant 0$, the formula

$$S(t)\Phi = \int_{\mathscr{G}} U(g)\Phi\mu_t(dg), \qquad \Phi \in E \times E,$$

defines a continuous linear operator of $E\times E$ into itself. Moreover, from the fact that U is a strongly continuous representation of $\mathscr{G}$ and from relations (3)–(6) it follows that $S(t), t\geqslant 0$, is a one-parameter strongly continuous semi-group of operators in $E\times E$. The crucial point in our proof of M. Kac's formula is the following

LEMMA 1. *Let $\mathscr{B}$ be the infinitesimal generator of the group $G(\xi)$ and let $D(\mathscr{B})$ be the domain of $\mathscr{B}$. Let $\mathscr{A}$ be the infinitesimal generator of the semi-group $S(t)$ and let $D(\mathscr{A})$ be the domain of $\mathscr{A}$. Then*

$$D(\mathscr{A}) = D(\mathscr{B})\times D(\mathscr{B})$$

and

$$\mathscr{A}(\varphi,\psi) = \big(v\mathscr{B}\varphi+a(\psi-\varphi);\; -v\mathscr{B}\psi+a(\varphi-\psi)\big)$$

for any pair $(\varphi,\psi)\in D(\mathscr{B})\times D(\mathscr{B})$.

Proof. For any $t\geqslant 0$ define the Borel measures μ_t^+ and μ_t^- on R^1 by the formulas

$$\mu_t^+(Z) = \mu_t\big((Z,1)\big)-e^{-at}\delta_t(Z),\qquad \mu_t^-(Z) = \mu_t\big((-Z,-1)\big). \tag{7}$$

Then

$$\begin{aligned} S(t)\cdot(\varphi,\psi) &= e^{-at}\big(G(vt)\varphi, G(-vt)\psi\big)+\\ &\quad+\Big(\int_{R^1} G(v\xi)(\varphi\mu_t^+ +\psi\mu_t^-)(d\xi),\ \int_{R^1} G(-v\xi)(\psi\mu_t^+ +\varphi\mu_t^-)(d\xi)\Big), \end{aligned}\tag{8}$$

so that the lemma will follow if we show that

$$\lim_{t\to+0}\frac{1}{t}\int_{R^1} G(\pm v\xi)\varphi\mu_t^+(d\xi) = 0$$

and

$$\lim_{t\to+0}\frac{1}{t}\int_{R^1} G(\pm v\xi)\varphi\mu_t^-(d\xi) = a\varphi$$

in the sense of the norm topology in E, for any $\varphi\in E$. But the above two relations follow from the fact that

$$\operatorname{supp}\mu_t^\pm \subset [-t,t],$$

$$\int_{R^1}\mu_t^+(d\xi) = P\big(N_a(t) = 2,4,\ldots\big) = e^{-at}\sum_{k=1}^{\infty}\frac{(at)^{2k}}{(2k)!} = o(t)\quad \text{as } t\to+0,$$

and

$$\int_{R^1}\mu_t^-(d\xi) = P\big(N_a(t) = 1,3,\ldots\big) = e^{-at}\sum_{k=1}^{\infty}\frac{(at)^{2k-1}}{(2k-1)!} = at+o(t)\quad \text{as } t\to+0.$$

Lemma 1 is proved.

Let us represent the operators $S(t)$, $t \geqslant 0$, as the matrices

(9) $$S(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix},$$

whose elements S_{ij} are continuous linear operators of E into itself. Since $D(\mathscr{A}) = D(\mathscr{B}) \times D(\mathscr{B})$, the infinitesimal generator of the semigroup $S(t)$ may also be represented in the matricial form

$$\mathscr{A} = \begin{pmatrix} v\mathscr{B} - a & a \\ a & -v\mathscr{B} - a \end{pmatrix} = \begin{pmatrix} v\mathscr{B} & 0 \\ 0 & -v\mathscr{B} \end{pmatrix} + \begin{pmatrix} -a & a \\ a & -a \end{pmatrix}.$$

On the right-hand side of this formula the operator $\begin{pmatrix} v\mathscr{B} & 0 \\ 0 & -v\mathscr{B} \end{pmatrix}$, defined on $D(\mathscr{B}) \times D(\mathscr{B})$, is the infinitesimal generator of the one-parameter strongly continuous group

$$T(t) = \begin{pmatrix} G(vt) & 0 \\ 0 & G(-vt) \end{pmatrix}, \quad t \in R^1,$$

of diagonal automorphisms of $E \times E$, while $\begin{pmatrix} -a & a \\ a & -a \end{pmatrix}$ is a continuous linear operator of $E \times E$ into itself. Thus, by the Phillips perturbation theorem (see [6]), $\mathscr{A}$ is the infinitesimal generator of a strongly continuous one-parameter group of continuous linear automorphisms of $E \times E$. Therefore the semigroup $S(t)$, $t \geqslant 0$, may be extended (of course in a unique manner) to a one-parameter group. For the operators of this group we shall preserve the symbols $S(t)$ and $S_{ij}(t)$, but now t may be an arbitrary real number.

Consider the operator-valued function

$$V(t) = \tfrac{1}{2}\big(S_{11}(t) + S_{12}(t) + S_{21}(t) + S_{22}(t)\big), \quad t \in R^1.$$

From formulas (7), (8) and (9) we see that, for $t \geqslant 0$, we have

$$V(t)\varphi = \tfrac{1}{2} \int_{R^1} \big(G(v\xi) + G(-v\xi)\big)\varphi m_t(d\xi), \quad \varphi \in E,$$

where, for any $t \geqslant 0$, m_t is the probabilistic Borel measure on R^1 defined by

(10) $$m_t(Z) = \mu_t(Z \times \{1, -1\})$$

for any Borel subset T of R^1. In other words, for $t \geqslant 0$ we have

$$V(t) = E\tfrac{1}{2}\Big(G\big(v\xi_a(t)\big) + G\big(-v\xi_a(t)\big)\Big).$$

An easy consequence of Lemma 1 is the following

LEMMA 2. *Let $D(\mathscr{B}^2)$ be the domain of the square $\mathscr{B}^2$ of the infinitesimal generator $\mathscr{B}$ of the one-parameter group $G(\xi)$. Then $V(t)D(\mathscr{B}^2) \subset D(\mathscr{B}^2)$*

for every $t \in R^1$ and, moreover, if $\varphi \in D(\mathscr{B}^2)$, then $V(t)\varphi$ is an E-valued function of t, twice strongly continuously differentiable on R^1 and such that

$$\frac{d^2}{dt^2} V(t)\varphi + 2a \frac{d}{dt} V(t)\varphi = v^2 \mathscr{B}^2 V(t)\varphi, \quad t \in R^1,$$

$$V(0)\varphi = \varphi, \quad \frac{d}{dt}\Big|_{t=0} V(t)\varphi = 0.$$

Proof. If $S(t)$ is represented as the matrix (9), then

$$V(t)\varphi = (\tfrac{1}{2}, \tfrac{1}{2}) S(t) \begin{pmatrix} \varphi \\ \varphi \end{pmatrix} \tag{11}$$

in the sense of the common rule of multiplication of matrices. The operator $\mathscr{A}^2$ has the domain

$$D(\mathscr{A}^2) = D(\mathscr{B}) \times D(\mathscr{B}) \tag{12}$$

and is represented by the matrix

$$\mathscr{A}^2 = \begin{pmatrix} v^2\mathscr{B}^2 - 2av\mathscr{B} + 2a^2 & -2a^2 \\ -2a^2 & v^2\mathscr{B}^2 - 2av\mathscr{B} + 2a^2 \end{pmatrix},$$

so that $\mathscr{A}^2 + 2a\mathscr{A} = \begin{pmatrix} v^2\mathscr{B}^2 & 0 \\ 0 & v^2\mathscr{B}^2 \end{pmatrix}$ and

$$(\tfrac{1}{2}, \tfrac{1}{2})(\mathscr{A}^2 + 2a\mathscr{A}) = v^2\mathscr{B}^2 (\tfrac{1}{2}, \tfrac{1}{2}), \tag{13}$$

where both sides are defined on $D(\mathscr{B}^2) \times D(\mathscr{B}^2)$.

According to a well-known semi-group theoretical lemma we have $S(t)D(\mathscr{A}^n) = D(\mathscr{A}^n)$ for every $t \in R^1$ and $n = 1, 2, \ldots$ and, moreover, if $\Phi = (\varphi, \psi) \in D(\mathscr{A}^n)$, then $S(t)\Phi$ is an $E \times E$-valued function of t, n times strongly continuously differentiable on R^1 and such that

$$\frac{d^n}{dt^n} S(t)\Phi = \mathscr{A}^n S(t)\Phi. \tag{14}$$

If $\varphi \in D(\mathscr{B}^2)$, then, by (11) and (12), $V(t)\varphi$ is an E-valued function of t, twice strongly continuously differentiable on R^1 and, by (11), (14) and (13) we have

$$\frac{d^2}{dt^2} V(t)\varphi + 2a \frac{d}{dt} V(t)\varphi = (\tfrac{1}{2}, \tfrac{1}{2})(\mathscr{A}^2 + 2a\mathscr{A}) S(t) \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$$
$$= v^2\mathscr{B}^2 (\tfrac{1}{2}, \tfrac{1}{2}) S(t) \begin{pmatrix} \varphi \\ \varphi \end{pmatrix} = v^2\mathscr{B}^2 V(t)$$

for any $t \in R^1$. Moreover,

$$V(0)\varphi = (\tfrac{1}{2}, \tfrac{1}{2}) S(0) \begin{pmatrix} \varphi \\ \varphi \end{pmatrix} = (\tfrac{1}{2}, \tfrac{1}{2}) \begin{pmatrix} \varphi \\ \varphi \end{pmatrix} = \varphi$$

and

$$\left.\frac{d}{dt}\right|_{t=0} V(t)\varphi = (\tfrac{1}{2}, \tfrac{1}{2})\mathscr{A}\begin{pmatrix}\varphi\\ \varphi\end{pmatrix} = 0.$$

This completes the proof.

The Kac's formula for the solution of the telegraphist's equation follows immediately from Lemma 2 if we choose as $G(\xi)$ the group of left translations in a Banach space of functions on R^1 such that the group of translations is strongly continuous in that space. Indeed, in this case Lemma 2 states that the Kac expression is the solution of the Cauchy problem (1) for the telegraphist's equation with $u_1(x) \equiv 0$.

In order to derive an analogous formula for the case of arbitrary $u_1(x)$, we shall need some other implications of Lemma 1. Namely, consider the operator-valued function

$$W(t) = \tfrac{1}{2}\big(S_{11}(t) - S_{12}(t) + S_{21}(t) - S_{22}(t)\big) = (\tfrac{1}{2}, \tfrac{1}{2})S(t)\begin{pmatrix}1\\ -1\end{pmatrix}.$$

It follows from (7), (8) and (10) that

$$W(t)\varphi = \tfrac{1}{2}\int_{R^1}\big(G(v\xi) - G(-v\xi)\big)\varphi m_t(d\xi) = \tfrac{1}{2}E\Big(G\big(v\xi_a(t)\big)\varphi - G\big(-v\xi_a(t)\big)\varphi\Big)$$

and, moreover, by the same argumentation as in the case of Lemma 2, it is easy to prove the following

LEMMA 3. *For any $t\in R^1$ we have $W(t)D(\mathscr{B}^2) \subset D(\mathscr{B}^2)$ and, if $\varphi\in D(\mathscr{B}^2)$, then $W(t)\varphi$ is an E-valued function of t, twice strongly continuously differentiable on R^1 and such that*

$$\frac{d^2}{dt^2}W(t)\varphi + 2a\frac{d}{dt}W(t)\varphi = v^2\mathscr{B}^2 W(t)\varphi, \qquad t\in R^1,$$

$$W(0)\varphi = 0, \qquad \left.\frac{d}{dt}\right|_{t=0} W(t)\varphi = v\mathscr{B}\varphi.$$

3. Cosine operator functions and the Kac's formula for the second order Cauchy problem in a Banach space. Let E_0 be a Banach space. Consider the Cauchy problem

$$\begin{aligned}&\varphi''(t) + 2a\varphi'(t) = v^2 A\varphi(t),\\ &\varphi(0) = \varphi_0, \quad \varphi'(0) = \varphi_1,\end{aligned} \tag{15}$$

where A is a linear operator from E_0 into E_0 with the domain $D(A)$ dense in E_0, v and a are positive constants, the initial data φ_0 and φ_1 are elements of E_0 and the solution $\varphi(t)$ is an E_0-valued function of the real variable t.

We want to show how the solution of (15) may be obtained, by means of a generalized Kac formula, from the solution of the Cauchy problem

$$\varphi''(t) = A\varphi(t), \qquad \varphi(0) = \varphi_0, \quad \varphi'(0) = 0. \tag{16}$$

3.1. The connection of problem (16) with cosine operator functions. We shall assume that problem (16) is well posed in the following sense: there are real constants $T > 0$ and $C \geqslant 1$ such that

1° for any $\varphi_0 \in D(A)$ problem (16) has exactly one solution $\varphi(t) = \varphi(t; \varphi_0)$ twice strongly continuously differentiable on $[-T, T]$, whose restriction is the unique twice strongly continuously differentiable solution of (16) on every interval $[T_1, T_2]$, such that $-T \leqslant T_1 \leqslant 0 \leqslant T_2 \leqslant T$

and

2° $\|\varphi(t; \varphi_0)\| \leqslant C\,\|\varphi_0\|$ for every $t \in [-T, T]$ and $\varphi_0 \in D(A)$.

It follows from 1° and 2° that there is a unique strongly continuous operator-valued function $[-T, T] \ni t \to \mathscr{C}(t) \in \mathscr{L}(E_0; E_0)$ such that

$$\mathscr{C}(t)\varphi_0 = \varphi(t; \varphi_0) \quad \text{for } \varphi_0 \in D(A) \text{ and } t \in [-T, T]. \tag{17}$$

We obviously have $\mathscr{C}(0) = 1$, $\mathscr{C}(t) = \mathscr{C}(-t)$ and, moreover, $\mathscr{C}(t)$ satisfies for $|t| + |s| \leqslant T$ the d'Alembert functional equation (18)

$$\mathscr{C}(t+s) + \mathscr{C}(t-s) = 2\mathscr{C}(t)\mathscr{C}(s). \tag{18}$$

Indeed, since $D(A)$ is dense in E_0, in order to prove (18) it is sufficient to show that, for any $s \in [-T, T]$ and any $\varphi_0 \in D(A)$ fixed, the E_0-valued function $\psi(t) = \mathscr{C}(t+s)\varphi_0 + \mathscr{C}(t-s)\varphi_0 - 2\mathscr{C}(t)\mathscr{C}(s)\varphi_0$ vanishes identically on $[-T+|s|, T-|s|]$. We obviously have $\psi(0) = 0$ and since, by (17), $\mathscr{C}(s)D(A) \subset D(A)$, we have $\left.\frac{d}{dt}\right|_{t=0} \mathscr{C}(t)\mathscr{C}(s)\varphi_0 = 0$, so that

$$\psi'(0) = \left.\frac{d}{dt}\right|_{t=0} \big(\mathscr{C}(t+s)\varphi_0 + \mathscr{C}(t-s)\varphi_0\big) = 0,$$

since $\mathscr{C}(t) = \mathscr{C}(-t)$. Moreover, $\psi''(t) = A\psi(t)$. Form all the above properties of $\psi(t)$ it follows by 1° that $\psi(t) \equiv 0$. The d'Alembert identity (18) is proved for $|t| + |s| \leqslant T$.

We shall extend $\mathscr{C}(t)$ to a $\mathscr{L}(E_0; E_0)$-valued function defined on the whole R^1, assuming that, for $t \in (0, T]$ and $n = 1, 2, \ldots$,

$$\mathscr{C}(nT+t) = 2\mathscr{C}(t)\mathscr{C}(nT) - \mathscr{C}(nT-t) \tag{19}$$

and that $\mathscr{C}(t) = \mathscr{C}(-t)$. As is easy to see, the extended function is strongly continuous on R^1 and we have $\mathscr{C}(t)D(A) \subset D(A)$ for every $t \in R^1$. If $\varphi_0 \in D(A)$, then it follows by induction in n that on any of the intervals $[-T, T]$, $(nT, (n+1)T]$ and $[-(n+1)T, -nT)$, $n = 1, 2, \ldots$, the function $t \to \mathscr{C}(t)\varphi_0$ is twice strongly continuously differentiable and that $\frac{d^2}{dt^2}\mathscr{C}(t)\varphi_0 = A\mathscr{C}(t)\varphi_0$ on any of these intervals. Moreover, if $\varphi_0 \in D(A)$, then, by (19), for every $n = 1, 2, \ldots$ we have

$$\lim_{t\to+0}\frac{d}{dt}\mathscr{C}(nT+t)\varphi_0 = 2\frac{d}{dt}\Big|_{t=0}\mathscr{C}(t)\mathscr{C}(nT)\varphi_0 + D_-\mathscr{C}(nT)\varphi_0 = D_-\mathscr{C}(nT)\varphi_0$$

and

$$\lim_{t\to+0}\frac{d^2}{dt^2}\mathscr{C}(nT+t)\varphi_0 = 2A\mathscr{C}(nT)\varphi_0 - D_-^2\mathscr{C}(nT)\varphi_0 = D_-^2\mathscr{C}(nT)\varphi_0,$$

where D_- stands for the left-hand derivative. Together with the strong continuity of the function $t \to \mathscr{C}(t)\varphi_0$ on $[0, \infty)$, these equalities imply that, if $\varphi_0 \in D(A)$, then the function $t \to \mathscr{C}(t)\varphi_0$ is twice strongly continuously differentiable on $[0, \infty)$ and that

$$\frac{d^2}{dt^2}\mathscr{C}(t)\varphi_0 = A\mathscr{C}(t)\varphi_0 \quad \text{for } t \in [0, \infty).$$

Since $\mathscr{C}(t) = \mathscr{C}(-t)$, the same is true for the negative halfaxis $(-\infty, 0]$. Therefore, repeating the argumentation used above in the proof of (18) for $|s| + |t| \leqslant T$, we infer that (18) holds for arbitrary real t and s.

According to the terminology introduced by M. Sova [7], any operator-valued solution $\mathscr{C}(t)$ of the d'Alembert functional equation defined on R^1 and such that $\mathscr{C}(0)$ is the identity operator is called an *operator cosine function*. Thus, starting from the Cauchy problem (16) and from assumptions 1° and 2°, we have constructed a $\mathscr{L}(E_0; E_0)$-valued strongly continuous cosine function $\mathscr{C}(t)$. Following Sova, we define the infinitesimal generator A_0 of $\mathscr{C}(t)$ as the linear operator from E_0 into E_0 such that

$$A_0\varphi = \lim_{t\to 0} 2t^{-2}\big(\mathscr{C}(t)\varphi - \varphi\big) \tag{19}$$

for any φ in the domain $D(A_0)$ of A_0, which consists of all the elements φ of E_0 such that the limit in the right-hand side of (19) exists in the sense of the norm in E_0. With our construction of the cosine operator function $\mathscr{C}(t)$ we clearly have

$$A \subset A_0. \tag{20}$$

The operator A_0, as the infinitesimal generator of a strongly continuous cosine operator function, is closed and, moreover, for any $t \in R^1$, we have $\mathscr{C}(t)D(A_0) \subset D(A_0)$, and if $\varphi \in D(A_0)$, then $\mathscr{C}(t)\varphi$ is an E-valued function of t, twice strongly continuously differentiable on R^1 and such that

$$\frac{d^2}{dt^2}\mathscr{C}(t)\varphi = A_0\mathscr{C}(t)\varphi = \mathscr{C}(t)A_0\varphi \tag{21}$$

for any $t \in R^1$ (see [7]). Consequently it follows from (20) that the operator A is closable, and for its closure $\bar{A}$ we have the inclusion

$$\bar{A} \subset A_0. \tag{22}$$

As we have sen, $\mathscr{C}(t)D(A) \subset D(A)$ for any $t \in R^1$ and therefore, by (21), if $\varphi \in D(A)$, then $A\mathscr{C}(t)\varphi = A_0\mathscr{C}(t)\varphi = \mathscr{C}(t)A_0\varphi = \mathscr{C}(t)A\varphi$. From the former properties of A it easily follows that

$$\mathscr{C}(t)D(\bar{A}) \subset D(\bar{A}) \tag{23}$$

for every $t \in R^1$. Now, the inclusions (22) and (23) imply that $\bar{A} = A_0$ (see [4], p. 95, Lemma 1.3.3). Thus we have the following

LEMMA 4. *If the domain $D(A)$ of the operator A is dense in the Banach space E_0 and if the Cauchy problem* (16) *is well posed in the sense that* 1° *and* 2° *holds, then the operator A has the closure $\bar{A}$, which is the infinitesimal generator of a strongly continuous $\mathscr{C}(E_0, E_0)$-valued cosine function.*

If we modify the Cauchy problem (16) substituting $\bar{A}$ in place of A, then conditions 1° and 2° remain valid. Indeed, the existence part of 1° follows from (21) and the uniqueness part of 1° follows by a reasoning given in [4], p. 93. Property 2° follows also from (21). Consequently, if we consider the Cauchy problem (16) under assumptions 1° and 2° and under the assumption that

3° $D(A)$ is dense in E_0,

then, without essential loss of generality, we may assume that

4° the operator A is closed.

But the conjunction $1^\circ \cap 2^\circ \cap 3^\circ \cap 4^\circ$ is equivalent to the assumption that A is the infinitesimal generator of a strongly continuous $\mathscr{C}(E_0, E_0)$-valued cosine function.

3.2. The one-parameter group constructed from a cosine operator function. Let E_0 be a Banach space and let $\mathscr{C}(t)$ be a strongly continuous $\mathscr{C}(E_0; E_0)$-valued cosine function with the infinitesimal generator A. Let E_1 be the set of all the elements φ of E_0 such that $\mathscr{C}(t)\varphi$ is an E_0-valued function strongly continuously differentiable on R^1. By (21) we have $D(A) \subset E_1$ and, by the main theorem of [4], E_1 under the norm

$$\|\varphi\|_{E_1} = \|\varphi\|_{E_0} + \sup_{0 \leqslant t \leqslant 1} \left\| \frac{d}{dt} \mathscr{C}(t)\varphi \right\|_{E_0}$$

is a Banach space. Consider the product $E = E_1 \times E_0$, represent the elements of E as the columns $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$, $\varphi \in E_1$, $\psi \in E_0$ and represent the linear operators B from E into E with domains of the type $D(B) = D_1 \times D_0$,

$D_i \in E_i$, as matrices

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where B_{ik} maps D_{2-k} into E_{2-i}. According to the same main theorem of [4], the operator B with domain $D(B) = D(A) \times E_1$, represented by the matrix

$$B = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix},$$

is the infinitesimal generator of a strongly continuous one-parameter group $\{G(t)\colon t \in R^1\} \subset \mathscr{C}(E; E)$, whose operators are represented by the matrices

$$G(t) = \begin{pmatrix} \mathscr{C}(t) & \int_0^t \mathscr{C}(\tau)\,d\tau \\ \dfrac{d}{dt}\mathscr{C}(t) & \mathscr{C}(t) \end{pmatrix}.$$

3.3. The Kac formula for the solution of (15). Let E_0 be a Banach space and let A be the infinitesimal generator of a strongly continuous $\mathscr{C}(E_0; E_0)$-valued cosine function $\mathscr{C}(t)$. Define the Banach space E_1, continuously imbedded in E_0, as in Section 3.2. Let a and v be positive constants. Let $N_a(t)$, $t \geqslant 0$, be a homogeneous Poisson process with the mean value $EN_a(t) = at$ and let

$$\xi_a(t) = \int_0^t (-1)^{N_a(\tau)}\,d\tau.$$

THEOREM. *Under the above assumptions for any $\varphi_0 \in D(A)$ and $\varphi_1 \in E_1$ the Cauchy problem* (15) *has the solution $\varphi(t)$, which is an E_0-valued function twice strongly continuously differentiable on R^1. This solution is unique on any interval $[T_1, T_2]$, $T_1 \leqslant 0 \leqslant T_2$, in the class of E_0-valued twice strongly continuously differentiable functions on $[T_1, T_2]$. Moreover, for $t \geqslant 0$ this solution is expressed by the formula*

$$\varphi(t) = E\mathscr{C}\big(v\xi_a(t)\big)\varphi_0 + E\int_0^{\xi_a(t)} \mathscr{C}(v\tau)\varphi_1\,d\tau. \tag{24}$$

Proof. *Existence of the solution.* According to Section 3.2, the operator $B = \begin{pmatrix} 0 & 1 \\ v^2 A & 0 \end{pmatrix}$ with the domain $D(B) = D(A) \times E_1$ generates the strongly continuous one-parameter group

$$G(t) = \begin{pmatrix} \mathscr{C}(vt) & \int_0^t \mathscr{C}(v\tau)\,d\tau \\ \dfrac{d}{dt}\mathscr{C}(vt) & \mathscr{C}(vt) \end{pmatrix}, \quad t \in R^1, \tag{25}$$

of continuous linear automorphisms of the Banach space $E = E_1 \times E_0$. The operator

$$B_{v,a} = \begin{pmatrix} 0 & 1 \\ v^2 A & -2a \end{pmatrix}$$

with the domain $D(B_{v,a}) = D(A) \times E_1$ is the sum of B and of the bounded linear operator $\begin{pmatrix} 0 & 0 \\ 0 & -2a \end{pmatrix}$ of E into E. Therefore, by the Phillips perturbation theorem (see [6]), $B_{v,a}$ is also the infinitesimal generator of a strongly continuous one-parameter group $G_{v,a}(t)$, $t \in R^1$, of continuous linear automorphisms of E. If $\varphi_0 \in D(A)$ and $\varphi_1 \in E_1$, then $\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \in D(B_{v,a})$ and the E_0-valued twice strongly continuously differentiable solution $\varphi(t)$ of (15) is given by the formula

$$\varphi(t) = (1, 0) G_{v,a}(t) \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}.$$

Uniqueness of the solution. Let $\varphi(t)$ be an E_0-valued solution of the equation $\varphi''(t) + 2a\varphi'(t) = v^2 A\varphi(t)$, twice strongly continuously differentiable on $[T_1, T_2]$, $T_1 \leqslant 0 \leqslant T_2$, and such that $\varphi(0) = \varphi'(0) = 0$. Let λ be a regular value of the operator A. Then $(\lambda - A)^{-1} \in \mathscr{L}(E_0; E_0)$ and its range is $D(A) \subset E_1$. Therefore $(\lambda - A)^{-1}$ is a closed operator on the Banach space E_0 with values in the Banach space E_1, and consequently, by the closed graph theorem, $(\lambda - A)^{-1} \in \mathscr{L}(E_0; E_1)$. Put

$$\Phi(t) = \begin{pmatrix} (\lambda - A)^{-1} \varphi(t) \\ (\lambda - A)^{-1} \varphi'(t) \end{pmatrix}.$$

Then $\Phi(t)$ is an E-valued function continuously differentiable on $[T_1, T_2]$ and such that $\Phi'(t) = B_{v,a}\Phi(t)$ on $[T_1, T_2]$ and that $\Phi(0) = 0$. Since $B_{v,a}$ is the infinitesimal generator of a strongly continuous one-parameter group of continuous linear automorphisms of E, it follows that $\Phi(t) \equiv 0$ on $[T_1, T_2]$ (see [5], p. 212). But this imples that $\varphi(t) \equiv 0$ on $[T_1, T_2]$.

Formula (24). For the group $G(t)$ defined by (25) consider the $\mathscr{L}(E; E)$-valued functions $V(t)$ and $W(t)$, defined as in Section 2. The square B^2 of the infinitesimal generator $B = \begin{pmatrix} 0 & 1 \\ v^2 A & 0 \end{pmatrix}$ of the group (25) has the domain

$$D(B^2) = D(A_1) \times D(A)$$

and is represented by the matrix

$$B^2 = \begin{pmatrix} v^2 A_1 & 0 \\ 0 & v^2 A \end{pmatrix}, \tag{26}$$

where A_1 is the restriction of A with the domain

$$D(A_1) = \{\varphi : \varphi \in D(A),\ A\varphi \in E_1\}.$$

Let $\tilde{\varphi}_0, \tilde{\varphi}_1 \in D(A)$; then $\begin{pmatrix} 0 \\ \tilde{\varphi}_i \end{pmatrix} \in D(B^2)$. Since $\mathscr{C}(t)$ is a pair function of t, it follows from (25) that, for $t \geqslant 0$,

$$V(t)\begin{pmatrix} 0 \\ \tilde{\varphi}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ E\mathscr{C}\big(v\xi\ (t)\big)\tilde{\varphi}_0 \end{pmatrix} \quad \text{and} \quad W(t)\begin{pmatrix} 0 \\ \tilde{\varphi}_1 \end{pmatrix} = \begin{pmatrix} E\int\limits_0^{\xi_a(t)} \mathscr{C}(v\tau)\tilde{\varphi}_1 d\tau \\ 0 \end{pmatrix}.$$

From (26) and from Lemmas 2 and 3 it follows that

(a) the function $\varphi(t) = E\mathscr{C}\big(v\xi_a(t)\big)\tilde{\varphi}_0$ is an E_0-valued solution of (15), twice strongly continuously differentiable on $[0, \infty)$ and corresponding to the initial data $\varphi_0 = \tilde{\varphi}_0$ and $\varphi_1 = 0$ and

(b) the function $\varphi(t) = E\int\limits_0^{\xi_a(t)} \mathscr{C}(v\tau)\varphi_1 d\tau$ is a solution of (15), E_1-valued twice strongly continuously differentiable on $[0, \infty)$ (and therefore also E_0-valued twice strongly continuously differentiable on $[0, \infty)$) and corresponding to the initial data $\varphi_0 = 0$ and $\varphi_1 = (1, 0)\, B\begin{pmatrix} 0 \\ \tilde{\varphi}_1 \end{pmatrix} = \tilde{\varphi}_1$.

Consequently, by the already complete existence and uniqueness parts of this proof, it follows that if $\varphi_0 \in D(A)$ and $\varphi_1 \in D(A)$, then, for $t \geqslant 0$, the E_0-valued twice strongly continuously differentiable solution of (15) is expressed by

$$\varphi(t) = (1, 0)G_{v,a}(t)\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = E\mathscr{C}\big(v\xi_a(t)\big)\varphi_0 + E\int\limits_0^{\xi_a(t)} \mathscr{C}(v\tau)\varphi_1 d\tau. \tag{27}$$

Since, for any fixed $t \geqslant 0$, the operators $E_0 \ni \varphi \rightarrow (1, 0)G_{v,a}(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in E_0$ and $E_0 \ni \varphi \rightarrow E\int\limits_0^{\xi_a(t)} \mathscr{C}(v\tau)\varphi d\tau \in E_0$ are continuous, and since $D(A)$ is dense in E_0, it follows that the right-hand equality in (27) remains true for $\varphi_0 \in D(A)$ and $\varphi_1 \in E_1$. But as we already know, in such a case $\varphi(t) = (1, 0) \times \times G_{v,a}(t)\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$ is an E_0-valued twice strongly continuously differentiable solution of (15). This completes the proof.

References

[1] R. Courant, *Partial differential equations*, 1962.
[2] U. Grenander, *Probabilities on algebraic structures*, 1968.
[3] K. Mac, *Some stochastic problems in physics and mathematics*, 1956.
[4] J. Kisyński, *On cosine operator functions and one parameter groups of operators*, Studia Math. 44 (1972), p. 93–105.
[5] K. Maurin, *Methods of Hilbert spaces*, 1967.
[6] R. S. Phillips, *Perturbation theory for semi-groups of linear operators*, Trans. Amer. Math. Soc. 74 (1953), p. 199–221.
[7] M. Sova, *Cosine operator functions*, Diss. Math. 49 (1966), p. 1–47.

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