ANNALES
POLONICI MATHEMATICI
XVI (1964)

Remarks on A. B. Turowicz's and Z. Mikołajska's notes on approximations to roots of positive numbers

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1. In a recent note [3] Turowicz proved the following result (called T, hereafter):

Let A and x_0 be positive and n be an integer >1. Then the sequence $\{x_r\}$ $(r \geqslant 0)$, where

$$x_{k+1} = \frac{(n-1)x_k^n+(n+1)A}{(n+1)x_k^n+(n-1)A}x_k \qquad (k \geqslant 0), \tag{1}$$

converges to $A^{1/n}$.

In another recent note [1] Mikołajska, using Turowicz's method, generalized his result to the following (called M, hereafter):

Let A and x_0 be positive. Then the sequence $\{x_r\}$, where

$$x_{k+1} = \frac{\psi(x_k)-\varphi(x_k^n-A)}{\psi(x_k)+\varphi(x_k^n+A)}x_k \qquad (k \geqslant 0), \tag{2}$$

converges to $A^{1/n}$ if, for non-negative x and real u, $\psi(x)$ and $\varphi(u)$ are continuous,

$$\psi(x)+\varphi(x^n-A) > 0, \tag{3}$$

$$\psi(x)-\frac{x+A^{1/n}}{x-A^{1/n}}\varphi(x^n-A) > 0 \qquad (x^n \neq A), \tag{4}$$

$$\varphi(u) = -\varphi(-u), \quad \varphi(u) \textit{ is monotonic and } \operatorname{sgn}\varphi(u) = \operatorname{sgn} u. \tag{5}$$

The condition $\operatorname{sgn}\varphi(u) = \operatorname{sgn} u$ is not explicitly mentioned by Mikołajska but it is used in the derivation of the set of equations (9) of her note [1].

In this note we shall prove the following result (called O, hereafter):

Let a and x_0 be positive. Then the sequence $\{x_r\}$, where

$$x_{k+1} = \frac{\psi(x_k)-\chi(x_k)}{\psi(x_k)+\chi(x_k)}x_k \qquad (k \geqslant 0), \tag{6}$$

converges to a, the unique positive zero of $\chi(x)$, if, for positive x,

(7) $$\psi(x) \text{ and } \chi(x) \text{ are continuous},$$

(8) $$\psi(x) > \chi(x)$$

and

(9) $$\psi(x) > \frac{a}{x-a}\chi(x) > 0 \qquad (x \neq a).$$

It is not difficult to prove that ***O*** is a generalization of ***M*** and thus of ***T*** too. For this purpose, set

(10) $$a = A^{1/n} \qquad \text{and} \qquad \chi(x) = \varphi(x^n - A)$$

and let the conditions in ***M*** be satisfied. Then we clearly have (6) from (2), and also (7) from the continuity of $\psi(x)$ and $\varphi(u)$.

From (3) to (5) and (10), we have, for positive x,

(11) $$\psi(x) + \chi(x) > 0,$$

(12) $$\psi(x) - \frac{x+a}{x-a}\chi(x) > 0 \qquad (x \neq a)$$

and

(13) $$\operatorname{sgn}\chi(x) = \operatorname{sgn}(x-a).$$

Adding (11) and (12) and using (13), we clearly have (9). We have $\psi(x) > -\chi(x)$, from (11); and $\chi(x) \leqslant 0$ for $x \leqslant a$, from (13). Thus

$$\psi(x) > -\chi(x) \geqslant 0 \geqslant \chi(x) \qquad (x \leqslant a).$$

Hence (8) is true for $x \leqslant a$. For $x > a$, (8) follows from (12) and (13).

Thus (6) to (9) are all satisfied. Hence ***O*** generalizes ***M*** and ***T***.

2. To prove ***O***, we first prove (11) and (13). From (7) and (9), we have (13) and, in particular,

(14) $$\chi(a) = 0.$$

For $x = a$, (11) follows from (8) and (14). Also

$$\psi(x) + \chi(x) > \frac{x}{x-a}\chi(x) > 0 \qquad (x \neq a),$$

from (9). Thus (11) is true. Note that (13) implies that a is the unique positive zero of $\chi(x)$.

We next prove that if $x_s = a$ for some $s \geqslant 0$ then $x_r = a$ for all $r \geqslant s$. This follows, by induction on r, since $x_k = a$ implies $x_{k+1} = a$ from (6), (11) and (14). Hence ***O*** follows if some $x_s = a$, the unique positive zero of $\chi(x)$.

In what follows we therefore assume that no $x_s = a$. From (6),

$$\frac{x_{k+1}-a}{x_k-a} = \frac{\psi(x_k)-\dfrac{x_k+a}{x_k-a}\chi(x_k)}{\psi(x_k)+\chi(x_k)} \quad (k \geqslant 0), \tag{15}$$

as may be verified. Thus

$$|x_{k+1}-a| < |x_k-a| \quad (k \geqslant 0) \tag{16}$$

if, for positive x,

$$-\psi(x)-\chi(x) < \psi(x)-\frac{x+a}{x-a}\chi(x) < \psi(x)+\chi(x) \quad (x \neq a). \tag{17}$$

But (17) follows from (9), since (17) is equivalent to

$$\psi(x) > \frac{a}{x-a}\chi(x) \quad \text{and} \quad \frac{x}{x-a}\chi(x) > 0 \quad (x \neq a).$$

Hence (16) is true. Thus $\{|x_r-a|\}$ is a positive decreasing sequence. Therefore it converges to a non-negative limit L. $\boldsymbol{O}$ follows if L is zero, since a is the unique positive zero of $\chi(x)$.

Suppose that L is non-zero. Then L is positive and the sequence $\{x_r\}$ has a subsequence $\{x_{u_r}\}$ $(0 \leqslant u_0 < u_1 < \ldots)$, which converges to a finite limit $c = a \pm L \neq a$. Also c is positive since the x_r and a are all positive and $\{|x_r-a|\}$ is a decreasing sequence.

The sequence $\{x_{u_r}\}$ converges to c, positive and not a. Hence using (7), (11), (15) and considerations of continuity, we obtain that the sequence $\{x_{u_r+1}\}$ converges to a limit d satisfying

$$\frac{d-a}{c-a} = \frac{\psi(c)-\dfrac{c+a}{c-a}\chi(c)}{\psi(c)+\chi(c)}.$$

Since c is positive and not a, it follows, in the same way as (16), that

$$|d-a| < |c-a|. \tag{18}$$

But the sequence $\{|x_r-a|\}$ converges to L and therefore its subsequences $\{|x_{u_r}-a|\}$ and $\{|x_{u_r+1}-a|\}$ also converge to L. Hence $|c-a| = L$ and $|d-a| = L$. This contradicts (18). Thus our supposition that L is non-zero is false. Hence L is zero.

3. Turowicz and Mikołajska proved that the sequences $\{x_r\}$, in $\boldsymbol{T}$ and $\boldsymbol{M}$, not merely converge, but converge monotonically, to $A^{1/n}$. The conditions assumed in $\boldsymbol{O}$ do not ensure that the sequence $\{x_r\}$, in $\boldsymbol{O}$, is monotonic. However, this can be ensured if we replace the conditions (8) and (9) by

$$\psi(a) > 0 \quad \text{and} \quad \psi(x) > \frac{x+a}{x-a}\chi(x) > 0 \quad (x \neq a). \tag{19}$$

To prove this, suppose that (8) and (9) in O have been replaced by (19). Clearly (19) implies (9); (7) and (19) imply (13); and (13) and (19) imply (8). Thus the conditions of O are all satisfied. Consequently, the sequence $\{x_r\}$ converges to a. Also, as in § 2, (11) and (15) are true. From (11), (15) and (19) we obtain that all the $x_r - a$ have the same sign. Thus, since the sequence $\{x_r\}$ converges to a, the convergence is monotonic.

4. In M, n is not restricted to be an integer while, in T, there is this restriction. We can remove it by calculus methods, noting that identification of (1), (2) and (6) yields

$$\psi(x) = n(x^n + A) \quad \text{and} \quad \varphi(x^n - A) = \chi(x) = x^n - A\,,$$

where $A = a^n$, and using M or our result of § 3.

Turowicz also proved that the sequence $\{x_r\}$, defined by

$$(20)\quad x_{k+1} = \frac{(n-1)(2n-1)x_k^{2n} + 2(4n^2-1)Ax_k^n + (n+1)(2n+1)A^2}{(n+1)(2n+1)x_k^{2n} + 2(4n^2-1)Ax_k^n + (n-1)(2n-1)A^2}\, x_k \quad (k \geqslant 0)\,,$$

where x_0 and A are positive and n is an integer >1, converges to $A^{1/n}$. For this sequence, identification of (2), (6) and (20) yields

$$\psi(x) = (2n^2+1)(x^{2n} + A^2) + 2(4n^2-1)Ax^n$$

and

$$\varphi(x^n - A) = \chi(x) = 3n(x^{2n} - A^2)\,,$$

where $A = a^n$. It is seen that $\varphi(u) = 3n(u^2 + 2Au) \neq -\varphi(-u)$. Thus (5) is not satisfied and M is inapplicable. However, we can apply our result of § 3 to prove that the sequence converges monotonically to $A^{1/n}$ for real (not merely integral) $n > 1$.

The extensions of Turowicz's results for non-integral n are not of practical importance since x_k^n has to be first computed to find x_{k+1}.

5. Turowicz and Mikołajska proved results on the rapidity of convergence of the sequences $\{x_r\}$ in T and M. Following them, using (15), we can easily prove, for the convergent sequence $\{x_r\}$ in O, the following:

Let $\inf_E |\psi(x) + \chi(x)| = \sigma > 0$, *where* $E = \{x\colon x > 0, |x-a| \leqslant |x_0 - a|\}$, *and* $\sup_E |(x-a)\psi(x) + (x+a)\chi(x)||x-a|^{-\nu} = \tau(\nu) < \infty$ (ν *real*). *Then*

$$|x_{k+1} - a| \leqslant \frac{\tau(\nu)}{\sigma} |x_k - a|^\nu \quad (k \geqslant 0)$$

and the convergence of $\{x_r\}$ *is of degree* $\geqslant \nu$. *Further, this convergence is of degree* $\geqslant N$ *if* N *is the largest* ν *for which* $\tau(\nu) < \infty$.

For the monotonically convergent sequence $\{x_r\}$ in our result of § 3, omitting the trivial case $x_0 = a$, we can improve the above by replacing the set E by the set $F = \left\{x\colon\ 0 < \frac{x-a}{x_0-a} \leqslant 1\right\}$.

6. Richmond [2] had, previously to Turowicz, studied the sequence in $\boldsymbol{T}$, given references to earlier work and stated that the case $n = 3$ was given by R. Burrow in his *Theory of Gunnery* published in 1779. Richmond also studied the sequence $\{x_r\}$, where x_0 is suitably chosen and

$$x_{k+1} = x_k - \frac{2\chi(x_k)\chi'(x_k)}{2\{\chi'(x_k)\}^2 - \chi(x_k)\chi''(x_k)} \qquad (k \geqslant 0)\,.$$

This sequence results from the application of Newton's method of approximation to find a zero of the function $f(x) = \chi(x)\,\{\chi'(x)\}^{-1/2}$. Richmond has shown that, under certain conditions, the convergence of this sequence to a zero of $\chi(x)$ is of degree 3. Further, if x_0 is positive, this sequence (i) results from that in $\boldsymbol{O}$ if

$$\psi(x) = \frac{x^{-1}\chi'(x)}{2\{\chi'(x)\}^2 - \chi(x)\chi''(x)} - \chi(x)$$

and (ii) as noted by Richmond, reduces to that in $\boldsymbol{T}$ if $\chi(x) = x^n - A$.

References

[1] Z. Mikołajska, *Remarque sur la note de A. B. Turowicz sur l'approximation des racines de nombres positifs*, Ann. Polon. Math. 8 (1960), pp. 285-289.

[2] H. W. Richmond, *On certain formulae for numerical approximation*, J. London Math. Soc. 19 (1944), pp. 31-38.

[3] A. B. Turowicz, *Sur l'approximation des racines de nombres positifs*, Ann. Polon. Math. 8 (1960), pp. 265-269.

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Reçu par la Rédaction le 2. 5. 1963